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On a particular analytical solution to the 3D Navier-Stokes equations and its peculiarity for high Reynolds numbers

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A special class of axially symmetric nonstationary flows of incompressible viscous fluids is examined. For it, the 3D Navier-Stokes equations are reduced to a nonlinear partial differential equation of the third order and a linear partial differential equation of the second order. These equations are studied and their particular analytical solutions are found. The obtained particular solution to the Navier-Stokes equations could be used to describe some types of turbulent flows of viscous fluids in the case of high Reynolds numbers. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4929845]

I. INTRODUCTION

Consider the Navier-Stokes equations describing a homogeneous incompressible fluid with no bulk forces. They can be represented in the form[1–3]

$$\frac{\partial \mathbf{v}}{\partial t} + v_1\frac{\partial \mathbf{v}}{\partial x} + v_2\frac{\partial \mathbf{v}}{\partial y} + v_3\frac{\partial \mathbf{v}}{\partial z} = -\text{grad}\left(\frac{p}{\rho}\right) + \nu\Delta\mathbf{v}, \tag{1}$$

$$\text{div}\,\mathbf{v} = 0, \tag{2}$$

where $\mathbf{v} = \mathbf{v}(t, x, y, z)$ is the vector of velocity, $p = p(t, x, y, z)$ is pressure, v_1, v_2, v_3 are the projections of the vector $\mathbf{v}$ onto the orthogonal axes x, y, z, t is time, ρ is the density of the considered fluid, and ν is its kinematic viscosity.

The Navier-Stokes equations are basic equations of fluid mechanics and extensive analytical and numerical studies are devoted to them.[1–21] However, because of their substantial nonlinearity, a number of important problems of fluid dynamics still remain unsolved. One of them is the problem of describing turbulent flows of viscous fluids in the case of high Reynolds numbers.[22–34] Our aim is to find particular solutions in this case to the Navier-Stokes equations that could reflect some features of turbulent fluid motions.

We will consider the case of axial symmetry and seek the components v_1, v_2, v_3 of the vector function $\mathbf{v}$ and the function p/ρ in the following form:

$$\begin{gathered} v_1 = -\alpha y + \beta x, \quad v_2 = \alpha x + \beta y, \quad v_3 = \gamma, \quad \alpha = \alpha(t,r,z), \\ \beta = \beta(t,r,z), \quad \gamma = \gamma(t,r,z), \quad p/\rho = q(t,r,z), \quad r = \sqrt{x^2+y^2}. \end{gathered} \tag{3}$$

Here, the function α presents the angular velocities of rotations about the axis z of points of a fluid and the functions β and γ describe changing its shape.

Substituting expressions (3) into Eq. (2), we find

$$r\beta_r + 2\beta + \gamma_z = 0, \tag{4}$$

where $\beta_r \equiv \partial\beta/\partial r$, $\gamma_z \equiv \partial\gamma/\partial z$.

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Using expressions (3), after simple calculations, we obtain

$$\begin{aligned}
v_1\frac{\partial v_1}{\partial x}+v_2\frac{\partial v_1}{\partial y}+v_3\frac{\partial v_1}{\partial z} &= -(r\beta\alpha_r+\gamma\alpha_z+2\alpha\beta)y+(r\beta\beta_r+\gamma\beta_z+\beta^2-\alpha^2)x,\\
v_1\frac{\partial v_2}{\partial x}+v_2\frac{\partial v_2}{\partial y}+v_3\frac{\partial v_2}{\partial z} &= (r\beta\alpha_r+\gamma\alpha_z+2\alpha\beta)x+(r\beta\beta_r+\gamma\beta_z+\beta^2-\alpha^2)y,\\
v_1\frac{\partial v_3}{\partial x}+v_2\frac{\partial v_3}{\partial y}+v_3\frac{\partial v_3}{\partial z} &= r\beta\gamma_r+\gamma\gamma_z.
\end{aligned} \tag{5}$$

