Operads in algebra and topology

Alexey Gorinov (gorinov@mccme.ru,agorinov@hse.ru),
Anton Khoroshkin (akhoroshkin.hse@gmail.com)

The notion of an operad was first introduced in topology in the 1960's for studying loop spaces. But it quickly became clear that there are many examples which come from algebra. E.g., if one has an algebra of some type (e.g. associative or Lie), which is a set with some number of operations on it, then one could forget about the underlying set and just remember the relations that various compositions of the operations must satisfy. E.g., in the associative case, the statement that the maps $(x, y, z) \mapsto x \cdot (y \cdot z)$ and $(x, y, z) \mapsto (x \cdot y) \cdot z$ are the same is really a statement about maps, not about x, y, z. The resulting structure is an operad.

In the late 20th and early 21st centuries many questions in algebra, topology and mathematical physics were reformulated in terms of operads, which led to a lot of progress in understanding these questions. Nevertheless it should be noted that the combinatorics of operads can be quite complicated.

Historically, one of the first examples of operads provides a convenient way to formalise the notion of a "collection of compatible homotopies". For example, multiplication of loops is not associative on the nose, but only up to homotopy. If X is a topological space, then the maps

$$\begin{array}{l}(\gamma_1, \gamma_2, \gamma_3) \mapsto (\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \\ (\gamma_1, \gamma_2, \gamma_3) \mapsto \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)\end{array} : \Omega(X) \times \Omega(X) \times \Omega(X) \to \Omega(X)$$

are not equal, but only homotopic. There are 5 similar maps from $(\Omega X)^4$ to ΩX; these correspond to the ways to put brackets in a product of 4 factors. Any two such homotopies are themselves homotopic via rearranging the brackets, but there is always more than one such homotopy. These homotopies themselves turn out to be homotopic. By generalising this to an arbitrary number of factors one arrives at the notion of the operad of associahedra, which allows one to characterise spaces homotopy equivalent to loop spaces.

The purpose of the seminar is to discuss the definition, main examples, basic properties and applications of operads.

Here is a tentative list of topics:

- The definition, examples and basic properties of operads. Algebras over operads.
- Topological operads. Recognising (multiple) loop spaces.
- Stasheff polyhedra and ∞-structures. Massey products in cohomology. Modelling rational homotopy type.
- Resolutions and homotopy transfer formulas.
- Operads given by generators and relations. Koszul operads.
- Operads, configuration spaces and moduli spaces of curves.
- Model categories. The Hinich machinery: the model structure on the categories of all operads, algebras over a given operad and modules over a given algebra over a given operad in characteristic 0.
- (*) E_∞-operads и E_∞-algebras. Modelling the homotopy type modulo a prime and integral homotopy type.

Prerequisites:

- topology as covered e.g. in the first three chapters of Hatcher's Algebraic topology;
- linear algebra;
- some knowledge of the representation theory of symmetric groups;
- some knowledge of homological algebra (complexes, exact sequences, resolutions, homology).