Syllabus for MATHEMATICAL METHODS FOR ECONOMISTS

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Course description

Mathematical Methods for Economists is a two-semester course for the second year students studying at ICEF which specialize in "Mathematics and Economics".
This course is an important part of the bachelor stage in education of the future economists. It has give students skills for implementation of the mathematical knowledge and expertise to the problems of economics. Its prerequisite is the knowledge of the single variable calculus.
In the fall semester this course is split into two self-contained sub courses: namely "Multivariate Calculus and Optimization" (MCO) and "Linear Algebra" (LA). LA lasts for one fall module and completes with the final exam in the end of October. MCO continues beyond and from January onwards incorporates also the chapters of "Methods of Optimization" course. Also note that starting from January and through the end of April UoL "Algebra MT1 173" will be taught to the students of 'Math and Econ".
The final assessment will be provided by the University of London (UoL) examinations in May. Students will sit two exams: on Calculus and Algebra. The table below shows the distribution of contact hours weekly throughout the academic year:

	I module	II module	III module	IV module
Lectures	4 hrs	2 hrs	6 hrs	6 hrs
Classes	4 hrs	2 hrs	4 hrs	4 hrs

The sub course "Multivariate Calculus and Optimization" covers multi-dimensional calculus, optimization theory and the selected topics drawn from the theory of differential and difference equations are taught in spring. That course is aimed at teaching students to master comparative statics problems, optimization problems and dynamic models using the acquired mathematical tools. The course is taught in English. The students are also studying for the Russian degree in Economics, and that implies knowing Russian terminology is also a must.
"Linear Algebra" is a half-semester (8 weeks long) sub course that is obligatory for the second-year ICEF students. The course was originally designed as an instrumental supplement to the principal quantitative block subjects such as "Methods of optimization", "Time series analysis", and "Econometrics". Linear Algebra shares many exam topics with the program of UoL. At the same time, the class of Linear Algebra at ICEF is taught on its own to deliver basic principles of matrix calculus. From a broader prospective, the aim of the course is to deliver one of the most general mathematical concepts - the idea of linearity.

LA splits naturally into the following three parts:

1. Problems related to systems of linear equations and to the extension of the 2D-

and 3D- intuition to linear spaces of higher dimensions. This part includes the concepts of basis, rank, dimension, linear hull, linear subspace, etc.

2. Problems that involve antisymmetric polylinear forms (determinants) and also problems from the geometry of linear operators such that eigenvectors and eigenvalues, matrix diagonalization, etc.
3. Problems from the calculus of bilinear forms: quadratic forms, orthogonalization, and other geometric problems in higher-dimensional Eucledian spaces.

In Linear Algebra it is critically important to teach not only the technique of manipulations with matrices and vectors, but also general algebraic concepts that are used, for instance, in problems that involve linear differential equations or linear difference equations.

The latter means the thinking over the theoretical material, working on the home assignments given by the lecturer. Over the examination week in late October LA will be completed with the final exam and the students will also sit the mock on MC.

Teaching objectives.

Students are expected to develop an understanding of basic algebraic concepts such as linear vector space, linear independence, bases, coordinate systems, dimension, matrix algebra, linear operators, dot product, orthogonality. On the practical side, among other skills, students are expected to be able to solve systems of linear equations, find fundamental system of solutions, invert matrices, find eigenvalues, and do orthogonal projections.

Students are supposed:

to acquire knowledge in the field of higher mathematics and become ready to analyze simulated as well as real economic situations;

to develop ability to apply the knowledge of the differential and difference equations which will enable them to analyze dynamics of the processes.

Teaching Methods

The course program consists of:

- lectures,
- classes,
- regular self-study and working on home assignments.

Assessment and control

- Home assignments (weekly)
- Midterm exams
- Final exams

The midterm exam on LA will be held approximately after the 4th or 5th lecture. The final exam on LA will be held when the course is completed that is in late October. Both exams consist of two parts, multiple choice and free response. The midterm exam takes **60** minutes; the final exam takes **90** minutes. The final exam is not cumulative, i.e., it covers the part of the course that was not covered by the midterm exam.