For the components of the Laplacian $\Delta\mathbf{v}$, we find

$$\begin{aligned}
\Delta v_1 &= -(\alpha_{rr}+3\alpha_r/r+\alpha_{zz})y+(\beta_{rr}+3\beta_r/r+\beta_{zz})x,\\
\Delta v_2 &= (\alpha_{rr}+3\alpha_r/r+\alpha_{zz})x+(\beta_{rr}+3\beta_r/r+\beta_{zz})y,\\
\Delta v_3 &= \gamma_{rr}+\gamma_r/r+\gamma_{zz},
\end{aligned} \tag{6}$$

where $\alpha_{rr}\equiv\partial^2\alpha/\partial r^2$, $\alpha_{zz}\equiv\partial^2\alpha/\partial z^2$.

Let us now substitute formulas (3), (5), and (6) into Navier-Stokes Equation (1). Then, we come to the following three nonlinear partial differential equations:[35,36]

$$\alpha_t+\beta(r\alpha_r+2\alpha)+\gamma\alpha_z-\nu(\alpha_{rr}+3\alpha_r/r+\alpha_{zz})=0,\quad \alpha_t\equiv\partial\alpha/\partial t, \tag{7}$$

$$\beta_t+\beta(r\beta_r+\beta)+\gamma\beta_z-\alpha^2-\nu(\beta_{rr}+3\beta_r/r+\beta_{zz})=q_r/r, \tag{8}$$

$$\gamma_t+r\beta\gamma_r+\gamma\gamma_z-\nu(\gamma_{rr}+\gamma_r/r+\gamma_{zz})=q_z. \tag{9}$$

Derived Equations (4) and (7)-(9) were studied in our papers,[35,36] where some exact solutions to them were found. In Sec. II, we will examine these equations in a special case in which the components v_1 and v_2 of the vector of velocity are independent of the coordinate z and the component v_3 is zero when $z=0$. Then, we will reduce Eqs. (4) and (7)-(9) to a nonlinear partial differential equation of the third order and a linear partial differential equation of the second order. In Sec. III, we will examine the nonlinear differential equation of the third order and find its particular solutions. In Sec. IV, using them, we will obtain a particular analytical solution to the Navier-Stokes equations. It will be shown that in the case of high Reynolds numbers, this solution can reflect some features of turbulent flows of viscous fluids.

II. A SPECIAL AXIALLY SYMMETRIC CASE FOR THE NAVIER-STOKES EQUATIONS

Let us examine axially symmetric solutions to Navier-Stokes (1) and (2) satisfying the following properties in some spatial region:

(1) The components v_1 and v_2 of the vector of velocity are independent of the coordinate z.
(2) The component v_3 is zero when $z=0$.

Then, as follows from (3) and (4), the functions α, β, γ should have the form

$$\alpha=a(t,r),\quad \beta=b(t,r),\quad \gamma=zc(t,r), \tag{10}$$

where $a(t,r)$, $b(t,r)$, and $c(t,r)$ are some differentiable functions.

It is assumed that the range of z in the spatial region under consideration is finite. As to all the space or half-space $z\geq 0$, for them, formulas (10) give infinite energies.

From Eqs. (8) and (9) and formulas (10), we find

$$q_r=rs_0(t,r),\quad q_z=zs(t,r), \tag{11}$$

where s_0 and s are some functions.

As follows from (11), $q_{rz}=0$ and hence

$$s=s(t),\quad q=d(t,r)+\tfrac{1}{2}z^2s(t),\quad d_r=rs_0(t,r), \tag{12}$$

where s is some function of t.

