JOURNAL OF MATHEMATICAL PHYSICS **55**, 093102 (2014)

On some classes of nonstationary axially symmetric solutions to the Navier-Stokes equations

Alexander S. Rabinowitch[a)]
Department of Applied Mathematics and Mechanics, Moscow State University of Instrument Engineering and Computer Science, Moscow 107996, Russia

(Received 15 March 2014; accepted 30 August 2014; published online 15 September 2014)

In the paper, the Navier-Stokes equations are studied in axially symmetric cases of nonstationary motion with rotation of incompressible viscous fluids. The problem is reduced to a nonlinear system of three partial differential equations for three unknown functions of the cylindrical coordinates r and z and time t. The three functions are sought in the form of power series in r with coefficients depending on t and z. For the unknown coefficients recurrence relations are obtained which contain three arbitrary functions. The relations are examined in three particular cases in which they give analytical solutions to the Navier-Stokes equations for any values of the coordinates t, z, and r. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4895463]

I. INTRODUCTION

Consider the Navier-Stokes equations describing a homogeneous incompressible fluid. They can be represented in the form[1–3]

$$\frac{\partial \mathbf{v}}{\partial t} + v_1 \frac{\partial \mathbf{v}}{\partial x} + v_2 \frac{\partial \mathbf{v}}{\partial y} + v_3 \frac{\partial \mathbf{v}}{\partial z} = -\frac{1}{\rho}\,\mathrm{grad}\, p + \mathbf{f} + \nu \Delta \mathbf{v}, \tag{1}$$

$$\mathrm{div}\,\mathbf{v} = 0, \quad \rho = \mathrm{const}, \quad \nu = \mathrm{const}, \tag{2}$$

where $\mathbf{v} = \mathbf{v}(t, x, y, z)$ is the vector of velocity, $p = p(t, x, y, z)$ is pressure, v_1, v_2, v_3 are the projections of the vector $\mathbf{v}$ onto the orthogonal axes x, y, z, t is time, $\mathbf{f} = \mathbf{f}(t, x, y, z)$ is the force per unit mass in the considered fluid, ρ is its density, and ν is its kinematic viscosity.

The Navier-Stokes equations are basic equations of fluid mechanics and extensive studies are devoted to them. The studies consider general questions of fluid dynamics,[1–21] problems of hydrodynamic stability, instabilities, and turbulence,[22–37] special cases of viscous flow,[38–50] and numerical methods for fluid dynamics.[2,51–65] However, because of substantial nonlinearity of these equations, only a small number of classes of analytical solutions to them were found. Our aim is to study some new analytical solutions in the axially symmetric case to the Navier-Stokes equations.

Further we will study the case in which the force $\mathbf{f}$ is potential. Then for its potential Φ we have the equality

$$\mathbf{f} = -\mathrm{grad}\,\Phi. \tag{3}$$

In this case, Eqs. (1) and (3) can be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t} + v_1 \frac{\partial \mathbf{v}}{\partial x} + v_2 \frac{\partial \mathbf{v}}{\partial y} + v_3 \frac{\partial \mathbf{v}}{\partial z} = \mathrm{grad}\, q + \nu \Delta \mathbf{v}, \quad q = -p/\rho - \Phi. \tag{4}$$

[a)]E-mail: arabinowitch17@gmail.com

Equations (4) are widely used in astrophysical studies where the function Φ is a gravitational potential.[66] It should be noted that the differential equations (2) and (4) describe the vector function $\mathbf{v}$ and, instead of the pressure p, the scalar function q. However, when the potential Φ is known and the function q is found, the pressure p can be determined by the equality $p = -\rho(q + \Phi)$.

Let us study axially symmetric solutions to the Navier-Stokes equations. For this purpose, we will seek the components $v_1,\ v_2,\ v_3$ of the vector function $\mathbf{v}$ and the function q in the following form:

$$v_1 = -\alpha y + \beta x, \qquad v_2 = \alpha x + \beta y, \qquad v_3 = \gamma, \qquad \alpha = \alpha(t,r,z),$$
$$\beta = \beta(t,r,z), \qquad \gamma = \gamma(t,r,z), \qquad q = q(t,r,z), \qquad r = \sqrt{x^2+y^2}. \tag{5}$$

Here the function α presents the angular velocities of points of a rotating fluid and the functions β and γ describe changing its shape.

