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The generalized knockout tournaments for an arbitrary number of participants in one match are designed. A combinatorial approach for generalized knockout tournament seedings is developed. Several properties of knockout tournament seedings are investigated. Several new knockout tournaments seedings are proposed and justified by the set of properties.

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1. Introduction[1]

There are two main types of sports tournaments: knockout tournaments and round robin tournaments. In knockout tournaments, after each round all losers are eliminated and all winners are promoted to the next round. In round robin tournaments, all participants play against each other. The main advantage of knockout tournaments is a sufficiently lower number of matches and rounds. As a result, the spectator interest increases from round to round. For example, if the number of participants in a tournament equals $N = 2^n$ then the number of games (with two players in one match) in a knockout tournament equals $2^n - 1$, and in a round robin tournament equals $2^{n-1}(2^n + 1)$, the number of rounds in a knockout tournament equals n, and in round robin tournament $2^n - 1$. The lower requirement for time, sports facilities, and increasing spectator interest are the main reasons for the popularity of knockout tournaments.

There are many single-winner games with a higher number of players in one match (e.g. some card games Blackjack, Poker). Football teams are typically divided into groups, with four teams in each group. A round robin subtournament, within one group, can be considered as one match with 4 teams.

Running tracks, swimming pools, bowling lanes and other sports facilities have limited capacities. It is not possible to organize one race for all athletes. Because of limited capacities of sports facilities, usually several rounds of races are organized: – e.g. a qualification round, regular races, the final. In cases of a high number of participants, the knockout tournament structure of the competition is applied. In our setting, the lane position does not matter. Only the set of race (match) participants matters. Real sports tournaments have own specific rules (e.g. not only relative, but also absolute results matter), but in this paper we develop a general theory of such tournaments, which can be applied for all tournaments.

This paper generalizes knockout tournaments model considering tournaments with the number of participants in one match higher than two. There are many ways of scheduling knockout tournaments. Different knockout tournament schedules are called *seedings* (assignment of players to tournament brackets, having

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information regarding the initial order of participants' strengths mainly from historical data). In computational social choice, knockout tournaments correspond with a voting tree or an agenda [Vassilevska Williams, 2016]. Finding a seeding with predefined properties is a combinatorial optimization problem, which is solved under different constraints [Dagaev, Suzdaltsev, 2017; Karpov, 2016].

In this paper, a combinatorial approach for generalized knockout tournament seedings is developed. We define several desirable properties of seedings and enumerate seedings, that satisfy these properties. Several new knockout tournament seedings are proposed and justified by the set of properties. Sports tournament organizers can easily apply the proposed seedings to real competitions.

Because of the novelty of the combinatorial object, all enumeration formulas are new. References to (OEIS) for sequences in the standard case of two participants in one match are quoted in the text.

The structure of the paper is as follows. Section 2 describes generalized knockout tournament seedings and its properties. Section 3 presents representation theorems for different seedings.

2. Framework

Let k be the number of participants in one match, n be the number of rounds and $X = \{1,2,\dots,k^n\}$ be the set of participants of the knockout tournament (henceforth in the text tournament). The indices of the participants represent the order of the participants' strengths, where participant 1 is the strongest and participant k^n is the weakest.

Knockout tournament seeding, or simply the seeding, is a hypergraph with k^n vertices labeled from 1 to k^n, described by a following set system (nested set system). There are k^{n-1} disjoint sets of k vertices (each such set is one match), k^{n-2} disjoint sets of k^2 vertices, such that each new set unites k sets of k vertices (each such set is a subtournament with two rounds), k^{n-3} disjoint sets of k^3 vertices, such that each new set unites k sets of k^2 vertices (each such set is a subtournament with three rounds), etc.

For example, a seeding of tournament with $k = 2$ participants in each match and $n = 3$ rounds is described by set system $\{1,4\},\{2,3\},\{5,8\}$, $\{6,7\},\{1,2,3,4\},\{5,6,7,8\},\{1,2,3,4,5,6,7,8\}$, but it is more convenient to describe this seeding as a nested set system, $\left\{\{\{1,4\},\{2,3\}\},\{\{5,8\},\{6,7\}\}\right\}$. This is called the *nested set representation of seeding.* There are two subtournaments

$\{\{1,4\},\{2,3\}\}$ and $\{\{5,8\},\{6,7\}\}$, each of them also contains two subtournaments. In each subsequent round, the winners of subtournaments meet. The order of sets inside subtournament does not matter. $\{\{\{1,4\},\{2,3\}\},\{\{5,8\},\{6,7\}\}\}$ and $\{\{\{1,4\},\{3,2\}\},\{\{6,7\},\{8,5\}\}\}$ represent the same seeding.