Substituting expressions (10) and (12) into Eqs. (4) and (7)-(9), we come to the equations

$$c = -rb_r - 2b, \tag{13}$$

$$a_t + b(ra_r + 2a) - \nu(a_{rr} + 3a_r/r) = 0, \tag{14}$$

$$b_t + b(rb_r + b) - a^2 - \nu(b_{rr} + 3b_r/r) = s_0(t,r), \tag{15}$$

$$c_t + rbc_r + c^2 - \nu(c_{rr} + c_r/r) = s(t). \tag{16}$$

Our objective is to find particular solutions to Eqs. (13), (14), and (16). Using them and formulas (12) and (15), we can also obtain expressions for the function $q = p/\rho$.

Let us put

$$a = A(t,r)/r^2, \quad b = B(t,r)/r^2, \quad c = C(t,r)/r^2, \tag{17}$$

where A, B, C are some differentiable functions of t and r.

Then, we find

$$\begin{aligned} &ra_r + 2a = A_r/r, \quad a_{rr} + 3a_r/r \equiv (ra_r + 2a)_r/r = (A_{rr} - A_r/r)/r^2, \\ &rb_r + 2b = B_r/r, \quad b_{rr} + 3b_r/r = (B_{rr} - B_r/r)/r^2, \\ &c_r = (C_r - 2C/r)/r^2, \quad c_{rr} + c_r/r = (C_{rr} - 3C_r/r + 4C/r^2)/r^2, \end{aligned} \tag{18}$$

and Eqs. (13)-(16) acquire the form

$$C = -rB_r, \tag{19}$$

$$A_t + (\nu + B)A_r/r - \nu A_{rr} = 0, \tag{20}$$

$$B_t + B(B_r - B/r)/r - A^2/r^2 - \nu(B_{rr} - B_r/r) = s_0(t,r)r^2, \tag{21}$$

$$C_t + B(C_r - 2C/r)/r + C^2/r^2 - \nu(C_{rr} - 3C_r/r + 4C/r^2) = s(t)r^2, \tag{22}$$

Substituting expression (19) for C into Eq. (22), we obtain

$$\nu B_{rrr} - (\nu + B)(B_{rr} - B_r/r)/r + B_r^2/r - B_{tr} = s(t)r, \tag{23}$$

where $B_r \equiv \partial B/\partial r$, $B_{tr} \equiv \partial^2 B/\partial t \partial r$, $B_{rr} \equiv \partial^2 B/\partial r^2$, $B_{rrr} \equiv \partial^3 B/\partial r^3$.

Let us choose the following dimensionless variables τ, η and dimensionless functions $P(\tau,\eta)$, $Q(\tau,\eta)$, $S_0(\tau,\eta)$, $S(\tau)$:

$$\begin{aligned} &\tau = (V_*/l_*)t, \quad \eta = (r/l_*)^2, \quad A(t,r) = \nu P(\tau,\eta), \quad B(t,r) = \nu Q(\tau,\eta), \\ &s_0(t,r) = (V_*/l_*)^2 S_0(\tau,\eta), \quad s(t) = (V_*/l_*)^2 S(\tau), \end{aligned} \tag{24}$$

where V_* is the mean absolute value of the velocity of a fluid and l_* is its characteristic linear dimension.

Then, we find

$$A_t = \frac{\nu V_*}{l_*}\frac{\partial P}{\partial \tau}, \quad A_r = 2\frac{\nu r}{l_*^2}\frac{\partial P}{\partial \eta}, \quad A_{rr} = 2\frac{\nu}{l_*^2}\left(2\eta\frac{\partial^2 P}{\partial \eta^2} + \frac{\partial P}{\partial \eta}\right), \tag{25}$$

$$\begin{aligned} &B_t = \frac{\nu V_*}{l_*}\frac{\partial Q}{\partial \tau}, \quad B_r = 2\frac{\nu r}{l_*^2}\frac{\partial Q}{\partial \eta}, \quad B_{tr} = 2\frac{\nu V_* r}{l_*^3}\frac{\partial^2 Q}{\partial \tau\, \partial \eta}, \\ &B_{rr} = 2\frac{\nu}{l_*^2}\left(2\eta\frac{\partial^2 Q}{\partial \eta^2} + \frac{\partial Q}{\partial \eta}\right), \quad B_{rrr} = 4\frac{\nu r}{l_*^4}\left(2\eta\frac{\partial^3 Q}{\partial \eta^3} + 3\frac{\partial^2 Q}{\partial \eta^2}\right). \end{aligned} \tag{26}$$