Grade determination of LA sub course

The weekly home works constitute **10%** of the final grade. The mid-term test contributes **40%** to the final grade. The final exam is **50%** of the final grade. If a student missed the mid-term test without a valid excuse (see school's schedule for valid excuses), he or she will be given zero grade, which contributes as zero with **40%** weight to the total grade. If a student missed the mid-term test with a valid excuse, the calculation of the final grade will be based on other types of assessment, using a formula that partially compensates for the lost points. In this case, the weights of all other components of the final grade are multiplied by (1+0,5a), where "a" is the weight of a grade for the missed midterm test in the final grade. The weight of the grades on retake will be decided based on the validity of excuse for missing the exam and also taking into account the grade received earlier.

Assessment and grade determination of the "Mathematical methods for Economists" course

Along with the assessment on LA sub course control on the "MME" course takes the following forms:

- written home assignments;
- two midterm exams (120 min) in the UoL examination format in the end of the I and III modules;
- written exam (120 min) at the end of the fall semester,
- University of London exams by the end of the spring semester (120 min), i.e. Algebra and Calculus.

The **fall semester** grade on "MME" course will be determined according to the formula:

Grade for "MME"=0.3*grade for LA+0.7*grade for MCO,
where grade for MCO is an accumulation based on the following:

- average grade for the home assignments (20%);
- fall semester midterm exam (20%);
- fall exam (60%).

The **final course grade** on "MME" course will be determined according to the formula:

Course grade for those students who specialize in **Mathematics and Economics** is determined by

- University of London exam grade for "Calculus MT1 174" (25%),
- University of London exam grade for "Algebra MT1 173" (25%),
- fall term grade (10%),

- two midterm exam grades on “M for E” and on “Algebra” both accounting for (15%+15%),
- average grade for the spring home assignments (10%).

Warning

As a general policy, personal computing devices such as laptops, calculators etc. are not supposed to be used in the course. They are *absolutely prohibited* in all exams. Students are expected to do all necessary arithmetic computations by hand.

Main Reading

1.Carl P. Simon and Lawrence Blume. Mathematics for Economists, W.W. Norton and Co, 1994.
2. A.C. Chiang. Fundamental Methods of Mathematical Economics, McGraw-Hill, 1984, 2008.
3. Pervouchine DD. Lecture Notes on Linear Algebra. ICEF 2011 (Pervouchine)
4. Chernyak V. Lecture Notes on Linear Algebra. Introductory course. Dialog, MSU, 1998, 2000 (Chernyak)
5. Anton, H. and C. Rorres. *Elementary Linear Algebra* (International Student Version). (John Wiley & Sons (Asia) Plc Ltd, 2010) tenth edition.
6. Anthony M., and Biggs N., *Mathematics for Economics and Finance,* Cambridge University Press, UK, 1996.
7. Study guide by M. Anthony and M. Harvey.

Additional Reading

1. B. P. Demidovich. Collection of problems and exercises on calculus, Moscow, “Nauka”, 1966.
2. A.F. Fillipov. Collection of problems on differential equations. Moscow, “Nauka”, 1973.
3. Anthony M., and Biggs N., Mathematics for Economics and Finance, Cambridge University Press, UK, 1996.
4. Anthony M., Reader in Mathematics, LSE, University of London; Mathematics for Economists, Study Guide, University of London.
5. R.O.Hill, Elementary Linear Algebra, Academic Press, 1986
6. Гельфанд И.М. Лекции по линейной алгебре Москва, Наука, 1999.
7. Кострикин А.И., Манин Ю.И., Линейная алгебра и геометрия, Москва, Наука 1986.
8. Проскуряков И.В. Сборник задач по линейной алгебре, Москва, Наука, 1985.

Internet resources

University of London Exam papers and Examiners reports for the last three years http://www.londonexternal.ac.uk/current_students/programme_resources/lse/index.shtml .
Current course materials are post at the ICEF information system http://mief.hse.ru

Course outline for Linear Algebra

1. **Systems of linear equations in matrix form**. Basic concepts and geometric interpretation. Consistency. Elementary transformations of equations. Gauss and Gauss-Jordan methods. (Pervouchine, ch. 1; Chernyak, ch. 1 - 5; Simon & Blume, ch. 7)

