JOURNAL OF MATHEMATICAL PHYSICS **57**, 083103 (2016)

On a particular solution to the 3D Navier-Stokes equations for liquids with cavitation

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(Received 11 January 2016; accepted 16 August 2016; published online 31 August 2016)

The 3D Navier-Stokes equations for incompressible viscous liquids are examined. In the axially symmetric case, they are represented in the form of three nonlinear partial differential equations. These equations are studied and their particular solution is found. In it, the velocity components are sinusoidal in the direction of their axis of symmetry. As to the pressure, it can reach a sufficiently small value at which the phenomenon of cavitation takes place in a liquid. The found solution describes some flows of viscous liquids outside vapor-filled regions in them. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4961905]

I. INTRODUCTION

Consider the Navier-Stokes equations describing a homogeneous incompressible liquid with no bulk forces. They can be represented in the form[1–3]

$$\frac{\partial \mathbf{v}}{\partial t} + v_1\frac{\partial \mathbf{v}}{\partial x} + v_2\frac{\partial \mathbf{v}}{\partial y} + v_3\frac{\partial \mathbf{v}}{\partial z} = -\frac{1}{\rho}\operatorname{grad} p + \nu\Delta\mathbf{v}, \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{2}$$

where $\mathbf{v} = \mathbf{v}(t,x,y,z)$ is the vector of velocity, $p = p(t,x,y,z)$ is pressure, v_1, v_2, v_3 are the projections of the vector $\mathbf{v}$ onto the orthogonal axes x, y, z, t is time, ρ is the density of the considered liquid, and ν is its kinematic viscosity.

The Navier-Stokes equations play a major role in fluid dynamics and a large number of works are devoted to their analytical and numerical investigations.[1–28] However, because of nonlinearity and complexity of these equations, many properties of their solutions are not well studied. Our aim is to find a particular analytical solution to the Navier-Stokes equations that could describe some flows of viscous liquids when the phenomenon of cavitation takes place in them.

We will consider the case of axial symmetry and seek the components v_1, v_2, v_3 of the vector function $\mathbf{v}$ and the pressure p in the following form:

$$\begin{gathered} v_1 = -\alpha y + \beta x, \quad v_2 = \alpha x + \beta y, \quad v_3 = \gamma, \quad \alpha = \alpha(t,r,z), \\ \beta = \beta(t,r,z), \quad \gamma = \gamma(t,r,z), \quad p = p(t,r,z), \quad r = \sqrt{x^2 + y^2}. \end{gathered} \tag{3}$$

Here the function α presents the angular velocities of rotations about the axis z of points of a liquid and the functions β and γ describe changing its shape.

Substituting expressions (3) into Eq. (2), we find

$$r\beta_r + 2\beta + \gamma_z = 0, \tag{4}$$

where $\beta_r \equiv \partial\beta/\partial r, \gamma_z \equiv \partial\gamma/\partial z$.

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Using expressions (3), after simple calculations, we obtain[29–31]

$$
\begin{aligned}
v_1\frac{\partial v_1}{\partial x}+v_2\frac{\partial v_1}{\partial y}+v_3\frac{\partial v_1}{\partial z} &= -(r\beta\alpha_r+\gamma\alpha_z+2\alpha\beta)y+(r\beta\beta_r+\gamma\beta_z+\beta^2-\alpha^2)x,\\
v_1\frac{\partial v_2}{\partial x}+v_2\frac{\partial v_2}{\partial y}+v_3\frac{\partial v_2}{\partial z} &= (r\beta\alpha_r+\gamma\alpha_z+2\alpha\beta)x+(r\beta\beta_r+\gamma\beta_z+\beta^2-\alpha^2)y,\\
v_1\frac{\partial v_3}{\partial x}+v_2\frac{\partial v_3}{\partial y}+v_3\frac{\partial v_3}{\partial z} &= r\beta\gamma_r+\gamma\gamma_z,
\end{aligned}
\tag{5}
$$