Substituting expressions (24)-(26) into Eqs. (20), (21), and (23), we derive

$$4\eta P_{\eta\eta} - 2QP_\eta - \mathrm{Re}\, P_\tau = 0, \quad \mathrm{Re} = \mathrm{V}_* l_*/\nu, \quad P = P(\tau,\eta), \tag{27}$$

$$\mathrm{Re}Q_\tau - 4\eta Q_{\eta\eta} + Q(2Q_\eta - Q/\eta) - P^2/\eta = \mathrm{Re}^2 S_0(\tau,\eta)\eta, \quad Q = Q(\tau,\eta), \tag{28}$$

$$4(\eta Q_{\eta\eta\eta} + Q_{\eta\eta}) - 2(QQ_{\eta\eta} - Q_\eta^2) - \mathrm{Re}Q_{\tau\eta} = \tfrac{1}{2}\mathrm{Re}^2 S(\tau), \tag{29}$$

where $Q_\tau \equiv \partial Q/\partial \tau$, $Q_{\tau\eta} \equiv \partial^2 Q/\partial \tau \partial \eta$, $Q_\eta \equiv \partial Q/\partial \eta$, $Q_{\eta\eta} \equiv \partial^2 Q/\partial \eta^2$, $Q_{\eta\eta\eta} \equiv \partial^3 Q/\partial \eta^3$, and the constant Re is the Reynolds number.

Thus, we have come to two partial differential equations (27) and (29) for the functions $P(\tau,\eta)$ and $Q(\tau,\eta)$ and expression (28) for the function $S_0(\tau,\eta)$.

In Sec. III, we will investigate particular solutions to nonlinear differential equation (29) for the function $Q(\tau,\eta)$. Using them, one can find the other function $P(\tau,\eta)$ by solving linear partial differential equation (27).

III. INVESTIGATION OF THE NONLINEAR PARTIAL DIFFERENTIAL EQUATION (29)

Consider Eqs. (27)-(29) and choose the following form for the functions $P(\tau,\eta)$ and $Q(\tau,\eta)$ in them,

$$P = F(\tau,\xi), \quad Q = G(\tau,\xi), \quad \xi = \mathrm{Re}\eta \tag{30}$$

Then, we obtain

$$\begin{aligned} &P_\eta = \mathrm{Re}\partial F/\partial\xi, \quad P_{\eta\eta} = \mathrm{Re}^2\partial^2 F/\partial\xi^2, \quad P_\tau = \partial F/\partial\tau, \\ &Q_\eta = \mathrm{Re}\partial G/\partial\xi, \quad Q_{\tau\eta} = \mathrm{Re}\partial^2 G/\partial\tau\partial\xi, \quad Q_{\eta\eta} = \mathrm{Re}^2\partial^2 G/\partial\xi^2, \quad Q_{\eta\eta\eta} = \mathrm{Re}^3\partial^3 G/\partial\xi^3. \end{aligned} \tag{31}$$

Substituting (30) and (31) into Eqs. (27)-(29), we find

$$4\xi F_{\xi\xi} - 2GF_\xi - F_\tau = 0, \quad F = F(\tau,\xi), \tag{32}$$

$$S_0(\tau,\xi)\xi = G_\tau - 4\xi G_{\xi\xi} + G(2G_\xi - G/\xi) - F^2/\xi, \quad G = G(\tau,\xi), \tag{33}$$

$$4(\xi G_{\xi\xi\xi} + G_{\xi\xi}) - 2(GG_{\xi\xi} - G_\xi'^2) - G_{\tau\xi} = \tfrac{1}{2}S(\tau), \tag{34}$$

where $G_\tau \equiv \partial G/\partial\tau, G_{\tau\xi} \equiv \partial^2 G/\partial\tau\partial\xi, G_\xi \equiv \partial G/\partial\xi, G_{\xi\xi} \equiv \partial^2 G/\partial\xi^2, G_{\xi\xi\xi} \equiv \partial^3 G/\partial\xi^3$.