Substituting expressions (5) into Eq. (2), we find

$$r\beta_r + 2\beta + \gamma_z = 0, \tag{6}$$

where $\beta_r \equiv \partial\beta/\partial r$, $\gamma_z \equiv \partial\gamma/\partial z$.

Using expressions (5), after simple calculations we obtain

$$\begin{aligned}
v_1\frac{\partial v_1}{\partial x} + v_2\frac{\partial v_1}{\partial y} + v_3\frac{\partial v_1}{\partial z} &= -(r\beta\alpha_r + \gamma\alpha_z + 2\alpha\beta)y + (r\beta\beta_r + \gamma\beta_z + \beta^2 - \alpha^2)x,\\
v_1\frac{\partial v_2}{\partial x} + v_2\frac{\partial v_2}{\partial y} + v_3\frac{\partial v_2}{\partial z} &= (r\beta\alpha_r + \gamma\alpha_z + 2\alpha\beta)x + (r\beta\beta_r + \gamma\beta_z + \beta^2 - \alpha^2)y,\\
v_1\frac{\partial v_3}{\partial x} + v_2\frac{\partial v_3}{\partial y} + v_3\frac{\partial v_3}{\partial z} &= r\beta\gamma_r + \gamma\gamma_z.
\end{aligned} \tag{7}$$

For the components of the Laplacian $\Delta\mathbf{v}$ we find

$$\begin{aligned}
\Delta v_1 &= -(\alpha_{rr} + 3\alpha_r/r + \alpha_{zz})y + (\beta_{rr} + 3\beta_r/r + \beta_{zz})x,\\
\Delta v_2 &= (\alpha_{rr} + 3\alpha_r/r + \alpha_{zz})x + (\beta_{rr} + 3\beta_r/r + \beta_{zz})y,\\
\Delta v_3 &= \gamma_{rr} + \gamma_r/r + \gamma_{zz},
\end{aligned} \tag{8}$$

where $\alpha_{rr} \equiv \partial^2\alpha/\partial r^2$, $\alpha_{zz} \equiv \partial^2\alpha/\partial z^2$.

Let us now substitute formulas (5), (7), and (8) into the Navier-Stokes equations (4). Then we come to the following three nonlinear partial differential equations:

$$\alpha_t + \beta(r\alpha_r + 2\alpha) + \gamma\alpha_z - \nu(\alpha_{rr} + 3\alpha_r/r + \alpha_{zz}) = 0, \quad \alpha_t \equiv \partial\alpha/\partial t, \tag{9}$$

$$\beta_t + \beta(r\beta_r + \beta) + \gamma\beta_z - \alpha^2 - \nu(\beta_{rr} + 3\beta_r/r + \beta_{zz}) = q_r/r, \tag{10}$$

$$\gamma_t + r\beta\gamma_r + \gamma\gamma_z - \nu(\gamma_{rr} + \gamma_r/r + \gamma_{zz}) = q_z. \tag{11}$$

In Sec. II, we will investigate the obtained four partial differential equations (6) and (9)–(11) for the unknown functions α, β, γ, and q and reduce them to a nonlinear system of three partial differential equations for the unknown functions α, β, and γ. In Sec. III, we will seek the functions α, β, and γ in the form of power series in r with coefficients depending on t and z and obtain recurrence relations for these coefficients. In Sec. IV, we will consider three particular cases in which the examined power series give analytical solutions to the Navier-Stokes equations for any values of the coordinates t, z, and r.

II. CONSEQUENCES OF THE NAVIER-STOKES EQUATIONS IN THE CASE OF AXIALLY SYMMETRY

Consider the obtained equations (9)–(11). First let us eliminate the function q in them. For this purpose, differentiating Eqs. (10) and (11) with respect to z and r, respectively, and using the evident equality $\partial q_r/\partial z = \partial q_z/\partial r$, we find

$$\frac{\partial}{\partial z}\left[\beta_t + \beta(r\beta_r + \beta) + \gamma\beta_z - \alpha^2 - \nu(\beta_{rr} + 3\beta_r/r + \beta_{zz})\right] = \frac{1}{r}\frac{\partial}{\partial r}\left[\gamma_t + r\beta\gamma_r + \gamma\gamma_z - \nu(\gamma_{rr} + \gamma_r/r + \gamma_{zz})\right]. \tag{12}$$