and for the components of the Laplacian $\Delta\mathbf{v}$, we find

$$
\begin{aligned}
\Delta v_1 &= -(\alpha_{rr}+3\alpha_r/r+\alpha_{zz})y+(\beta_{rr}+3\beta_r/r+\beta_{zz})x,\\
\Delta v_2 &= (\alpha_{rr}+3\alpha_r/r+\alpha_{zz})x+(\beta_{rr}+3\beta_r/r+\beta_{zz})y,\\
\Delta v_3 &= \gamma_{rr}+\gamma_r/r+\gamma_{zz},
\end{aligned}
\tag{6}
$$

where $\alpha_{rr}\equiv\partial^2\alpha/\partial r^2, \alpha_{zz}\equiv\partial^2\alpha/\partial z^2$.

Let us now substitute formulas (3), (5), and (6) into Navier-Stokes Equation (1). Then we come to the following three nonlinear partial differential equations:[29–31]

$$
\alpha_t+\beta(r\alpha_r+2\alpha)+\gamma\alpha_z-\nu(\alpha_{rr}+3\alpha_r/r+\alpha_{zz})=0,\quad \alpha_t\equiv\partial\alpha/\partial t,
\tag{7}
$$

$$
\beta_t+\beta(r\beta_r+\beta)+\gamma\beta_z-\alpha^2-\nu(\beta_{rr}+3\beta_r/r+\beta_{zz})=-(1/\rho)p_r/r,
\tag{8}
$$

$$
\gamma_t+r\beta\gamma_r+\gamma\gamma_z-\nu(\gamma_{rr}+\gamma_r/r+\gamma_{zz})=-(1/\rho)p_z.
\tag{9}
$$

The derived Equations (4) and (7)–(9) were studied in our papers[29–31] where some exact solutions to them were found. In Sec. II, we will examine these equations in the case of flows of viscous liquids with sinusoidal dependence on the coordinate z. In Sec. III, we will find an exact solution to the Navier-Stokes equations for liquids in the examined case. It will be shown that in this solution, the pressure in a liquid can reach a sufficiently small value at which the phenomenon of cavitation takes place. In Sec. IV, we will discuss the obtained results.

II. A SPECIAL AXIALLY SYMMETRIC CASE FOR THE NAVIER-STOKES EQUATIONS

Let us seek the functions $\alpha,\beta,$and γ in the following form:

$$
\alpha=\frac{A}{r^2},\quad \beta=-\frac{F_z}{r^2},\quad \gamma=\frac{F_r}{r},
\tag{10}
$$

where $A=A(t,r,z)$ and $F=F(t,r,z)$ are some differentiable functions.

Then, as can be readily verified, Eq. (4) is identically satisfied.

Substituting expressions (10) for α,β,γ into Eqs. (7)–(9), we obtain

$$
A_t-(A_rF_z-A_zF_r)/r-\nu(A_{rr}-A_r/r+A_{zz})=0,
\tag{11}
$$

$$
F_{tz}+(F_rF_{zz}-F_zF_{rz})/r+(A^2+F_z^2)/r^2-\nu(F_{rrz}-F_{rz}/r+F_{zzz})=\frac{1}{\rho}rp_r,
\tag{12}
$$

$$
F_{tr}+(F_rF_{rz}-F_zF_{rr})/r+F_rF_z/r^2-\nu(F_{rrr}-F_{rr}/r+F_r/r^2+F_{rzz})=-\frac{1}{\rho}rp_z.
\tag{13}
$$

Let us choose the variable $\eta=r^2$ instead of r. Then the functions $p,A,$and F can be represented as

$$
p=p(t,\eta,z),\quad A=A(t,\eta,z),\quad F=F(t,\eta,z),\quad \eta=r^2.
\tag{14}
$$

From expressions (14), we get

$$
\begin{aligned}
&p_r=2rp_\eta,\quad A_r=2rA_\eta,\quad A_{rr}=2(2\eta A_{\eta\eta}+A_\eta),\quad F_r=2rF_\eta,\quad F_{rr}=2(2\eta F_{\eta\eta}+F_\eta),\\
&F_{rz}=2rF_{\eta z},\quad F_{rrz}=2(2\eta F_{\eta\eta z}+F_{\eta z}),\quad F_{rzz}=2rF_{\eta zz},\quad F_{rrr}=4r(2\eta F_{\eta\eta\eta}+3F_{\eta\eta}).
\end{aligned}
\tag{15}
$$

