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Isomonodromic deformations and quantum field theory

Summary of the PhD thesis

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Introduction

In this thesis I consider the correspondence between isomonodromic deformations and conformal field theory with W-symmetry. First example of it was found by Gamayun, Iorgov and Lisovyy in 2012: they have found that the general tau-function of the Painlevé VI equation can be given as a series expansion over 4-point conformal blocks in $c = 1$ theory. From mathematical point of view this formula is a series over a pair of Young diagrams and one integer number, with coefficients given by some explicit factorized expressions (which come from the AGT formula for conformal block). The generalization of this correspondence to the case of multi-point conformal blocks and tau-functions of more Garnier systems with more that four points, which generalize Painlevé VI equation, was found later.

Cases which were known before are related to isomonodromic deformations of the linear 2×2 Fuchsian system of the form

$$\frac{d\Phi(z)}{dz} = \sum_{i=1}^{n} \frac{A_i}{z - z_i} \Phi(z)$$

Isomonodromic deformations of this system are such simultaneous changes of A_i and z_i that preserve monodromies of solutions around singular points. These transformations are governed by the system of non-linear Schlesinger equations:

$$\frac{\partial A_i}{\partial z_j} = \frac{[A_i, A_j]}{z_j - z_j}, \quad \frac{\partial A_i}{\partial z_i} = -\sum_{i \neq j} \frac{[A_i, A_j]}{z_i - z_j}$$

The tau-function of this system, in some sense, is the simplest object. It is correctly defined by its first derivatives:

$$\frac{\partial}{\partial z_i} \log \tau = \sum_{j \neq i} \frac{\operatorname{tr} A_i A_j}{z_i - z_j}$$

The natural question is the question about the generalization of isomonodromy-CFT correspondence to rank N higher than 2. This thesis deals exactly with such generalization: I show that the tau-functions of general $N \times N$ isomonodromic systems are related to correlation functions of the primary fields of W_N algebras. Some parts of the thesis are devoted solely to isomonodromic deformations in order to make investigation more rigorous and self-contained. Some parts are devoted to the study of the W-algebras only, but are inspired by their relation to isomonodromic deformations: namely, I present a construction of the primary fields of the W-algebra that generalize Zamolodchikov's field of dimension $\frac{1}{16}$. Correlation functions of such fields can be computed explicitly with the help of some constructions on the branch covers of the Riemann sphere.

Study of the isomonodromy-CFT correspondence for the higher rank case is interesting and important due to the several reasons. One reason is that as for rank two case, it gives explicit formulas for the isomonodromic tau-functions which were not known before. Another reason is that W-algebras are much more complicated than Virasoro algebra: for example, spaces of their conformal blocks start to become infinite-dimensional faster than in the Virasoro case. In the W-algebra case even 3-point conformal blocks form an infinite-dimensional space, and there is no algebraic way to pick some particular element from this space in order to use it in construction of multi-point conformal blocks. Isomonodormy-CFT correspondence gives a way to resolve this ambiguity by fixing monodromy properties of the vertex operators.

This thesis consists of six chapters and the bibliography. Chapter 1 is an introduction to the subject that hopefully should be clear for the non-experts. It gives the definition of basic objects of conformal field theory, such as infinitesimal conformal transformations, operator product expansions, Virasoro algebra. Then there are two simplest examples of conformal theories with W-symmetry, theory of N free massless bosonic fields and theory of N massless charged fermions. I explain the definition of W-algebras and the definition of their vertex operators together with the simplest examples. I also explain from utilitarian point of view what is the AGT relation in conformal field theory. Then there follows the explanation of what are Fuchsian systems and what are isomonodromic deformations. After this I present the dictionary of isomonoromy-CFT correspondence, then the definition of Zamolodchikov's twist fields and their generalization to W_N case. Last part of this chapter contains the outline of the thesis, the list of key results, the brief contents of each chapter, and the list of publications.

Chapter 2

In this chapter I formulate the main conjecture that tau-function of the general isomonodromic (Schlesinger) system can be given in terms of conformal blocks of the W-algebra. Then I check this conjecture for 3×3 case with 4 singular points. In order to do this first I study the structure of solution of the Schlesinger system in the limit when two singular points collide. It turns out that solution of the system is given by series in fractional powers of t (the distance between colliding points). They contain monomials of the form $t^{k+(\boldsymbol{w},\boldsymbol{\sigma})}$, where $\boldsymbol{w}$ is an integer vector, and $\boldsymbol{\sigma}$ is some arbitrary complex vector.