For $k = 2$, the most popular tournament seeding (called *standard seeding*) creates pairs in the first round of the strongest participant with the weakest participant, the second strongest participant with the second weakest participant, etc. The pairs in subsequent rounds are determined in a way that preserves the first two participants from the head-to-head match before the final and that delays the confrontations between other strong participants until later rounds. Strong participants are rewarded for their success through such a seeding. For $k = 2,\ n = 3$, it is $T_{2,3}^{\text{standard}} = \{\{\{1,8\},\{4,5\}\},\{\{2,7\},\{3,6\}\}\}$.

Each tournament with $n \geq 2$ rounds is a set which consists of k subtournaments. Each subtournament with $n \geq 2$ rounds is a set which also consists of k subtournaments. $T_{k,n}^{m,i}$ is a subtournament i with m rounds. It is a part of a tournament with n rounds. For notational convenience, let $T_{k,n}^{n} = T_{k,n}$ and $T_{k,n}^{0,i} = \{i\}$. Subtournaments $T_{k,n}^{m,i}, T_{k,n}^{m,j}$ are nonoverlapping if there is no participant that plays in both subtournaments. A tournament with n rounds is a set $T_{k,n} = \bigcup_{i=1}^{k} T_{k,n}^{n-1,i}$, where all subtournaments are nonoverlapping. A subtournament with m rounds is a set $T_{k,n}^{m,i} = \bigcup_{i=1}^{k} T_{k,n}^{m-1,i}$, where all subtournaments are nonoverlapping.

Let $\mathbb{T}_{k,n}$ be the set of all possible seedings with k participants in one match and n rounds. The cardinality of the set of all possible seedings is denoted as $\#\mathbb{T}_{k,n}$.

Proposition 1. *The number of seedings equals to*

$$\#\mathbb{T}_{k,n} = (k!)^{\frac{1-k^n}{k-1}}(k^n!). \tag{1}$$

Proof. There are $k^n!$ permutations of participants. Each permutation corresponds with the nested set representation of a seeding. There are $\sum_{i=0}^{n-1} k^i = \frac{1-k^n}{1-k}$ matches, subtournaments and tournament. For each of them, there are $k!$ permutations of participants (subtournaments), that do not change a tournament.

For $k = 2$, it is A067667 in (OEIS). Considering the tournament as the union of k subtournaments we obtain the recursive representation of formula (1):

$$\#\mathbb{T}_{k,n} = \frac{k^n!}{k!(k^{n-1}!)^k}\left(\#\mathbb{T}_{k,n-1}\right)^k. \tag{2}$$

A knockout tournament seeding is a purely combinatorial object, without any assumptions about participants' behavior. For the purpose of studying the properties of seedings, we assume that a stronger participant always wins in a match with weaker participants. We introduce several properties of tournaments. Some of them have the close prototype in [Karpov, 2016], where the case of $k = 2$ is considered. All combinatorial formulas are new; some sequences in case of $k = 2$ that are mentioned in (OEIS) are added by the author.

The first property makes a tournament invariant under strength/weakness ranking transformation. There are no special rules for weak or strong participants. A tournament designed for strength-ordered participants is equal to a tournament designed for weakness-ordered participants.

Symmetry. *A seeding is invariant under the point mapping $i \to k^n + 1 - i$.*

$\{\{1,4\},\{2,3\}\},\{\{5,8\},\{6,7\}\}$ and $\{\{1,8\},\{2,7\}\},\{\{3,4\},\{5,6\}\}$ are examples of symmetric seedings.

Proposition 2. *The number of seedings, that satisfy the symmetry property, equals to*
for odd k

$$\#\mathbb{T}^S_{k,n} = \frac{2^{\frac{k^n - nk + n - 1}{2}}}{(k!)^{\frac{1}{2}\left(\frac{k^n-1}{k-1} - n\right)}} \frac{\left(\frac{k^n-1}{2}\right)!}{\left[\left(\frac{k-1}{2}\right)!\right]^n}, \tag{3}$$

for even k

$$\#\mathbb{T}^S_{k,n} = \left(\frac{k^n}{2}\right)! \sum_{i=0}^{k/2} \frac{\left(\#\mathbb{T}^S_{k,n-1}\right)^{2i}}{\left(\left(\frac{k^{n-1}}{2}\right)!\right)^{2i} i!\left(\frac{k}{2}-i\right)!} \left(\frac{2^{k^{n-1}-1}}{(k!)^{\frac{k^{n-1}-1}{k-1}}}\right)^{\frac{k}{2}-i}. \tag{4}$$

Proof. A pair of sets $A, B \subseteq \{1, \dots, x\}$ are said to be symmetric if and only if $|A| = |B| = y$, $y < x$ and if $i \in A$, then $x + 1 - i \in B$. A set $A \subseteq \{1, \dots, x\}$ are said to be self-symmetric if and only if $|A| = y$, $y < x$ and if $i \in A$, then $x + 1 - i \in A$.