Let us turn to Eq. (34) and seek its particular solutions in the following form:

$$G = K_0(\tau)e^{f(\tau)\xi} + K_1(\tau)\xi + K_2(\tau), \tag{35}$$

where f and K_i are some differentiable functions of the argument τ.

From (35), we have

$$\begin{aligned} &G_\xi = K_1(\tau) + K_0(\tau)f(\tau)e^{f(\tau)\xi}, \\ &G_{\tau\xi} = \dot{K}_1(\tau) + [\dot{K}_0(\tau)f(\tau) + K_0(\tau)\dot{f}(\tau)(1 + f(\tau)\xi)]e^{f(\tau)\xi}, \\ &G_{\xi\xi} = K_0(\tau)f^2(\tau)e^{f(\tau)\xi}, \quad G_{\xi\xi\xi} = K_0(\tau)f^3(\tau)e^{f(\tau)\xi}, \end{aligned} \tag{36}$$

where $\dot{f}(\tau) \equiv df/d\tau$.

Using (35) and (36), we find

$$GG_{\xi\xi} - G_\xi^2 = -K_1^2(\tau) + K_0(\tau)f(\tau)[(K_2(\tau) + K_1(\tau)\xi)f(\tau) - 2K_1(\tau)]e^{f(\tau)\xi}. \tag{37}$$

Substituting formulas (35)-(37) into Eq. (34), we obtain

$$2K_1^2 - \dot{K}_1 + \{K_0[2f^2(2 - K_2) + 4K_1 f - \dot{f}] - \dot{K}_0 f + K_0 f\xi[2f(2f - K_1) - \dot{f}]\}e^{f\xi} = \tfrac{1}{2}S. \tag{38}$$

This equation gives

$$2f(2f - K_1) - \dot{f} = 0, \tag{39}$$

$$K_0[2f^2(2 - K_2) + 4K_1 f - \dot{f}] - \dot{K}_0 f = 0, \tag{40}$$

$$2K_1^2 - \dot{K}_1 = \tfrac{1}{2}S. \tag{41}$$

After using (39), Eq. (40) acquires the simpler form

$$\dot{K}_0/K_0 = 2(3K_1 - K_2 f). \tag{42}$$

Obtained Equations (39), (41), and (42) are three ones for the five functions $S(\tau), f(\tau)$ and $K_0(\tau), K_1(\tau), K_2(\tau)$. Using their solutions and formula (35) for the function $G(\tau,\xi)$, one can determine particular solutions to Eq. (34). After the function $G(\tau,\xi)$ is determined, the function $F(\tau,\xi)$ can be found by solving linear differential equation (32).

From formulas (3), (10), (17), (19), (24), (30), and the expression in (27) for the constant Re, we derive the following expressions for the components v_i of the vector of velocity:

$$v_1 = \nu[-F(\tau,\xi)y + G(\tau,\xi)x]/r^2, \quad v_2 = \nu[F(\tau,\xi)x + G(\tau,\xi)y]/r^2, \\ v_3 = -2V_* G_\xi(\tau,\xi)z/l_*. \tag{43}$$

IV. A PARTICULAR SOLUTION TO THE NAVIER-STOKES EQUATIONS AND ITS PECULIARITY IN THE CASE OF HIGH REYNOLDS NUMBERS

Let us apply the results obtained above to find a particular solution for $t \geq 0$ to the Navier-Stokes equations that satisfies the following initial conditions:

$$v_i = 0, \quad t = 0, \quad r \neq 0, \tag{44}$$

and v_i are singular when $r = 0$.