As can be readily verified, this equation can be represented in the form

$$\begin{aligned}&\beta_{tz} + \beta_z(r\beta_r + 2\beta) + r\beta\beta_{rz} + \gamma_z\beta_z + \gamma\beta_{zz} - 2\alpha\alpha_z - \nu(\beta_{rrz} + 3\beta_{rz}/r + \beta_{zzz})\\ &= (\gamma_r/r)_t + (r\beta_r + 2\beta)(\gamma_r/r) + r\beta(\gamma_r/r)_r + \gamma_z(\gamma_r/r) + \gamma(\gamma_r/r)_z\\ &-\nu[(\gamma_r/r)_{rr} + (3/r)(\gamma_r/r)_r + (\gamma_r/r)_{zz}].\end{aligned} \tag{13}$$

After eliminating γ_z by using equality (6): $\gamma_z = -r\beta_r - 2\beta$, Eq. (13) acquires the following form:

$$\begin{aligned}2\alpha\alpha_z &= (\beta_z - \gamma_r/r)_t + r\beta(\beta_z - \gamma_r/r)_r + \gamma(\beta_z - \gamma_r/r)_z\\ &-\nu[(\beta_z - \gamma_r/r)_{rr} + (3/r)(\beta_z - \gamma_r/r)_r + (\beta_z - \gamma_r/r)_{zz}].\end{aligned} \tag{14}$$

The obtained equations (6), (9), and (14) can be rewritten as

$$\begin{aligned}\gamma_z &= -r\beta_r - 2\beta,\\ -2\alpha\beta &= \alpha_t + r\beta\alpha_r + \gamma\alpha_z - \nu(\alpha_{rr} + 3\alpha_r/r + \alpha_{zz}),\\ 2\alpha\alpha_z &= \varphi_t + r\beta\varphi_r + \gamma\varphi_z - \nu(\varphi_{rr} + 3\varphi_r/r + \varphi_{zz}), \quad \varphi = \beta_z - \gamma_r/r.\end{aligned} \tag{15}$$

In order to determine the function $q = -p/\rho - \Phi$ and hence the pressure p, let us turn to Eqs. (10) and (11). From them we have

$$\begin{aligned}&q_r = \Psi(t, r, z), \quad q_z = \Theta(t, r, z),\\ &\Psi = r[\beta_t + \beta(r\beta_r + \beta) + \gamma\beta_z - \alpha^2 - \nu(\beta_{rr} + 3\beta_r/r + \beta_{zz})],\\ &\Theta = \gamma_t + r\beta\gamma_r + \gamma\gamma_z - \nu(\gamma_{rr} + \gamma_r/r + \gamma_{zz}).\end{aligned} \tag{16}$$

From (16), we derive the following equality which is equivalent to equality (12):

$$\partial\Psi/\partial z = \partial\Theta/\partial r. \tag{17}$$

This gives that in any singly connected region the expression $\Psi dr + \Theta dz$ is a total differential and the function q can be determined as follows:

$$q = \int_{(r_0,z_0)}^{(r,z)} (\Psi dr + \Theta dz) + q_0(t), \tag{18}$$

where the integral is taken along an arbitrary line connecting a fixed point (r_0, z_0) and point (r, z) and $q_0(t)$ is some function which gives values of q at the point (r_0, z_0).

Thus, the problem under consideration consists in finding solutions to the system of three equations (15) which will be further examined.

III. INVESTIGATION OF AXIALLY SYMMETRIC SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

Let us seek solutions to Eqs. (15) in the following form:

$$\alpha = \sum_{n=0}^{\infty} a_n(t, z)r^{2n}, \quad \beta = \sum_{n=0}^{\infty} b_n(t, z)r^{2n}, \quad \gamma = \sum_{n=0}^{\infty} c_n(t, z)r^{2n}, \tag{19}$$

where a_n, b_n, c_n are some functions of t and z, in a region in which the three power series are convergent.