Odd k. For each tournament, there is only one self-symmetric set of the cardinality of k^{n-1}. There are $\#\mathbb{T}^S_{k,n-1}$ ways to define a symmetric subtournament generated by the self-symmetric set. There are $\frac{k-1}{2}$ symmetric pairs of sets of the cardinality of k^{n-1}. There are $2^{k^{n-1}-1}\#\mathbb{T}_{k,n-1}$ ways to define two subtournaments generated by the symmetric pair. Considering a tournament as the union of k subtournaments we obtain

$$\#\mathbb{T}^S_{k,n} = \frac{\left(\frac{k^n-1}{2}\right)!}{\left(\frac{k^{n-1}-1}{2}\right)!(k^{n-1}!)^{\frac{k-1}{2}}\left(\frac{k-1}{2}\right)!}\#\mathbb{T}^S_{k,n-1}\left(2^{k^{n-1}-1}\#\mathbb{T}_{k,n-1}\right)^{\frac{k-1}{2}}. \qquad (5)$$

Having $\#\mathbb{T}^S_{k,1} = 1$, we obtain

$$\#\mathbb{T}^S_{k,n} = \prod_{i=2}^{n}\frac{\left(\frac{k^i-1}{2}\right)!}{\left(\frac{k^{i-1}-1}{2}\right)!\left(\frac{k-1}{2}\right)!}\frac{2^{\frac{\left(k^{i-1}-1\right)(k-1)}{2}}}{(k!)^{\frac{k^{i-1}-1}{2}}}. \qquad (6)$$

Simplifying we obtain the result.

Even k. We have an even number of self-symmetric sets of the cardinality of k^{n-1}. Thus, we have

$$\#\mathbb{T}^S_{k,n} = \sum_{i=0}^{k/2}\frac{\left(\frac{k^n}{2}\right)!}{\left(\left(\frac{k^{n-1}}{2}\right)!\right)^{2i}(k^{n-1}!)^{\frac{k}{2}-i}(2i)!\left(\frac{k}{2}-i\right)!}\left(2^{k^{n-1}-1}\#\mathbb{T}_{k,n-1}\right)^{\frac{k}{2}-i}\left(\#\mathbb{T}^S_{k,n-1}\right)^{2i}. \qquad (7)$$

Substituting $\#\mathbb{T}_{k,n-1}$, we obtain the result.

For $k = 2$, it is A261187 in (OEIS). In this case the formula (4) has a simpler representation

$$\#\mathbb{T}^S_{2,n} = 2^{\mathrm{n}-1}!\,y_n, \qquad (8)$$

where $y_n = 0.5(y_{n-1})^2 + 1$, with $y_1 = 1$.

For an odd k there is another representation of the recurrence (5). For each tournament, there is only one self-symmetric set of the cardinality k (one match set). There are $\frac{k^{n-1}-1}{2}$ symmetric pairs of sets of the cardinality of k. There are 2^{k-1} ways to define a symmetric pair of sets from a self-symmetric set of the cardinality of $2k$. Considering the tournament as the union of k^{n-1} matches we obtain

$$\#\mathbb{T}_{k,n}^{S} = \frac{\left(\frac{k^n-1}{2}\right)!}{\left(\frac{k-1}{2}\right)!(k!)^{\frac{k^{n-1}-1}{2}}\left(\frac{k^{n-1}-1}{2}\right)!}(2^{k-1})^{\frac{k^{n-1}-1}{2}}\#\mathbb{T}_{k,n-1}^{S}. \quad (9)$$

Having $\#\mathbb{T}_{k,1}^{S} = 1$, we obtain formula (6). These two representations of tournament (tournament as the union of k subtournaments (formula (5)) or the union of k^{n-1} matches (formula (9))) are applied to all derivations of subsequent combinatorial formulas.