2. **Linear space. Linear independence.** Rank. Linear span. Bases and dimension of a linear space. Ordered bases and coordinates. Transition from one basis to another. Properties of linearly dependent and linearly independent vectors. Examples. (Pervouchine, ch. 2; Chernyak, ch. 9-11; Simon & Blume, ch. 7,11)

3. **Linear subspace**. The set of solutions as a linear subspace. General and particular solutions. Fundamental set of solutions. (Pervouchine, ch. 2-3; Chernyak, ch. 11; Simon & Blume, ch. 11)

4. **Matrix as a set of columns and as a set of rows.** Linear operations on matrices. Transpose matrix and matrix algebra. Special types of matrices. Matrices of elementary transformations. (Pervouchine, ch. 5; Chernyak, ch. 2-3; Simon & Blume, ch. 8)

5. **Determinant of a set of vectors.** Geometric interpretation. Determinant of a matrix. Computation and basic properties of determinants. Cramer's rule. Applications to rank computation. (Pervouchine, ch. 4; Chernyak, ch. 6-8; Simon & Blume, ch. 9)

6. **Inverse matrix.** Degenerate matrices. Computation of the inverse matrix by the extended Gauss algorithm and by using algebraic complements. (Pervouchine, ch. 4-5; Chernyak, ch. 12; Simon & Blume, ch. 8)

7. **Linear operator as a geometric object.** Matrix of a linear operator. Examples, including linear operators in functional spaces. Transformations of vectors and matrices of linear operators induced by a change of coordinates. Conjugate matrices. (Pervouchine, ch. 6; Chernyak, ch. 15)

8. **Eigenvalues, eigenvectors and their properties**. Characteristic equation. Basis and dimension of eigenspaces. Diagonalization and its applications. (Pervouchine, ch. 6; Chernyak, ch. 13-14; Simon & Blume, ch. 23)

9. **Bilinear and quadratic forms.** Canonical representation. Full squares method. Symmetric matrices and quadratic forms. Definite, indefinite, and semidefinite forms. Silvester's criterion. (Pervouchine, ch. 7; Simon & Blume, ch. 16)

10. **Dot product in linear spaces.** Norm of a vector. Metric properties: distances

and angles. Projection onto a subspace. Orthogonal bases. Orthogonalization. Equations of lines and planes. (Pervouchine, ch. 8; Chernyak, ch. 16, Simon & Blume, ch. 10)

Distribution of hours

№	Topic	Total	In-class hours		
			Lectures	**Seminars**	**Self-study**
1.	Systems of linear equations in matrix form	16	2	2	12
2.	Linear space. Linear independence	8	1	1	6
3.	Linear subspace	12	2	2	8
4.	Matrix as a set of columns and as a set of rows	16	2	2	12
5.	Determinant of a set of vectors	4	2	2	0
6.	Inverse matrix	4	2	2	0
7.	Linear operator as a geometric object	8	2	2	4
8.	Eigenvalues, eigenvectors and their properties	12	2	2	10
9.	Bilinear and quadratic forms	8	2	2	6
10	Dot product in linear spaces	12	1	1	10
	Total:	**108**	**18**	**18**	**72**

Course outline for Mathematics for Economists

Part I. Multi-dimensional calculus

1. Main concepts of set theory. Operations on sets. Direct product of sets. Relations and functions. Level sets and level curves.
(SL Sections 2.1-2.2; C Sections1.1-2.7)

2. Space R^n. Metric in *n*-dimensional space. The triangle inequality. Euclidean spaces. Neighborhoods and open sets in R^n, Sequences and their limits. Close sets. The closure and the boundary of a set.
(SL Sections 10.1-10.4; C Sections12.1-12.6)

3. Functions of several variables. Limits of functions. Continuity of functions.
(SL Sections 13.1-13.5; C Sections 6.4-6.7)

4. Partial differentiation. Economic interpretation, marginal products and elasticities. Chain rule for partial differentiation.
(SL Sections 14.1-14.3; C Section 7.4)

5. Total differential. Geometric interpretation of partial derivatives and the differential. Linear approximation. Differentiability. Smooth functions. Directional derivatives and gradient.
(SL Sections 14.4-14.6; C Sections 8.1-8.7)