Substituting expressions (15) into Eqs. (11)–(13), we come to the following equations:

$$A_t - 2(A_\eta F_z - A_z F_\eta) - \nu(4\eta A_{\eta\eta} + A_{zz}) = 0, \tag{16}$$

$$F_{tz} + 2(F_\eta F_{zz} - F_z F_{\eta z}) + (A^2 + F_z^2)/\eta - \nu(4\eta F_{\eta\eta z} + F_{zzz}) = (2/\rho)\eta p_\eta, \tag{17}$$

$$F_{t\eta} + 2(F_\eta F_{\eta z} - F_z F_{\eta\eta}) - \nu(4\eta F_{\eta\eta\eta} + 4F_{\eta\eta} + F_{\eta zz}) = -(1/\rho)p_z/2. \tag{18}$$

Let us seek particular solutions to Eqs. (16)–(18) in the form

$$p = p(t,\eta), \quad A(t,\eta,z) = \lambda F(t,\eta,z), \quad \lambda = \text{const} \neq 0. \tag{19}$$

Then substituting the expression for A in (19) into Eq. (16), we obtain

$$F_t - \nu(4\eta F_{\eta\eta} + F_{zz}) = 0. \tag{20}$$

After differentiating this equation with respect to z and η, we find

$$F_{tz} - \nu(4\eta F_{\eta\eta z} + F_{zzz}) = 0, \quad F_{t\eta} - \nu(4\eta F_{\eta\eta\eta} + 4F_{\eta\eta} + F_{\eta zz}) = 0. \tag{21}$$

Substituting equalities (19) and (21) into Eqs. (17) and (18), we obtain

$$2(F_\eta F_{zz} - F_z F_{\eta z}) + (\lambda^2 F^2 + F_z^2)/\eta = (2/\rho)\eta p_\eta, \tag{22}$$

$$F_\eta F_{\eta z} - F_z F_{\eta\eta} = 0. \tag{23}$$

Thus, we come to three Equations (20), (22), and (23).

Let us seek their particular solutions in the form

$$F = U(t,\eta)\sin(\lambda z + \delta), \quad \delta = \text{const}, \tag{24}$$

where $U(t,\eta)$ is some differentiable function.

Substituting expression (24) into Eqs. (20), (22), and (23), we obtain

$$U_t - \nu(4\eta U_{\eta\eta} - \lambda^2 U) = 0, \tag{25}$$

$$(2/\rho)\eta p_\eta = -\lambda^2 U(2U_\eta - U/\eta), \tag{26}$$

$$U_\eta{}^2 - UU_{\eta\eta} = 0. \tag{27}$$

Consider Eq. (26). From it, we find

$$p_\eta = -\frac{\rho\lambda^2}{2}\left(\frac{(U^2)_\eta}{\eta} - \frac{U^2}{\eta^2}\right) = -\frac{\rho\lambda^2}{2}(U^2/\eta)_\eta. \tag{28}$$

Taking into account (19) and the relation $\eta = r^2$ in (14), from Eq. (28), we come to the following expression for the pressure p:

$$p = p_*(t) - \frac{\rho\lambda^2 U^2}{2r^2}, \tag{29}$$

where $p_*(t)$ is some continuous and positive function.

In Sec. III, we will seek a particular analytical solution to the obtained partial differential equations (25) and (27).

III. A PARTICULAR SOLUTION TO THE NAVIER-STOKES EQUATIONS

First, consider Eq. (27). It can be rewritten as

$$U_{\eta\eta}/U_\eta = U_\eta/U, \quad U = U(t,\eta). \tag{30}$$

Integrating this equation with respect to η, we find

$$\ln|U_\eta| = \ln|U| + D(t), \tag{31}$$

where $D(t)$ is an arbitrary function of t.

From (31), we get

$$U_\eta = f(t)U, \tag{32}$$

where $f(t)$ is determined by $D(t)$.