Following Wright [2014], competitive intensity is a key property for sports competition design. The closer the strength of the participants the higher the competitive intensity is. The two strongest participants of the match are main rivals. From round to round, the two strongest participants of each match become stronger and strengths of participants become closer. The intensity of competition increases, supporting spectator interest. In the final match, the two strongest participants play against each other.

Increasingly competitive intensity. *In each subsequent round, a winner faces at least one stronger rival than the strongest rival in the previous round.*

Proposition 3. *The number of seedings, that satisfy the increasingly competitive intensity property, equals to*

$$\#\mathbb{T}_{k,n}^{ICI} = \left((k-2)!\right)^{\frac{1-k^{n-1}}{k-1}} \prod_{i=2}^{n} \left[\frac{(k^i-2)!}{\left((k^{i-1}-1)!\right)^2\left((k^{i-1})!\right)^{k-2}}\right]^{k^{n-i}}. \quad (10)$$

Proof. The strongest participant and the second strongest participant should be in different subtournaments. Thus, we have

$$\#\mathbb{T}_{k,n}^{ICI} = \frac{1}{(k-2)!}\frac{(k^n-2)!}{\left((k^{n-1}-1)!\right)^2\left((k^{n-1})!\right)^{k-2}}\left(\#\mathbb{T}_{k,n-1}^{ICI}\right)^k. \quad (11)$$

Having $\#\mathbb{T}_{k,1}^{ICI} = 1$, we obtain the result.

For $k = 2$, formula (11) is also the number of binary heaps (A056972 in (OEIS)). Increasingly competitive intensity is a very weak condition, with $\lim_{k\to\infty}\frac{\#\mathbb{T}_{k,2}^{ICI}}{\#\mathbb{T}_{k,2}} = 1$ and $\lim_{k\to\infty}\frac{\#\mathbb{T}_{k,3}^{ICI}}{\#\mathbb{T}_{k,3}} = e^{-1}$. The next property strengthens the increasingly competitive intensity property guaranteeing the strongest final match, the strongest semifinal, etc.

Delayed confrontation (Schwenk 2000). *Participants rated among the top* k^j *participants shall never meet until the number of participants has been reduced to* k^j *or fewer.*

It is a core property for tournament design. This property is aimed to support spectator interest. Matches with, and between the strongest participants draw the interest of spectators. These participants should not be dropped out at the beginning of the tournament. This property allocates strong participants equally between subtournaments. Thus, there is no subtournament with only weak participants or only with strong participants.

Proposition 4. *The number of seedings, that satisfy the delayed confrontation property, equals to*

$$\#\mathbb{T}_{k,n}^{DC} = \left((k-1)!\right)^{\frac{k-k^n}{k-1}} \prod_{i=2}^{n} (k^i - k^{i-1}). \tag{12}$$

Proof. From delayed confrontation property, participants $\{k^{n-1}+1, \dots, k^n\}$ should lose in round 1, participants $\{k^{n-2}+1, \dots, k^{n-1}\}$ should lose in round 2, etc. Thus we have

$$\#\mathbb{T}_{k,n}^{DC} = \frac{(k^n - k^{n-1})!}{((k-1)!)^{k^{n-1}}} \#\mathbb{T}_{k,n-1}^{DC}. \tag{13}$$

Having $\#\mathbb{T}_{k,1}^{DC} = 1$, we obtain the result.

For $k = 2$, it is A261125 in (OEIS). Delayed confrontation property does not require assumption about the deterministic result of each match. Strong participants are divided between different subtournaments and do not play against each other. We introduce several refinements of the delayed confrontation property: sincerity rewarded, equal difference, equal sums, balance and equal partition of losers properties.

The sincerity rewarded property goes back to [Schwenk, 2000]. We should encourage strong participants, otherwise, they have incentives to lose in pretournament games and get a weaker rival (a model with such incentives is developed in [Dagaev, Sonin, 2017].

Sincerity rewarded. *In addition to the delayed confrontation property, in each round r, the absolute value of the difference between the two strongest par-*

ticipants ranks in the match among top k^{n-r} participants strictly increases with the strength of the top participant.

The standard seeding satisfies this property. The strongest participant plays against the weakest participant, guaranteeing the highest absolute value of the difference between participants ranks.

The weakest violation of the sincerity rewarded property leads to the equal difference property. It implements an idea of favoritism minimize property from [Schwenk, 2000]. We generalize competitive intensity measure of Dagaev, Suzdaltsev [2017] for k higher than 2. The competitive intensity is an absolute value of the difference between the strongest participant rank and the second strongest participant rank in the match. Equalizing competitive intensities of all matches of the round we obtain the equal differences property.