Then for the function φ in (15) we find

$$\varphi = \beta_z - \gamma_r/r = \sum_{n=0}^{\infty} d_n(t, z) r^{2n}, \tag{20}$$

where

$$d_n(t, z) = b'_n - 2(n + 1)c_{n+1}, \quad b'_n \equiv \partial b_n/\partial z. \tag{21}$$

Let us substitute formulas (19) and (20) into the three equations (15). Then equating the terms containing r^{2n} in their left-hand and right-hand sides, we obtain

$$c'_n = -2(n + 1)b_n, \tag{22}$$

$$-2\sum_{k=0}^{n} a_k b_{n-k} = \dot{a}_n + \sum_{k=0}^{n}(2ka_k b_{n-k} + a'_k c_{n-k}) - \nu[4(n + 1)(n + 2)a_{n+1} + a''_n], \tag{23}$$

$$2\sum_{k=0}^{n} a'_k a_{n-k} = \dot{d}_n + \sum_{k=0}^{n}(2kd_k b_{n-k} + d'_k c_{n-k}) - \nu[4(n + 1)(n + 2)d_{n+1} + d''_n], \tag{24}$$

where $a'_n \equiv \partial a_n/\partial z$, $a''_n \equiv \partial^2 a_n/\partial z^2$, $\dot{a}_n \equiv \partial a_n/\partial t$.

From Eqs. (23) and (24), using (21) and (22), we come to the following recurrence relations:

$$a_{n+1} = \frac{1}{4\nu(n + 1)(n + 2)}\left(\dot{a}_n - \nu a''_n + \sum_{k=0}^{n}\frac{(n - k + 1)a'_k c_{n-k} - (k + 1)a_k c'_{n-k}}{n - k + 1}\right), \tag{25}$$

$$d_{n+1} = \frac{1}{4\nu(n + 1)(n + 2)}\left(\dot{d}_n - \nu d''_n - \sum_{k=0}^{n}\frac{(n - k + 1)(2a'_k a_{n-k} - d'_k c_{n-k}) + kd_k c'_{n-k}}{n - k + 1}\right), \tag{26}$$

where $n = 0, 1, 2, \ldots$ and

$$c_{n+1} = -\frac{1}{2(n + 1)}\left(d_n + \frac{c''_n}{2(n + 1)}\right). \tag{27}$$

It should be noted that in the recurrence relations (25)–(27) the three functions $a_0(t, z)$, $c_0(t, z)$, and $d_0(t, z)$ are arbitrary differentiable ones.

As is seen from recurrence relations (25)–(27), they contain the multipliers $(n + 1)^{-1}$, $(n + 2)^{-1}$, and $(n + 1)^{-1}$, respectively, which tend to zero as $n \to \infty$. This circumstance is important for the power series (19) to be absolutely convergent in a number of cases, as will be shown later on.

IV. THREE CLASSES OF PARTICULAR SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

Further we will consider three cases in which the recurrence relations (25)–(27) give solutions to the Navier-Stokes equations for any values of t, z, and r.

A. A class of solutions of closed form

Examine the case

$$a_0 = a(t), \quad c_0 = g(t) + zh(t), \quad d_0 = 0, \tag{28}$$

where $a(t)$, $g(t)$, and $h(t)$ are some differentiable functions. Then from (25)–(27), we derive

$$a_n = a_n(t), \quad c_{n+1}(t) = 0, \quad d_n = 0, \quad n = 0, 1, 2, \ldots, \tag{29}$$

where the functions $a_n(t)$ satisfy the recurrence relation

$$a_{n+1} = \frac{\dot{a}_n - (n+1)ha_n}{4\nu(n+1)(n+2)}, \quad n = 0, 1, 2, \ldots., \quad h = h(t), \quad a_n = a_n(t). \tag{30}$$

Consider now the case $a_{N+1} = 0$, where N is some non-negative integer. Then from (30), we obtain

$$a_n = 0, \quad n \geq N + 1 \tag{31}$$

and sequentially find $a_N, a_{N-1}, \ldots, a_0$ by the recurrence relation

$$a_n = 4\nu(n+1)(n+2)A_n(t)\left(\int_0^t \frac{a_{n+1}(\tau)}{A_n(\tau)}d\tau + C_n\right), \quad A_n(t) = \exp\left((n+1)\int_0^t h(\tau)d\tau\right), \tag{32}$$

where $n = N, N-1, \ldots, 0$, $a_{N+1} = 0$ and C_n are arbitrary constants.

From formulas (19), (22), (28), (29), and (31) we obtain the following particular solutions to Eqs. (15):

$$\alpha = \sum_{n=0}^{N} a_n(t)r^{2n}, \quad \beta = -\tfrac{1}{2}h(t), \quad \gamma = g(t) + zh(t), \tag{33}$$

where $a_n(t)$ are determined by formulas (32), N is an arbitrary non-negative integer, $g(t)$ and $h(t)$ are arbitrary differentiable functions, and the expressions for α and γ contain $N + 1$ arbitrary constants $C_0, C_1, \ldots, C_N$.