Equation (32) gives

$$U = h(t)\exp\left(f(t)\eta\right), \tag{33}$$

where $h(t)$ is an arbitrary function.

Let us now substitute expression (33) for the function $U(t,\eta)$ into Eq. (25). Then, we obtain

$$\dot{h} + h\dot{f}\eta - \nu h(4\eta f^2 - \lambda^2) = 0. \tag{34}$$

From this equation, we find

$$\dot{h} + \lambda^2 \nu h = 0, \quad \dot{f} = 4\nu f^2. \tag{35}$$

Equations (35) give

$$h = h_0 \exp(-\lambda^2 \nu t), \quad f = -\frac{1}{4\nu t}, \quad h_0 = \text{const}, \tag{36}$$

where $t = 0$ corresponds to the singularity of the function $f(t)$.

Since $\eta = r^2$, as indicated in (14), from formulas (33) and (36), we obtain

$$U = h_0 \exp\left(-\lambda^2 \nu t - \frac{r^2}{4\nu t}\right). \tag{37}$$

Formulas (19), (24), and (37) give

$$F = h_0 \exp\left(-\lambda^2 \nu t - \frac{r^2}{4\nu t}\right)\sin(\lambda z + \delta), \quad A = \lambda F. \tag{38}$$

Using formulas (3), (10), and (38), we come to the following formulas for the components v_i of the vector of velocity:

$$v_1 = -\frac{\lambda h_0}{r^2}\left[\sin(\lambda z + \delta)y + \cos(\lambda z + \delta)x\right]\exp\left(-\lambda^2 \nu t - \frac{r^2}{4\nu t}\right), \tag{39}$$

$$v_2 = \frac{\lambda h_0}{r^2}\left[\sin(\lambda z + \delta)x - \cos(\lambda z + \delta)y\right]\exp\left(-\lambda^2 \nu t - \frac{r^2}{4\nu t}\right), \tag{40}$$

$$v_3 = -\frac{h_0}{2\nu t}\sin(\lambda z + \delta)\exp\left(-\lambda^2 \nu t - \frac{r^2}{4\nu t}\right). \tag{41}$$

From formulas (29) and (37), we obtain

$$p = p_*(t) - \frac{\rho \lambda^2 h_0^2}{2r^2}\exp\left(-2\lambda^2 \nu t - \frac{r^2}{2\nu t}\right), \tag{42}$$

where $p_*(t)$ can be an arbitrary continuous and positive function.

The found particular analytical solution (39)–(42) has the following properties for $t > 0$:

1) The functions v_i exponentially tend to zero as $r \to \infty$ and as $t \to +\infty$ when $\lambda \neq 0$ and have sinusoidal dependence on the coordinate z.
2) As $t \to 0+$ and $r \neq 0$, the functions v_i tend to zero.
3) As follows from (42), the function $p \to -\infty$ as $r \to 0$. This implies that the radial coordinate r should be positive in order to have positive pressures p in the considered liquids. Moreover, the coordinate r in them should satisfy the inequality $r \geq r_0(t) > 0$. Here the values $r = r_0(t)$ correspond to a sufficiently small positive value of the pressure p in a liquid below which the phenomenon of cavitation occurs. In this case, the process of rupturing a liquid and forming a vapor-filled cavity in it takes place.[32–34]

Therefore, the obtained solution (39)–(42) can be applied to the region occupied by a liquid which satisfies the inequality $r \geq r_0(t) > 0$. As to the region $r < r_0(t)$, it corresponds to the cavity filled by the liquid vapor.

It should be noted that in the region $r \geq r_* > 0$, where r_* is an arbitrary positive radius, the obtained particular solution to the Navier-Stokes equations is smooth. This property is consistent with the following important result proved in Ref. 35: weak solutions of the three-dimensional incompressible Navier-Stokes equations are smooth if the negative part of the pressure p is controlled, or if the positive part of the quantity $|\mathbf{v}|^2 + 2p/\rho$ is controlled.