Equal differences. *In addition to the delayed confrontation property, all matches of one round should have an equal absolute value of the difference between the strongest participant rank and the second strongest participant rank in the match.*

Proposition 5. *The number of seedings, that satisfy the equal difference property, equals to*

$$\#\mathbb{T}_{k,n}^{ED} = \left((k-2)!\right)^{\frac{k-k^n}{k-1}} \prod_{i=2}^{n} (k^i - 2k^{i-1}). \tag{14}$$

Proof. From the equal differences property, participants $\{1, \dots, k^{n-1}\}$ should be matched with participants $\{k^{n-1}+1, \dots, 2k^{n-1}\}$. Thus, we have

$$\#\mathbb{T}_{k,n}^{ED} = \frac{[(k^n - 2k^{n-1})!]}{\left((k-2)!\right)^{k^{n-1}}} \#\mathbb{T}_{k,n-1}^{ED}. \tag{15}$$

Having $\#\mathbb{T}_{k,1}^{ED} = 1$, we obtain the result.

The subsequent property equates qualities of matches [Dagaev, Suzdaltsev, 2017] and supports spectator interest to all matches.

Equal sums. *In addition to the delayed confrontation property, all matches of one round should have equal sum of ranks of match's participants.*

The subsequent property simplifies symmetry property in presence of delayed confrontation property.

Balance. *In addition to the delayed confrontation property, all matches of one round should be invariant under the point mapping* $i \to k^{n-r+1} + 1 - i$*, where r is the number of the round.*

Proposition 6. *The number of seedings, that satisfy the balance property, equals to*
for odd k

$$\#\mathbb{T}_{k,n}^{B} = 0, \tag{16}$$

for even k

$$\#\mathbb{T}_{k,n}^{B} = \left(\left(\frac{k}{2} - 1\right)!\right)^{\frac{k-k^n}{k-1}} \prod_{i=2}^{n} \left(\frac{k^i - 2k^{i-1}}{2}\right). \tag{17}$$

Proof. Odd k. Only one match can be invariant under the point mapping $i \to k^{n-r+1} + 1 - i$.

Even k. The strongest k^{n-1} participants play in different matches against the weakest k^{n-1} participants. There are $\left(\frac{k^n - 2k^{n-1}}{2}\right)!\,((0.5k-1)!)^{-k^{n-1}}$ ways to assign all other participants to k^{n-1} matches consistent with the balance property. Thus, we have

$$\#\mathbb{T}_{k,n}^{B} = \frac{\left(\frac{k^n - 2k^{n-1}}{2}\right)!}{((0.5k-1)!)^{k^{n-1}}} \#\mathbb{T}_{k,n-1}^{B}. \tag{18}$$

Having $\#\mathbb{T}_{\mathrm{k},1}^{\mathrm{B}} = 1$ we obtain the result.

The balance property implies the equal sums property. Sincerely rewarded, equal difference, equal sums, and balance properties are quite strong, with

$$\#\mathbb{T}_{2,n}^{SR} = \#\mathbb{T}_{2,n}^{ED} = \#\mathbb{T}_{2,n}^{ES} = \#\mathbb{T}_{2,n}^{B} = 1. \tag{19}$$

The next property equates matches by the presence of the weakest participants. We eliminate advantages of having many weak competitors.

Equal partition of losers. *In addition to the delayed confrontation property, in all matches of one round there should be only one participant from the set of participants* $\{k^{n-r+1} - k^{n-r} + 1, \ldots, k^{n-r+1}\}$ *where r is the number of the round.*

Proposition 7. *The number of seedings, that satisfy the equal partition of losers property, equals to*

$$\#\mathbb{T}_{k,n}^{EPL} = \left((k-2)!\right)^{\frac{k-k^n}{k-1}} \prod_{\mathrm{i}=2}^{\mathrm{n}} k^{i-1}!\,(k^i - 2k^{i-1}). \tag{20}$$

Proof. From the equal differences property, participants $\{1, \dots, k^{n-1}\}$ should be matched with participants $\{k^n - k^{n-1} + 1, \dots, k^n\}$. Thus, we have

$$\#\mathbb{T}_{k,n}^{EPL} = k^{n-1}!\,\frac{[(k^n - 2k^{n-1})!]}{((k-2)!)^{k^{n-1}}}\,\#\mathbb{T}_{k,n-1}^{EPL}. \tag{21}$$

Having $\#\mathbb{T}_{k,1}^{EPL} = 1$, we obtain the result.