I check numerically that the structure of expansion of the tau-function matches exactly the CFT prediction for the growth of powers of t and has the form

$$\tau(t)=\sum_{\boldsymbol{w}\in Q} e^{(\boldsymbol{\beta},\boldsymbol{w})}C^{(0t)}_{\boldsymbol{w}}(\boldsymbol{\theta}_0,\boldsymbol{\theta}_t,\boldsymbol{\sigma}_{0t},\mu_{0t},\nu_{0t})C^{(1\infty)}_{\boldsymbol{w}}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_\infty,\boldsymbol{\sigma}_{0t},\mu_{1t},\nu_{1t})\times$$
$$\times t^{\frac12(\boldsymbol{\sigma}_{0t}+\boldsymbol{w},\boldsymbol{\sigma}_{0t}+\boldsymbol{w})-\frac12(\boldsymbol{\theta}_0,\boldsymbol{\theta}_0)-\frac12(\boldsymbol{\theta}_t,\boldsymbol{\theta}_t)}\mathcal{B}_{\boldsymbol{w}}(\{\boldsymbol{\theta}_i\},\boldsymbol{\sigma}_{0t},\mu_{0t},\nu_{0t},\mu_{1\infty},\nu_{1\infty};t)$$

I also check that in the cases when we have the definition and explicit formula for W-conformal block, it coincides with function $\mathcal{B}$. In this case I conjecture and check that 3-point functions are given by explicit formula that generalizes Gamayun-Iorgov-Lisovyy result:

$$C^{(0t)}_{\boldsymbol{w}}(\boldsymbol{\theta}_0,a_t,\boldsymbol{\sigma})C^{(1\infty)}_{\boldsymbol{w}}(\boldsymbol{\sigma},a_1,\boldsymbol{\theta}_\infty)=$$
$$=\frac{\prod_{ij}G[1-\frac{a_t}{N}+(e_i,\boldsymbol{\theta}_0)-(e_j,\boldsymbol{\sigma}+\boldsymbol{w})]G[1-\frac{a_1}{N}+(e_i,\boldsymbol{\sigma}+\boldsymbol{w})+(e_j,\boldsymbol{\theta}_\infty)]}{\prod\limits_i G[1+(\alpha_i,\boldsymbol{\sigma}+\boldsymbol{w})]}$$

where G is a Barnes function: $G(z+1)=\Gamma(z)G(z)$.

In the next chapters I present the proofs of these statements by different methods.

Chapter 3

In this chapter we develop the free-fermionic formalism for the W-algebras. We represent W-currents in terms of fermions $\psi, \tilde{\psi}$ by the following formula:

$$\sum_{\alpha} \tilde{\psi}^{\boldsymbol{\sigma}}_{\alpha}(z+\frac{t}{2})\psi^{\boldsymbol{\sigma}}_{\alpha}(z-\frac{t}{2}) = \frac{N}{t} + \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} U^{\boldsymbol{\sigma}}_{k}(z)$$

Then we construct vertex operators for the W-algebra in axiomatic way. Namely, we postulate that:

1) Vertex operator is a fermionic group-like element.

2) Its two-particle matrix elements are expressed through the solution of 3-point Fuchsian system.

Then we prove that such operators are W-primaries – it relates them to conformal side of the correspondence. We also prove that solution of the Fuchsian system with n singular points is given by $(z-w)\mathfrak{K}_{\alpha\beta}(z,w)$, where $\mathfrak{K}_{\alpha\beta}(z,w)$ is a two-fermionic correlator in presence of the vertex operators:

$$\mathfrak{K}_{\alpha\beta}(z,w) = \frac{\langle \boldsymbol{\theta}_{\infty}| V_{\boldsymbol{\theta}_{n-2}}(t_{n-2}) \ldots V_{\boldsymbol{\theta}_1}(t_1) \tilde{\psi}^{\boldsymbol{\theta}_0}_{\alpha}(z) \psi^{\boldsymbol{\theta}_0}_{\beta}(w) |\boldsymbol{\theta}_0\rangle}{\langle \boldsymbol{\theta}_{\infty}| V_{\boldsymbol{\theta}_{n-2}}(t_{n-2}) \ldots V_{\boldsymbol{\theta}_1}(t_1) |\boldsymbol{\theta}_0\rangle}$$

At the same time isomonodromic tau-function is given by denominator:

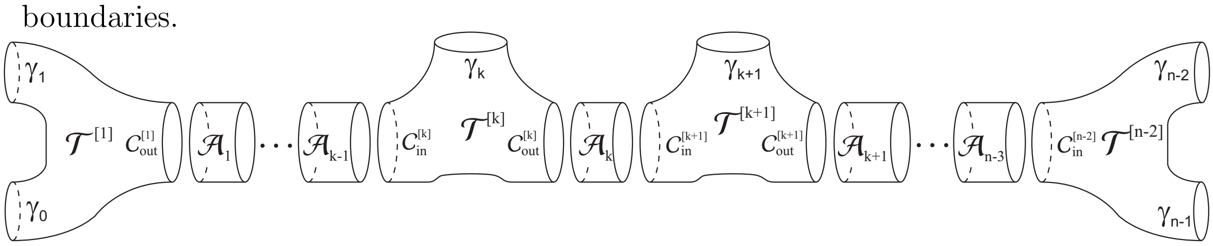
$$\tau(t_1, \ldots, t_{n-2}) = \langle \boldsymbol{\theta}_{\infty}| V_{\boldsymbol{\theta}_{n-2}}(t_{n-2}) \ldots V_{\boldsymbol{\theta}_1}(t_1) |\boldsymbol{\theta}_0\rangle$$

In this way we relate constructed free-fermionic operators to the both side of the correspondence, W-algebras and isomonodromic deformations. This gives the free-fermionic proof of the conjectures from the Chapter 2.

We also show that 4-point tau-function can be written as a Fredholm determinant with some explicit kernel given in terms of hypergeometric functions: $\tau(t) = \det(1+\mathcal{R}_t)$. This formula will be the main object of the study in the next chapter.

Chapter 4

This chapter is written in a pure mathematical language and absolutely rigorously, so it does not require any field-theory background. Here we develop the framework in which the Fredholm determinant formula, that was obtained in the previous chapter from the field-theoretic considerations, and even more general n-point version of it, can be proved. To do this first we cut the sphere with n punctures into $n-2$ three-punctured sphere, like on the picture below, and then introduce the spaces of functions on the obtained boundaries.



Then we construct two projectors onto the space of functions that can be continued between the different boundaries, $\mathcal{P}_\Sigma$ and $\mathcal{P}_\oplus$. These projectors are given explicitly by the formulas of the following kind:

$$\mathcal{P}_\Sigma f(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}_\Sigma} \frac{\hat{\Psi}_+(z)\,\hat{\Psi}_+(z')^{-1} f(z')\,dz'}{z-z'}, \qquad \mathcal{C}_\Sigma := \bigcup_{k=1}^{n-3} \mathcal{C}_{\text{out}}^{[k]} \cup \mathcal{C}_{\text{in}}^{[k+1]}$$

This formula involves the solution of n-point problem given by $\hat{\Psi}_+(z)$, and the formula for $\mathcal{P}_\oplus$ involves solutions of auxiliary 3-point problems related to different pants. We restrict constructed projectors to another space $\mathcal{H}_+$ (some combination of spaces of positive and negative Laurent series): $\mathcal{P}_{\Sigma,+} = \mathcal{P}_\Sigma|_{\mathcal{H}_+}$, $\mathcal{P}_{\oplus,+} = \mathcal{P}_\oplus|_{\mathcal{H}_+}$. Then we define an infinite-dimensional determinant

$$\tau = \det \mathcal{P}_{\Sigma,+}^{-1}\mathcal{P}_{\oplus,+}$$

which is then proved to coincide with isomonodromic tau-function. After that we show that this determinant can be rewritten as a Fredholm determinant with the kernel given by 3-point solutions. It makes it absolutely explicit in the cases when these solutions are known.

We also expand found Fredholm determinant as a series of principal minors and compute each minor explicitly in the 2×2 case. This combinatorial expansion has the form of a series over the collection of Young diagrams and integer A_{N-1} lattices:

$$\tau = \sum_{\vec{Q}_1,\ldots\vec{Q}_{n-3}\in\mathfrak{Q}_N}\ \sum_{\vec{Y}_1,\ldots\vec{Y}_{n-3}\in\mathbb{Y}^N}\ \prod_{k=1}^{n-2} Z_{\vec{Y}_k,\vec{Q}_k}^{\vec{Y}_{k-1},\vec{Q}_{k-1}}\left(\mathcal{T}^{[k]}\right),$$

Furthermore, expressions $Z_{\vec{Y}_k,\vec{Q}_k}^{\vec{Y}_{k-1},\vec{Q}_{k-1}}\left(\mathcal{T}^{[k]}\right)$ in the 2×2 case can be written in terms of Nekrasov functions, which identifies expansion of the tau-function with the series over conformal blocks. We also find explicit reduction from the general Fredholm determinant to simpler scalar case considered by Borodin and Deift.