From (42), we find that the function $r_0(t)$ should satisfy the equality

$$p_*(t) - \frac{\rho\lambda^2 h_0^2}{2r_0^2(t)} \exp\left(-2\lambda^2 \nu t - \frac{r_0^2(t)}{2\nu t}\right) = p_{\text{cav}}, \tag{43}$$

where p_{cav} is the small positive pressure below which the cavitation in a liquid takes place. The value p_{cav} is equal to the saturated vapor pressure of a liquid.[32–34]

Consider $t > 0$ and assume that $p_*(t) > p_{\text{cav}}$. Then Eq. (43) has a solution $r_0(t) > 0$ since the function p in (42) is larger than p_{cav} for sufficiently large r and smaller than p_{cav} for sufficiently small r. Moreover, the positive function $r_0(t)$ satisfying Eq. (43) is unique since expression (42) for the pressure p is an increasing function of r for any $t > 0$.

Let us determine the asymptotic behavior of the function $r_0(t)$ for small positive values of t. For this purpose, consider the function $p(t,r)$, given by (42), for $r = \sqrt{2(1+\varepsilon)\nu t \ln(1/t)}$, where $\varepsilon = \text{const} > 0$. Then from (42), we find

$$p(t,\bar{r}(t)) = p_*(t) - \frac{\rho\lambda^2 h_0^2 t^{\varepsilon}}{4(1+\varepsilon)\nu \ln(1/t)} \exp(-2\lambda^2 \nu t), \quad \bar{r}(t) = \sqrt{2(1+\varepsilon)\nu t \ln(1/t)}. \tag{44}$$

Therefore, when t takes sufficiently small positive values, $\varepsilon > 0$, and $p_*(0) > p_{\text{cav}}$, we get

$$p(t,\bar{r}(t)) > p_{\text{cav}}, \quad t \to 0+. \tag{45}$$

On the other hand, from (42) and (43), we have

$$p(t, r_0(t)) = p_{\text{cav}}. \tag{46}$$

As stated above, $p(t,r)$ is an increasing function of r for any $t > 0$. That is why from (45) and (46), for small positive t, we derive

$$r_0(t) < \bar{r}(t), \quad t \to 0+. \tag{47}$$

Taking into account the expression for $\bar{r}(t)$ in (44), from (47), we find

$$r_0(t) = O\left(\sqrt{t \ln(1/t)}\right), \quad t \to 0+, \tag{48}$$

and hence, $r_0(0) = 0$.

Consider now $r_0(t)$ for large positive values of t. From (43), we readily find that the following inequality holds when $t > 0$:

$$2r_0^2(t)(p_*(t) - p_{\text{cav}}) \le \rho\lambda^2 h_0^2 \exp(-2\lambda^2 \nu t). \tag{49}$$

Using (49), we obtain that when $p_*(+\infty)$ is finite and $p_*(+\infty) > p_{\text{cav}}$, the function $r_0(t) \to 0$ as $t \to +\infty$ and equality (43) gives

$$r_0(t) = \sqrt{\frac{\rho}{2(p_*(+\infty) - p_{\text{cav}})}}\, |\lambda h_0| \exp(-\lambda^2 \nu t), \quad t \to +\infty. \tag{50}$$

IV. CONCLUSION

We have examined the 3D Navier-Stokes equations for incompressible viscous liquids in the case of axial symmetry. The problem under examination was represented as a system of three nonlinear partial differential equations. These equations were studied and in the particular case in which the sought functions have forms (19) and (24). In this case, we came to Eqs. (25)–(27) which were exactly solved. As a result, a particular analytical solution to the Navier-Stokes equations was found. This solution is given by formulas (39)–(42) and describes flows of incompressible viscous liquids sinusoidal in the direction of their axis z of symmetry. As follows from formula (42), in order to have positive values of the pressure p, the radial coordinate r should be positive. Besides, it should satisfy the inequality $r \geq r_0(t)$, where $r_0(t)$ is some positive function. This function should correspond to a sufficiently small positive value of the pressure p in a liquid below which the phenomenon of cavitation can take place. The found solution can be applied to the region occupied by a liquid, which satisfies the inequality $r \geq r_0(t) > 0$, and the region $r < r_0(t)$ should be filled by the liquid vapor.

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