For $k = 2$, the equal partition of losers coincides with the delayed confrontation property. The balance property implies the equal partition of losers. Sincerely rewarded, equal differences, equal sums, balance, equal partition of losers properties can be reformulated saving constrain only for the first match. In this case, these properties can be applied for the tournament design without the deterministic assumption about the result of the match. We definitely know only participants of all matches in the first round. All recursive combinatorial formulas can be rewritten through the substitution of $\#\mathbb{T}_{k,n-1}^{Property}$ by $\#\mathbb{T}_{k,n-1}$. By such substitution, recursive formulas become explicit formulas. Even without certain knowledge about all matches in the tournament, the application of above-mentioned properties for all round adds consistency for the tournament design.

3. Representation theorems

3.1. Standard seeding

For $k = 2$, the most popular seeding is the *standard seeding.* It is defined recursively. For any *m* from 1 to n, we have

$$T_{2,n}^{m,i} = \left\{T_{2,n}^{m-1,i}, T_{2,n}^{m-1,2^{n-m+1}-i+1}\right\}, i = \overline{1, 2^{n-m}}.$$

Thus, for $n = 3$, we have

$$T_{2,3}^{standard} = \left\{\{\{1,8\},\{4,5\}\},\{\{2,7\},\{3,6\}\}\right\}.$$

There are several justifications of the standard seeding.

Proposition 8. [Karpov, 2016] *For $k = 2$, the standard seeding is an unique seeding that satisfies the equal rank sums property.*

Proposition 9. *For $k = 2$, the standard seeding is an unique seeding that satisfies the sincerely rewarded property.*

Proof. Participant $2^{n-1} - 1$ has a weaker rival than participant 2^{n-1}, etc. Because participant $2^{n-1} + 1$ should have a rival, we should have a match $\{2^{n-1}, 2^{n-1}\}$. The standard seeding is the only way to pair all other participants.

Proposition 10. *For $k = 2$, the standard seeding is an unique seeding that satisfies the balance property.*

Proposition 10 follows from Proposition 6. The standard seeding also satisfies the equal partition of losers property. There is no direct generalization of the standard seeding for arbitrary k. For $k = 3$ and $n = 2$, $\{\{1,6,8\},\{2,4,9\},\{3,5,7\}\}$ and $\{\{1,5,9\},\{2,6,7\},\{3,4,8\}\}$ satisfies symmetry and equal rank sums properties, but not the sincerely rewarded property, $\{\{1,6,7\},\{2,5,8\},\{3,4,9\}\}$ satisfies symmetry and sincerity rewarded properties, but not the equal rank sums property. We develop two seedings, for $k = 3$ and $k = 4$, that satisfy properties of the standard seeding.

For $k = 3$ the *modified standard seeding* is defined recursively. For any *m* from 1 to *n*, we have

$$T_{3,n}^{m,i} = \left\{T_{3,n}^{m-1,i}, T_{3,n}^{m-1,2\cdot 3^{n-m}-i+1}, T_{3,n}^{m-1,2\cdot 3^{n-m}+i}\right\}, i = \overline{1, 3^{n-m}}.$$

Thus, for $k = 3$ and $n = 3$, we have

$$T_{3,3}^{MS} = \left\{ \begin{matrix} \{\{1,18,19\},\{6,13,24\},\{7,12,25\}\},\{\{2,17,20\},\{5,14,23\},\{8,11,26\}\}, \\ \{\{3,16,21\},\{4,15,22\},\{9,10,27\}\} \end{matrix} \right\}.$$

Proposition 11. *For $k = 3$ the modified standard seeding is an unique seeding that satisfies sincerity rewarded and symmetry properties.*

Proof. It is true for $n = 1$. Suppose in is true for $\mathrm{n} - 1$. Let us prove for n.

Because the sincerity rewarded property leads to delayed confrontation, it is sufficient to define only first-round matches. By the sincerely rewarded property the strongest 3^{n-1} participants play in different matches. By the symmetry property the weakest 3^{n-1} participants play in different matches. By the sincerely rewarded property the strongest participant among participants $\{1, \ldots, 3^{n-1}\}$ plays against the weakest participant among participants $\{3^{n-1}+1, \ldots, 2 \cdot 3^{n-1}\}$, the second strongest with the second weakest, etc. we have the following matches $\{i, 2 \cdot 3^{n-1} - i + 1, ?\}$. By the symmetry property, the participant $2 \cdot 3^{n-1} - i + 1$ corresponds to the participant $3^n - 2 \cdot 3^{n-1} + i - 1 + 1 = 2 \cdot 3^{n-1} - 3^{n-1} + i$. It is a second weakest participant of a match. Thus there is only one way to assign the third participant of the match (it is an image of the participant $i' = 3^{n-1} - i + 1$ of the symmetric match, $3^{n-1} - i + 1 \rightarrow 2 \cdot 3^{n-1} + i$). We design an unique seeding for a tournament with n rounds.