Such a solution could be used to describe an explosion along some axis z at $t = 0$ in an incompressible viscous fluid.

It should be noted that the question of possible solutions to the Navier-Stokes equations singular at $t = 0$ was studied in Refs. 12 and 13.

Let us require that for $t > 0$ the functions v_i should satisfy the following boundary conditions at infinity:

$$r v_i \to 0, \quad r \to \infty, \quad i = 1, 2, 3. \tag{45}$$

Then from (43), we find

$$F(\tau,\infty) = G(\tau,\infty) = G_\xi(\tau,\infty) = 0. \tag{46}$$

Using formulas (35) and (46), we obtain the following solutions to Eqs. (39), (41), and (42):

$$K_1 = K_2 = S = 0, \quad K_0 = \text{const}, \quad f = -\frac{1}{4\tau}, \tag{47}$$

singular at the initial time $t = 0$.

Formulas (35) and (47) give

$$G = K_0 \exp\left(-\frac{\xi}{4\tau}\right), \tag{48}$$

which tends to zero as $t \to 0+$ when $r \neq 0$.

Obtained formula (48), where K_0 is an arbitrary constant, presents a particular solution to Eq. (34) with $S = 0$ and satisfies conditions (46).

Let us now consider Eq. (32) and seek its solution $F(\tau,\xi)$ in the form

$$F = H(\theta), \quad \theta = \frac{\xi}{4\tau}, \tag{49}$$

where $H(\theta)$ is some differentiable function.

Then, we find

$$F_\tau = -\frac{\theta H'(\theta)}{\tau}, \quad F_\xi = \frac{H'(\theta)}{4\tau}, \quad F_{\xi\xi} = \frac{H''(\theta)}{16\tau^2}. \tag{50}$$

Substituting formulas (48)-(50) into Eq. (32) and multiplying it by τ, we obtain

$$\theta H''(\theta) + (\theta - \tfrac{1}{2}K_0 e^{-\theta})H'(\theta) = 0. \tag{51}$$

From Eq. (51), we readily find

$$H' = M_1 \exp\left(-\theta + \tfrac{1}{2}K_0 \text{Ei}(-\theta)\right), \quad \text{Ei}(x) = \int_\infty^{-x} \frac{e^{-s}}{s} ds, \tag{52}$$

where $M_1 = \text{const}$ and $\text{Ei}(x)$ is the exponential integral.

This gives the following formula for the function $F = H(\theta)$ satisfying the condition for it in (46):

$$F = H(\theta) = M_1 \int_{\infty}^{\theta} \exp\left(-\theta + \tfrac{1}{2}K_0 \mathrm{Ei}(-\theta)\right) d\theta, \tag{53}$$

Using (43), (48), (49), and formula for τ in (24), we obtain

$$v_1 = \frac{\nu}{r^2}\left(-H(\theta)y + K_0 e^{-\theta}x\right), \quad v_2 = \frac{\nu}{r^2}\left(H(\theta)x + K_0 e^{-\theta}y\right), \quad v_3 = \frac{K_0}{2t}e^{-\theta}z, \tag{54}$$

where K_0 is a constant, the function $H(\theta)$ is determined by formula (53), and, as follows from (49) and the expressions for τ, η, ξ, and Re in (24), (27), and (30), the variable θ has the form

$$\theta = \mathrm{Re}\frac{(r/l_*)^2}{4\tau} = \frac{r^2}{4\nu t}. \tag{55}$$

Formulas (54) and (55) are defined for $0 < t < \infty$ and satisfy boundary conditions (45).

As follows from (53), $H(\theta) \to 0$ as $\theta \to +\infty$. Hence, from (54) and (55), we find that for $r \neq 0$, the velocity components $v_i \to 0$ as $t \to 0+$.