Chapter 5

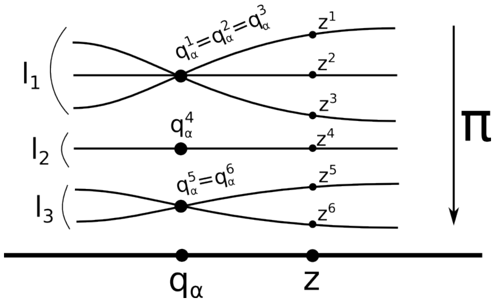
This chapter is devoted to the study of twist fields of W-algebra. We work in the bosonic realization of the W-algebra in terms of N free fields J_α, so its generators are elementary symmetric polynomials of bosonic fields:

$$W_k(z) \equiv \sum_{\alpha_1<\ldots<\alpha_k} :J_{\alpha_1}(z)\ldots J_{\alpha_k}(z):$$

Twist fields are labelled by the elements of the permutation group S_N. Rotation around the twist field permutes N bosonic currents, but leaves the W-generators untouched:

$$J_k(q_\alpha + \epsilon\cdot e^{2\pi i})\mathcal{O}_s(0) = J_{s(k)}(q_\alpha+\epsilon)\mathcal{O}_s(q_\alpha) \tag{1}$$

Such construction leads naturally to consideration of the branch cover above the Riemann sphere, whose branching structure is defined by the twist fields:



CFT considerations in this chapter lead us to explicit formula for the conformal block of such fields, which generalizes Zamolodchikov's exact conformal block:

$$\langle \mathcal{O}_{s_1}(q_1)\mathcal{O}_{s_1^{-1}}(q_2)...\mathcal{O}_{s_L}(q_{2L-1})\mathcal{O}_{s_L^{-1}}(q_{2L})\rangle = \tau_{SW}(\mathbf{q})\cdot\tau_B(\mathbf{q})$$

In this formula $\tau_B(\mathbf{q})$ is so called Bergmann tau-function, and it does not depend on W-charges. More interesting part is

$$\log\tau_{SW} = \tfrac{1}{2}\sum_{I,J} a_I\mathcal{T}_{IJ}a_J + \sum_I a_I U_I(\mathbf{r}) + \frac{1}{2}\sum_{q_\alpha^i\neq q_\beta^j} r_\alpha^i r_\beta^j \log\Theta_*(A(q_\alpha^i)-A(q_\beta^j)) -$$

$$-\frac{1}{2}\sum_{q_\alpha^i}(r_\alpha^i)^2 l_\alpha^i \log\frac{d(z(q)-q_\alpha)^{1/l_\alpha^i}}{h_*^2(q)}\Bigg|_{q=q_\alpha^i}$$

This expression contains the period matrix of the branch cover $\mathcal{T}_{IJ}$, W-charges in the intermediate channels a_I, some extra $U(1)$ charges r_α^i, some combinations of Abel maps U_I, odd Riemann theta-function Θ_*, and corresponding holomorphic 1-form h_*^2. This formula is obtained as a solution of system of integrable equations, so called Seiberg-Witten equations:

$$a_I = \oint_{A_I} dS, \quad \frac{\partial}{\partial a_I}\log\tau_{SW} = \oint_{B_I} dS$$

Similar equations describe low-energy behaviour of $\mathcal{N}=2$ supersymmetric gauge theories, that are also closely related to CFT due to AGT correspondence.

Another result of this chapter is identification between the Fourier transformation of conformal block and explicit formula for the tau-function for quasi-permutational monodromy known due to Korotkin:

$$\tau_{IM}(\mathbf{q}|\boldsymbol{a},\boldsymbol{b}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g}\mathcal{G}_0(\boldsymbol{q}|\boldsymbol{a}+\boldsymbol{n})e^{(\boldsymbol{n},\boldsymbol{b})} = \tau_B(\boldsymbol{q})\exp\left(\tfrac{1}{2}Q(\mathbf{r})\right)\Theta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{U})$$

This fact gives one more evidence for the correspondence between W-algebras and isomonodromic deformations.