For $k = 4$, the *modified standard seeding* is defined recursively. For any *m* from 1 to *n*, we have

$$T_{4,n}^{m,i} = \left\{ T_{4,n}^{m-1,i}, T_{4,n}^{m-1,\frac{4^{n-m+1}}{2}-i+1}, T_{4,n}^{m-1,\frac{4^{n-m+1}}{2}+i}, T_{4,n}^{m-1,4^{n-m+1}-i+1} \right\},$$
$$i = \overline{1, 4^{n-m}}.$$

Thus, for $k = 4$ and $n = 2$, we have

$$T_{4,2}^{MS} = \{\{1,8,9,16\},\{2,7,10,15\},\{3,6,11,14\},\{4,5,12,13\}\}.$$

Proposition 12. *For $k = 4$, the modified standard seeding is an unique seeding that satisfies sincerely rewarded and balance properties.*

Proof. By the balance property, for any *m* from 1 to *n*, we have $T_{4,n}^{m,i} = \left\{ T_{4,n}^{m-1,i}, ?, ?, T_{4,n}^{m-1,4^{n-m+1}-i+1} \right\}, i = \overline{1, 4^{n-m}}$. By sincerely rewarded property, we have $T_{4,n}^{m,i} = \left\{ T_{4,n}^{m-1,i}, T_{4,n}^{m-1,\frac{4^{n-m+1}}{2}-i+1}, T_{4,n}^{m-1,\frac{4^{n-m+1}}{2}+i}, T_{4,n}^{m-1,4^{n-m+1}-i+1} \right\}, i = \overline{1, 4^{n-m}}$.

3.2. Equal gap seeding

For $k = 2$, the equal gap seeding is investigated in (Karpov 2016). Here we generalize it. The *equal gap seeding* is defined recursively. For any *m* from 1 to n, we have

$$T_{k,n}^{m,i} = \bigcup_{j=0}^{k-1} T_{k,n}^{1,i+jk^{n-m}}, i = \overline{1, \mathrm{k}^{n-m}}.$$

Thus, for $k = 2$ and $n = 4$,we have

$$T_{2,3}^{EG} = \Big\{\big\{\{\{1,9\},\{5,13\}\},\{\{3,12\},\{7,15\}\}\big\},\big\{\{\{2,10\},\{6,14\}\},\{\{4,13\},\{8,16\}\}\big\}\Big\};$$

for $k = 3$ and $n = 3$, we have

$$T_{3,3}^{EG} = \left\{\begin{matrix}\{\{1,10,19\},\{4,13,22\},\{7,16,25\}\},\{\{2,11,20\},\{5,14,23\},\{8,17,26\}\}, \\ \{\{3,12,21\},\{6,15,24\},\{9,18,27\}\}\end{matrix}\right\};$$

for $k = 4$ and $n = 2$,we have

$$T_{4,2}^{EG} = \{\{1,5,9,13\},\{2,6,10,14\},\{3,7,11,15\},\{4,8,12,16\}\}.$$

There are several justifications of the equal gap seeding.

Proposition 13. [Karpov, 2016]. *For $k = 2$ the equal gap seeding is an unique seeding that satisfies the equal difference property.*

For $k = 2$ the equal gap tournament also satisfies the symmetry property.

Proposition 14. *For $k = 3$ the equal gap seeding is an unique seeding that satisfies equal difference and symmetry properties.*

Proof. It is true for $n = 1$. Suppose in is true for $\mathrm{n} - 1$. Let us prove for n. Because the equal differences property leads to the delayed confrontation, it is sufficient to define only first-round matches. By equal differences property, the strongest 3^{n-1} participants play in different matches against participants $\{3^{n-1} + 1, \dots, 2 \cdot 3^{n-1}\}$. The absolute difference between ranks of the strongest and the second strongest participant in the match equals 3^{n-1}. Because of the symmetry property the absolute difference between ranks of the strongest and the second strongest participant in the match also equals 3^{n-1}. Thus, we have

$$T_{3,n}^{1,i} = \cup_{j=0}^{2}\{i + j3^{n-1}\}, i = \overline{1, 3^{n-1}}.$$

The *modified equal gap seeding* is defined recursively. For any *m* from 1 to *n*, we have

$$T_{4,n}^{m,i} = \left\{T_{4,n}^{m-1,i}, T_{4,n}^{m-1,i+4^{n-m}}, T_{4,n}^{m-1,4^{n}-i+1-4^{n-m}}, T_{4,n}^{m-1,4^{n}-i+1}\right\}, i = \overline{1, 4^{n-m}}.$$