6. Higher-order derivatives. Young's theorem. Hessian matrix. Economic applications.
(SL Sections 14.8-14.9; C Sections 7.6, 9.3)
7. Implicit functions. Implicit function theorem.
(SL Sections 15.1-15.2; C Section 8.5)
8. Vector-valued functions. Jacobian.
(SL Section 14.7; C Section 8.5)
9. Implicit function theorem for the vector-valued functions.
(SL Sections 15.3, 15.5; C Section 8.5)
10. Economic applications of the IFT for the comparative statics problems.
(SL Section 15.4; C Section 8.6)

Part II. Optimization
11. Unconstrained optimization of the multi-dimensional functions. Stationary points. First-order conditions.
(SL Sections17.1-17.2; C Sections11.1-11.2)

12. Second differential. Quadratic forms and the associated matrices. Definiteness and semi-definiteness of the quadratic forms. Sylvester criterion. Second-order conditions for extrema.
(SL Sections 16.1-16.2, 17.3-17.4; C Sections11.3-11.7)
13. Constrained optimization. Lagrangian function and multiplier. First-order conditions for constrained optimization.
(SL Sections 18.1-18.2; C Sections 12.1-12.2)
14. Second differential for the function with the dependent variables. Definiteness of quadratic form under a linear constraint. Bordered Hessian. Second-order conditions for the constrained optimization.
(SL Sections 16.3-16.4, 19.3; C Section 12.3)
15. Economic meaning of a multiplier. Applications of the Lagrange approach in economics. Smooth dependence on the parameters. Envelope theorem.
(SL Sections 18.7-19.2, 19.4; C Section 12.5)

Part III. Differential and difference equations
16. Dynamics in economics. Simple first-order equations. Separable equations. Concept of stability of the solution of ODE. Exact equations. General solution as a sum of a general solution of homogeneous equation and a particular solution of a nonhomogeneous equation. Bernoulli equation.
(SL Sections 24.1-24.2; C Sections13.6, 14.1-14.3)
17. Qualitative theory of differential equations. Solow's growth model. Phase diagram.
(Section 24.5; C Sections14.6-14.7)
18. Second-order linear differential equations with constant coefficients.
(SL Section 24.3; C Section15.1)
19. Complex numbers and operations on them. Representation of a number. De Moivre and Euler formulae.
(SL Appendix A3; C Section 15.2)
20. Higher-order linear differential equation with constant coefficients. Characteristic equation. Method of undetermined coefficients for the search of a particular solution. Stability of solutions. Routh theorem (without proof).
(SL Section 24.3; C Sections15.3-15.7)

21. Discrete time economic systems. Difference equations. Method of solving first-order equations. Convergence and oscillations of a solution. Cobweb model. Partial equilibrium model with the inventory.
(SL Section 23.2; C Sections 16.2-16.6)
22. Second-order difference equations.
(C Sections 17.1-17.3)
23. Higher-order difference equations. Characteristic equation. Undetermined coefficients method. Conditions for the stability of solutions.
(C Section 17.4)

Distribution of hours

№	Topic	Total	Lectures	Classes	Self study
	Part I. Multi-dimensional calculus				
1.	Main concepts of set theory. Operations on sets. Direct product of sets. Relations and functions. Level sets and level curves.	14	4	4	6
2.	Space R^n. Metric in *n*-dimensional space. The triangle inequality. Euclidean spaces. Neighborhoods and open sets in R^n, Sequences and their limits. Close sets. The closure and the boundary of a set.	14	4	4	6
3.	Functions of several variables. Limits of functions. Continuity of functions	14	4	4	6
4	Partial differentiation. Economic interpretation, marginal products and elasticities. Chain rule for partial differentiation	12	4	2	6
5	Total differential. Geometric interpretation of partial derivatives and the differential. Linear approximation. Differentiability. Smooth functions. Directional derivatives and gradient	12	4	2	6
6	Higher-order derivatives. Young's theorem. Hessian matrix. Economic applications	12	4	2	6
7	Implicit functions. Implicit function theorem	12	4	2	6
8	Vector-valued functions. Jacobian	12	4	2	6
9	Implicit function theorem for the vector-valued functions	12	4	2	6
10	Economic applications of the IFT for the comparative statics problems.	12	4	2	6
	Part II. Optimization				
11	Unconstrained optimization of the multi-dimensional functions. Stationary points. First-order conditions	12	4	2	6
12	Second differential. Quadratic forms and the associated matrices. Definiteness and semi-definiteness of the quadratic forms. Sylvester criterion. Second-order conditions for extrema	10	2	2	6