Thus, formulas (5), (32), and (33) give a class of particular solutions of closed form to the Navier-Stokes equations. These solutions can be applied to a fluid occupying a cylindrical region with $0 \leq r \leq r_0$, where r_0 is some finite radius. The constants $C_0, C_1, \ldots, C_N$ in them determine an initial condition at $t = 0$ and the functions $h(t)$ and $g(t)$ correspond to certain boundary conditions at $r = r_0$.

It should be noted that in the obtained particular solutions the function α does not tend to zero as $\nu \to 0$ when $C_n = U_n/\nu$ for some indices n, where U_n are nonzero constants independent of the parameter ν.

Consider now the particular case $h(t) = -h_0$, where $h_0 = \text{const} > 0$. Then from formulas (32) we derive

$$A_n = e^{-(n+1)h_0 t}, \quad a_n = \sum_{k=n}^{N} a_{n,k}e^{-(k+1)h_0 t}, \tag{34}$$

where $a_{n,k}$ are some constants and $a_{N,N} = 4\nu(N+1)(N+2)C_N$.

Substituting (34) into formula (32), we obtain for $n = N-1, N-2, \ldots, 0$,

$$\sum_{k=n}^{N} a_{n,k}e^{-(k+1)h_0 t} = 4\nu(n+1)(n+2)\left[\left(C_n + \frac{1}{h_0}\sum_{k=n+1}^{N}\frac{a_{n+1,k}}{k-n}\right)e^{-(n+1)h_0 t} - \sum_{k=n+1}^{N}\frac{a_{n+1,k}}{(k-n)h_0}e^{-(k+1)h_0 t}\right]. \tag{35}$$

This gives the recurrence relations for $a_{n,k}$,

$$a_{n,n} = 4\nu(n+1)(n+2)\left(C_n + \frac{1}{h_0}\sum_{k=n+1}^{N}\frac{a_{n+1,k}}{k-n}\right), \quad a_{n,k} = -4\nu(n+1)(n+2)\frac{a_{n+1,k}}{(k-n)h_0}, \tag{36}$$

where $n + 1 \le k \le N$, $n = N - 1, N - 2, \ldots, 0$, $a_{N,N} = 4\nu(N + 1)(N + 2)C_N$.

Let us put

$$C_n = \frac{S_n}{(N+1)(N+2)}, \quad h_0 = (N+1)(N+2) f_0, \tag{37}$$

where $S_n = \text{const}, f_0 = \text{const} > 0$.

Then the recurrence relations (36) acquire the form

$$a_{n,n} = \frac{4\nu(n+1)(n+2)}{(N+1)(N+2)} \left(S_n + \frac{1}{f_0} \sum_{k=n+1}^{N} \frac{a_{n+1,k}}{k-n} \right), \quad a_{n,k} = -\frac{4\nu(n+1)(n+2)a_{n+1,k}}{(k-n)(N+1)\,N+2)\,f_0}, \tag{38}$$

where $n + 1 \le k \le N, n = N - 1, N - 2, \ldots, 0, a_{N,N} = 4\nu S_N$.

In the examined case $h(t) = -h_0$, particular solutions to the Navier-Stokes equations are determined by formulas (33), (34), and (38). Since the considered constant $h_0 > 0$, from (33) and (34) we find that the function $\alpha(t, r) \to 0$ as $t \to +\infty$.

B. A class of solutions independent of *z*

Examine the case

$$a_0 = a(t), \quad c_0 = c(t), \quad d_0 = d(t), \tag{39}$$

where $a(t)$, $c(t)$, and $d(t)$ are some differentiable functions. Then from (25)–(27), we find

$$a_n = a_n(t), \quad c_n = c_n(t), \quad d_n = d_n(t), \quad n \ge 0, \tag{40}$$

$$a_{n+1} = \frac{\dot{a}_n}{4\nu(n+1)(n+2)}, \quad d_{n+1} = \frac{\dot{d}_n}{4\nu(n+1)(n+2)}, \quad c_{n+1} = -\frac{d_n}{2(n+1)}. \tag{41}$$

Formulas (41) give

$$a_n = \frac{a^{(n)}(t)}{(4\nu)^n n!(n+1)!}, \quad d_n = \frac{d^{(n)}(t)}{(4\nu)^n n!(n+1)!}, \quad c_{n+1} = -\frac{1}{2}\frac{d^{(n)}(t)}{(4\nu)^n\,((n+1)!)^2}, \quad n \ge 0, \tag{42}$$

where $a^{(0)}(t) \equiv a(t)$.