Thus, for $k = 4$ and $n = 2$,we have

$$T_{4,2}^{modified\ equal\ gap} = \{\{1,5,12,16\},\{2,6,11,15\},\{3,7,10,14\},\{4,8,9,13\}\}.$$

The modified equal gap seeding satisfies equal sums and equal difference properties, uniting properties of the standard seeding and the equal gap seeding.

Proposition 15. For $k = 4$, *the modified equal gap seeding is an unique seeding that satisfies equal difference, balance properties.*

Proof. By the equal difference property in the round *m* the strongest 4^{n-m} participants play in 4^{n-m} matches against participants $\{4^{n-m}+1, \dots, 2 \cdot 4^{n-m}\}$. Because of the symmetry property, the absolute difference between ranks of the weakest and the second weakest participant in the match also equals 4^{n-m}. By the balance property the sum of ranks in each match equals $\frac{4^{n-m+1}(4^{n-m+1}+1)}{8}$. All strong and weak pairs considered above have different sums of ranks. There is only one way to define a tournament. For any *m* from 2 to *n*, we have

$$T_{4,n}^{m,i} = \left\{T_{4,n}^{m-1,i}, T_{4,n}^{m-1,i+4^{n-m}}, T_{4,n}^{m-1,4^n-i+1-4^{n-m}}, T_{4,n}^{m-1,4^n-i+1}\right\}, i = \overline{1, 4^{n-m}}.$$

Proposition 16. *For* $k = 5$, *there is no seeding that satisfies equal difference, symmetry, and equal sums properties.*

Proof. It is sufficient to consider the case of $n = 2$ to prove the impossibility result. It is the last two round of any tournament. By the equal difference property, the strongest 5 participants play in 5 matches against participants $\{6, \dots, 10\}$. Because of the symmetry property the absolute difference between ranks of the weakest and the second weakest participant in the match also equals 5. The sum of ranks of these four participants is even. The sum of participants ranks in one match equals 65. The rank of the middle participant should be odd in all matches, that is impossible.

For $k = 7$, there exists a seeding that satisfies equal difference, symmetry, equal sums, equal partition of losers properties:

$$T_{7,2} = \begin{Bmatrix} \{1,8,23,29,35,36,43\}, \{2,9,18,31,34,37,44\}, \{3,10,20,26,33,38,45\}, \\ \{4,11,22,24,28,39,46\}, \{5,14,17,24,30,40,47\}, \\ \{6,13,16,19,32,41,48\}, \{7,14,15,29,35,42,49\} \end{Bmatrix}.$$

The fourth match is self-symmetric. The first and the seventh matches, the second and the sixth matches, the third and the fifth matches generate symmetric pairs of matches. For even $k \geq 6$, there are many seedings that satisfy equal difference and balance properties.

Proposition 17. For even $k \geq 6$ *the number of tournaments that satisfy equal difference and balance properties equals to*

$$\#\mathbb{T}_{k,n}^{ED,B} = \left(\left(\frac{k}{2}-2\right)!\right)^{\frac{k-k^n}{k-1}} \prod_{i=2}^{n} \left(\frac{k^i-4k^{i-1}}{2}\right). \tag{22}$$

Proof. By equal difference, balance properties, the strongest k^{n-1} participants play in different matches with the weakest k^{n-1} participants, the second strongest k^{n-1} participants and the second weakest k^{n-1} participants. There are $\left(\frac{k^n-4k^{n-1}}{2}\right)!\,((0.5k-2)!)^{-k^{n-1}}$ ways to assign all other participants to k^{n-1} matches consistent with the balance property. Thus, we have

$$\#\mathbb{T}_{k,n}^{ED,B} = \frac{\left(\frac{k^n-4k^{n-1}}{2}\right)!}{((0.5k-2)!)^{k^{n-1}}} \#\mathbb{T}_{k,n-1}^{ED,B}. \tag{23}$$

Having $\#\mathbb{T}_{k,1}^{ED,B} = 1$, we obtain the result.

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