13	Constrained optimization. Lagrangian function and multiplier. First-order conditions for constrained optimization	10	2	2	6
14	Second differential for the function with the dependent variables. Definiteness of quadratic form under a linear constraint. Bordered Hessian. Second-order conditions for the constrained optimization	10	2	2	6
15	Economic meaning of a multiplier. Applications of the Lagrange approach in economics. Smooth dependence on the parameters. Envelope theorem	10	2	2	6
	Part III. Differential and difference equations				
16	Dynamics in economics. Simple first-order equations. Separable equations. Concept of stability of the solution of ODE. Exact equations. General solution as a sum of a general solution of homogeneous equation and a particular solution of a nonhomogeneous equation. Bernoulli equation.	10	2	2	6
17	Qualitative theory of differential equations. Solow's growth model. Phase diagram	10	2	2	6
18	Second-order linear differential equations with constant coefficients	10	2	2	6
19	Complex numbers and operations on them. Representation of a number. De Moivre and Euler formulae	10	2	2	6
20	Higher-order linear differential equation with constant coefficients. Characteristic equation. Method of undetermined coefficients for the search of a particular solution. Stability of solutions. Routh theorem (without proof).	12	2	2	8
21	Discrete time economic systems. Difference equations. Method of solving first-order equations. Convergence and oscillations of a solution. Cobweb model. Partial equilibrium model with the inventory	12	2	2	8
22	Second-order difference equations	14	2	2	10
23	Higher-order difference equations. Characteristic equation. Undetermined coefficients method. Conditions for the stability of solutions	14	2	2	10
	Total:	***270***	***68***	***52***	***150***

Additional topics taught in Spring on "Methods of optimization"

Distribution of the lecture hours

	Topics	Hours
1.	Homogeneous functions	2
2.	Optimization in 2 variables with the inequality constraints. First order conditions, generalization on the n-dimensional case	6
3.	Kuhn-Tucker formulation, applications from economics	4
4.	Meaning of Lagrange multipliers, envelope theorems (refreshment)	2
5.	Linear programming	6
6.	Introduction to the game theory. Bimatrix games. The notion of Nash equilibrium. Dominant and dominated strategies. Equilibrium in mixed strategies.	6
7.	Methods of finding equilibria in the zero sum games	2
	Total	**28**

Course outline of "Algebra MT1 173"

1. Lines, planes in R^2 and R^3. Lines and hyperplanes in R^n (Study Guide, pp. 37-50).
2. Homogeneous systems and null space. Consistent and inconsistent systems. Linear systems with free variables. Solution sets (Anton, H. and C. Rorres, Ch.1).
3. Matrix inversion and determinants (Anton, H. and C. Rorres, Ch.1 and Ch.2).
4. Rank, range and linear equations (Anton, H. and C. Rorres, Ch.4, Sections 4.7-4.8).
5. Vector spaces (Anton, H. and C. Rorres, Ch.4, Sections 4.1-4.2).
6. Linear independence, bases and dimension (Anton, H. and C. Rorres, Ch.4, Sections 4.3-4.5).
7. Linear transformations, change of basis (Anton, H. and C. Rorres, Ch.4, Section 4.6 and Ch.8).
8. Diagonalization (Anton, H. and C. Rorres, Ch.5, Sections 5.1-5.2).
9. Markov chains (Anton, H. and C. Rorres, Ch.10, Section 10.5).

Distribution of hours "Algebra MT1 173"

№	Topic	Total	Lectures	Classes	Self study
1.	Lines, planes in R^2 and R^3. Lines and hyperplanes in R^n	5	2	1	2
2.	Homogeneous systems and null space. Consistent and inconsistent systems. Linear systems with free variables. Solution sets	10	4	2	4
3.	Matrix inversion and determinants	5	2	1	2
4	Rank, range and linear equations	10	4	2	4
5	Vector spaces	13	4	3	6

6	Linear independence, bases and dimension	13	4	3	6
7	Linear transformations, change of basis	10	4	2	4
8	Diagonalization	10	4	2	4
9	Markov chains	8	2	2	4
	Total:	***84***	***30***	***18***	***36***