Therefore, the obtained components v_i of the vector of velocity satisfy initial conditions (44). As follows from (54) and (55), these components are singular at $r = 0$.

From (53)-(55), we obtain the following approximate formulas for v_i in the case of sufficiently small positive t:

$$\begin{gathered} v_1 = \frac{\nu(M_1 y + K_0 x)}{r^2}\exp\left(-\frac{r^2}{4\nu t}\right), \quad v_2 = \frac{\nu(-M_1 x + K_0 y)}{r^2}\exp\left(-\frac{r^2}{4\nu t}\right), \\ v_3 = \frac{K_0 z}{2t}\exp\left(-\frac{r^2}{4\nu t}\right), \quad r \neq 0, \quad t \to 0+. \end{gathered} \tag{56}$$

Consider now formulas (54) and (55) for v_i when $t \to +\infty$ or $r \to 0$, $t > 0$. Then, from (55), we have $\theta \to 0+$. In this case for v_3, we get

$$v_3 = \frac{K_0 z}{2t}\left[1 + O(\theta)\right], \quad \theta = \frac{r^2}{4\nu t}, \quad \theta \to 0+. \tag{57}$$

Let us examine the expressions for v_1 and v_2 in (54) as $\theta \to 0+$ in the two cases $K_0 < 0$ and $K_0 \geq 0$. As follows from formulas (54), the case $K_0 < 0$ corresponds to a fluid motion toward the axis z and the case $K_0 > 0$ corresponds to its motion outward from this axis.

It is well known[37] that the exponential integral can be represented as

$$\mathrm{Ei}(x) = \gamma_* + \ln|x| + \sum_{k=1}^{\infty}\frac{x^k}{k\cdot k!}, \quad x \neq 0, \tag{58}$$

where γ_* is the Euler-Mascheroni constant.

Using (58), we obtain the following asymptotic formula:

$$\exp\left(-\theta + \tfrac{1}{2}K_0\mathrm{Ei}(-\theta)\right) = \exp\left(\tfrac{1}{2}K_0\gamma_*\right)\theta^{K_0/2}[1 - (1 + K_0/2)\theta + O(\theta^2)], \quad \theta \to 0+. \tag{59}$$

Therefore, formulas (53), (59), and the expression for $\mathrm{Ei}(x)$ in (52) give

$$\begin{aligned} H(\theta) &= H(0) + M_1\int_0^{\theta}\exp\left(-\theta + \tfrac{1}{2}K_0\mathrm{Ei}(-\theta)\right)d\theta \\ &= H(0) + M_1\exp\left(\tfrac{1}{2}K_0\gamma_*\right)\frac{\theta^{1+K_0/2}}{1 + K_0/2}[1 + O(\theta)], \quad \theta \to 0+, \quad K_0 > -2, \end{aligned} \tag{60}$$

where $H(0)$ is a finite constant,

$$H(\theta) = M_1\exp\left(\tfrac{1}{2}K_0\gamma_*\right)[\ln\theta + O(1)], \quad \theta \to 0+, \quad K_0 = -2, \tag{61}$$

$$H(\theta) = M_1\exp\left(\tfrac{1}{2}K_0\gamma_*\right)\frac{\theta^{1+K_0/2}}{1 + K_0/2}(1 + \varepsilon(\theta)), \quad \theta \to 0+, \quad K_0 < -2, \tag{62}$$

where $\varepsilon(\theta) = O(\theta)$ when $K_0 < -4$, $\varepsilon(\theta) = O(\theta \ln \theta)$ when $K_0 = -4$, and $\varepsilon(\theta) = O\left(\theta^{-1-K_0/2}\right)$ when $-4 < K_0 < -2$.

