ALGEBRAIC GEOMETRY. HW1. (DUE SEPTEMBER, 28)

1. Show that, for $n > 1$, the scheme $\mathbb{A}_k^n - 0$ is not affine.

2. A topological space is quasi-compact if every open cover has a finite subcover.
(a) Show that a topological space is noetherian if and only if every open subset is quasi-compact.
(b) If X is an affine scheme, show that the underlying topological space $sp(X)$ is quasi-compact, but not in general noetherian. We say a scheme X is quasi-compact if $sp(X)$ is.
(c) If A is a noetherian ring, show that $sp(\operatorname{Spec} A)$ is a noetherian topological space.
(d) Give an example to show that $sp(\operatorname{Spec} A)$ can be noetherian even when A is not.

3. If X is a topological space, and Z an irreducible closed subset of X, a generic point for Z is a point η such that $Z = \overline{\{\eta\}}$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

Definition. Let $f : X \to Y$ be a morphism of schemes. We say that f is locally of finite type if X can be covered by open affines $U_j = \operatorname{Spec}(B_j)$'s and Y can be covered by open affines $V_i = \operatorname{Spec}(A_i)$'s with $f(U_j) \subset V_i$, such that each B_j is finitely generated as an A_i-algebra. We say that f is quasi-compact if Y can be covered by open affine subschemes V_i such that the pre-images $f^{-1}(V_i)$ are quasi-compact (as topological space). One says that f is of finite type if it's quasi-compact and locally of finite type.

4. Let X be a scheme of finite type over a field k (that is X is equipped with a morphism $X \to \operatorname{Spec} k$ which is of finite type) and let $|X|$ be the set of closed points of X. Denote by $\overline{k}$ an algebraic closure of k. Show that, for any morphism $\operatorname{Spec} \overline{k} \to X$ over $\operatorname{Spec} k$, its image is closed and the map

$$X(\overline{k}) := \operatorname{Mor}_{\operatorname{Spec} k}(\operatorname{Spec} \overline{k}, X) \to |X|$$

exibits $|X|$ as the set of $Gal(\overline{k}/k)$-orbits in $X(\overline{k})$.

5. Let X be a scheme of finite type over $\operatorname{Spec} \mathbb{Z}$. Prove that the residue field of every closed point is finite.

Definition. For a scheme X of finite type over $\operatorname{Spec} \mathbb{Z}$ its zeta function is defined to be

$$\zeta_X(s) = \prod_{x \subset |X|} \frac{1}{1 - |N(x)|^{-s}},$$

where the product is taken over all closed points of X and $N(x)$ stands for the cardinality of the residue field of a closed point.

6. Prove that the product converges absolutely for $\operatorname{Re} s > \dim X$. (Using the Noether Normalization reduce the claim to the case where $X = \operatorname{Spec} A[x_1, \cdots, x_n]$ and A is either $\mathbb{Z}$ or $\mathbb{F}_p$.)

7. Let X be a scheme of finite type over $\mathbb{F}_p$. For an integer $k > 0$, set $N_k = |X(\operatorname{Spec} \mathbb{F}_{p^k})|$. Show that

$$\zeta_X(s) = exp(\sum_{k \geq 1} \frac{N_k}{k} p^{-ks}).$$

8. Let $f : X \to Y$ be a morphism of schemes. Assume that f is locally of finite type. Show that in this case, for any open affine $U = \mathrm{Spec}(B) \subset X$ and $V = \mathrm{Spec}(A) \subset Y$ with $f(U) \subset V$, we have that B is f.g. as an A-algebra. Hint: Prove and use the following criterion on a A-algebra B to be finitely generated: for any A-algebra C and increasing family of subalgebras C_i such that $\bigcup_i C_i = C$, any A-algebra map $B \to C$ factors through a map $B \to C_i$ for some i.