Chapter 6

This chapter is also devoted to W-twist fields, but from more algebraic point of view. Here we consider the W-algebras for the orthogonal series, too. We start from the free-fermionic definition of W-algebras. Their generators in the B- and D-series may be written in terms of complexified real fermions as follows:

$$U_k(z) = \tfrac{1}{2}D_z^{k-1}\sum_{\alpha=1}^{N}\left(\psi^*_\alpha(z)\cdot\psi_\alpha(z) + \psi_\alpha(z)\cdot\psi^*_\alpha(z)\right) + \tfrac{1}{2}D_z^{k-1}\Psi(z)\cdot\Psi(z)$$

$$V(z) = \prod_{\alpha=1}^{N} :\psi^*_\alpha(z)\psi_\alpha(z): \Psi(z)$$

where D_z is Hirota derivative. We reformulate construction of the twist fields in terms of fermions. It turns out that now correct object, that parametrizes twist fields, is the normalizer of Cartan algebra $\mathrm{N}_G(\mathfrak{h})$. Different conjugacy classes in $\mathrm{N}_G(\mathfrak{h})$ give different twist fields.

Main subject of this chapter is the computation of characters of modules, constructed above the twist fields. Typical example of such character is the formula for the twist field which corresponds to element g consisting of K cycles of lengths l_i with extra diagonal multipliers r_i:

$$\chi_g(q) = q^{\sum\limits_{j=1}^{K}\frac{l_j^2-1}{24l_j}}\frac{\sum\limits_{n_1+\ldots+n_K=0} q^{\sum\limits_{i=1}^{K}\frac{1}{2l_i}(r_il_i+n_i)^2}}{\prod\limits_{j=1}^{K}\prod\limits_{k=1}^{\infty}(1-q^{k/l_j})}$$

In the numerator we see the lattice A_{K-1} theta-function.

One of the parts of this chapter is devoted to the situation when $g_1 \sim g_2$ are conjugated in G for inequivalent $g_1, g_2 \in \mathrm{N}_G(\mathfrak{h})$. We show that in this case two different characters coincide $\chi_{g_1}(q) = \chi_{g_2}(q)$. This gives a series of non-trivial character identities, which we also prove explicitly. Some of them coincide with Macdonald identity, and some of them seem to be new. One of the tools of character computations are exotic bosonization formulas that relate bosons and fermions with different boundary conditions: like bosonization of periodic and anti-periodic fermions into single anti-periodic boson.

In this chapter we also compute conformal blocks of the twist fields in D-series. Main feature of this case is the structure of the $2N$-fold cover, which can be shown on the following commutative diagram:

$$\sigma \circlearrowleft \overset{\pi_{2N}}{\overbrace{\Sigma \xrightarrow{\pi_2} \tilde{\Sigma} \xrightarrow{\pi_N} \mathbb{CP}^1}}$$

This cover has an involution σ, and its factor over this involution is smaller N-fold cover. Most of the objects that are used in the construction are σ-antisymmetric: for example, the only sufficient part of the period matrix is Prym period matrix. Desired formula for conformal block in this case has the structure similar to A-case:

$$\mathcal{G}_0(\boldsymbol{a},\boldsymbol{r},\boldsymbol{q}) = \tau_B(\Sigma|\boldsymbol{q})\tau_B^{-1}(\tilde{\Sigma}|\boldsymbol{q})\tau_{SW}(\boldsymbol{a},\boldsymbol{r},\boldsymbol{q})$$

In some cases it can also reduce to the formula for A-series.

Conclusion

This thesis contains some number of constructions that give explicit formulas for isomonodromic tau-functions, for conformal blocks of W-algebras, and relate some of them to each other. The main technical tools are free-field constructions of the vertex operators, use of Seiberg-Witten integrable system, and manipulations with projection-like operators in functional spaces.

References

The content of Chapters 2-5 is based on the following papers in order:

2. P. Gavrylenko, *Isomonodromic τ-functions and W_N conformal blocks*, JHEP09(2015)167, [hep-th/1505.00259]

3. P. Gavrylenko, A. Marshakov, *Free fermions, W-algebras and isomonodromic deformations*, Theor. Math. Phys. 2016, 187:2, 649–677, [hep-th/1605.04554]

4. P. Gavrylenko, O. Lisovyy, *Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions*, [math-ph/1608.00958], Under review in Communications in Mathematical Physics

5. P. Gavrylenko, A. Marshakov, *Exact conformal blocks for the W-algebras, twist fields and isomonodromic deformations*, JHEP02(2016)181,[hep-th/1507.08794]

6. M. Bershtein, P. Gavrylenko, A. Marshakov, *Twist-field representations of W-algebras, exact conformal blocks and character identities* , [hep-th/1705.00957], Under review in Communications in Mathematical Physics