From formulas (54) and (55) and (60)-(62), we derive the following asymptotic formulas for the components v_1 and v_2 of the vector of velocity:

$$
\begin{aligned}
v_1 &= \frac{\nu}{r^2}\left[-\left(H(0) + O(\theta^{1+K_0/2})\right) y + K_0\,(1 + O(\theta))\, x\right], \quad \theta = \frac{r^2}{4\nu t},\\
v_2 &= \frac{\nu}{r^2}\left[\left(H(0) + O(\theta^{1+K_0/2})\right) x + K_0\,(1 + O(\theta))\, y\right], \quad K_0 > -2, \quad \theta \to 0+,
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
v_1 &= -\frac{\nu}{r^2}\left[M_*\,(\ln\theta + O(1))\, y + 2\,(1 + O(\theta))\, x\right], \quad M_* = M_1 \exp(\tfrac{1}{2}K_0\gamma_*),\\
v_2 &= \frac{\nu}{r^2}\left[M_*\,(\ln\theta + O(1))\, x - 2\,(1 + O(\theta))\, y\right], \quad K_0 = -2, \quad \theta \to 0+,
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
v_1 &= \frac{\nu}{r^2}\left[-\frac{M_*}{1 + K_0/2}\theta^{1+K_0/2}\,(1 + \varepsilon(\theta))\, y + K_0\,(1 + O(\theta))\, x\right],\\
v_2 &= \frac{\nu}{r^2}\left[\frac{M_*}{1 + K_0/2}\theta^{1+K_0/2}\,(1 + \varepsilon(\theta))\, x + K_0\,(1 + O(\theta))\, y\right], \quad K_0 < -2, \quad \theta \to 0+.
\end{aligned}
\tag{65}
$$

From (57), we find that $v_3 \to 0$ as $t \to +\infty$. As follows from (63), when $K_0 > -2$, the velocity components v_1 and v_2 are finite for $r \neq 0$ and $t \to +\infty$. Formulas (64) and (65) give that when $K_0 \leq -2$ and $M_1 \neq 0$, $v_1 \to \infty$ and $v_2 \to \infty$ as $t \to +\infty$.

Therefore, obtained solution (54) and (55) could take place in a viscous fluid for sufficiently large t only when $K_0 > -2$ or $M_1 = 0$. When $K_0 \leq -2$, $M_1 \neq 0$, and t is sufficiently large, one should take into account compressibility of viscous fluids.

Let us study formulas (54) and (55) for the components v_i of the vector of velocity in the case of high Reynolds numbers. As follows from these formulas, the velocity components v_i of a viscous fluid could substantially change inside intervals $(r - \delta r, r + \delta r)$, where $r \sim l_*$ and $\delta r \sim l_*/\mathrm{Re}$. When the Reynolds number $\mathrm{Re} \sim 10^4$, $l_* \sim 1$ m, and $\tau \sim 1$, the value $\delta r \sim 0.1$ mm. Therefore, in this case, there can be substantial changes of velocities inside small radial intervals $\sim 10^{-4} l_*$. This peculiarity could be regarded as a manifestation of turbulent flows of viscous fluids with high Reynolds numbers. As follows from formulas (57) and (63), when $K_0 > -2$, such turbulent flows become absent as $t \to +\infty$.

V. CONCLUSION

We have examined a special class of axially symmetric solutions to the 3D Navier-Stokes equations in which the components v_1 and v_2 of the vector of velocity are independent of the coordinate z and the component v_3 is zero when $z = 0$. The problem under examination was reduced to a nonlinear partial differential equation of the third order and a linear partial differential equation of the second order. Studying the nonlinear differential equation of the third order, we found a class of its particular solutions which contained two arbitrary functions of time. Then, we considered initial conditions (44) at $t = 0$ and boundary conditions (45) at infinity and found a particular analytical solution to the examined nonlinear differential equation corresponding to these conditions. After that we studied the obtained linear partial differential equation of the second order and found an analytical expression for its particular solution. As a result, we obtained a particular analytical solution to the Navier-Stokes equations. This solution was examined in the case of high Reynolds numbers. It was shown that the found solution to the Navier-Stokes equations could reflect some features of fluid turbulent flows.

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