From (19), (22), and (42) we obtain

$$\alpha = a(t) + \sum_{n=1}^{\infty} \frac{a^{(n)}(t) r^{2n}}{(4\nu)^n n!(n+1)!}, \quad \beta = 0, \quad \gamma = c(t) - \frac{1}{2} \sum_{n=0}^{\infty} \frac{d^{(n)}(t) r^{2(n+1)}}{(4\nu)^n\,((n+1)!)^2}. \tag{43}$$

When $|a^{(n)}(t)| \le (A(t))^n$ and $|d^{(n)}(t)| \le (D(t))^n$, $n \ge 0$, where $A(t)$ and $D(t)$ are some positive functions, the two series in (43) are absolutely convergent for any r. In particular, these conditions are fulfilled when the functions $a(t)$ and $d(t)$ are finite sums of expressions of the form $Ce^{\lambda t}\sin(\vartheta t + \delta)$, where $C, \lambda, \vartheta, \delta$ are constants.

Thus, formulas (5) and (43) give one more class of particular solutions to the Navier-Stokes equations.

When the functions $a(t)$ and $d(t)$ are polynomials, from (43) we obtain finite sums for the functions $\alpha(t, r)$ and $\gamma(t, r)$.

Consider now the particular case

$$a(t) = \sum_{m=1}^{\infty} A_m e^{-\omega_m t}, \quad d(t) = \sum_{m=1}^{\infty} D_m e^{-\theta_m t}, \tag{44}$$

where $A_m, D_m, \omega_m, \theta_m$ are constants, $\omega_m > 0$, $\theta_m > 0$, $t \ge 0$, and the infinite consequences A_m and D_m are absolutely summable.

Then from formulas (43) we find

$$\alpha = \sum_{m=1}^{\infty} A_m e^{-\omega_m t} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!}\left(\frac{\omega_m r^2}{4\nu}\right)^n, \quad \beta = 0,$$
$$\gamma = c(t) + 2\nu \sum_{m=1}^{\infty} \frac{D_m}{\theta_m} e^{-\theta_m t} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{((n+1)!)^2}\left(\frac{\theta_m r^2}{4\nu}\right)^{n+1}. \tag{45}$$

Using the Bessel functions $J_0(x)$ and $J_1(x)$ of the first kind, these expressions can be represented, as

$$\alpha = \frac{2\sqrt{\nu}}{r} \sum_{m=1}^{\infty} \frac{A_m}{\sqrt{\omega_m}} J_1\left(\sqrt{(\omega_m/\nu)}r\right) e^{-\omega_m t},$$

$$\beta = 0,\ \ \gamma = c(t) + 2\nu \sum_{m=1}^{\infty} \frac{D_m}{\theta_m}\left[J_0\left(\sqrt{(\theta_m/\nu)}r\right) - 1\right] e^{-\theta_m t}. \tag{46}$$

It is well known that the functions $J_0(x)$ and $J_1(x)$ have the following form as $x \to +\infty$:

$$J_0(x) = (\pi x)^{-\frac{1}{2}}(\sin x + \cos x) + O(x^{-\frac{3}{2}}), \quad J_1(x) = (\pi x)^{-\frac{1}{2}}(\sin x - \cos x) + O(x^{-\frac{3}{2}}). \tag{47}$$

Therefore, in the considered case (44), formulas (5) and (46) give an infinite kinetic energy for a fluid which occupies the infinite region $0 \le r < \infty, \quad |z| \le z_0 = \text{const}$ and is not at rest.

Let us now examine a fluid occupying the cylindrical region $0 \le r \le r_0$, where r_0 is some finite radius, and let the boundary conditions for it be as follows:

$$\alpha(t, r_0) = 0, \quad \gamma(t, r_0) = 0. \tag{48}$$

It should be noted that in the particular case $\alpha(t, r) \equiv 0$, this problem is considered in Ref. 13. When $\alpha(t, r) \neq 0$, from (46) and (48) we derive

$$\sqrt{(\omega_m/\nu)}r_0 = k_m, \quad m = 1, 2, 3, \ldots, \tag{49}$$

where $k_1, k_2, k_3, \ldots$ is the infinite consequence of zeros of the Bessel function $J_1(x)$: $0 < k_1 < k_2 < k_3 < \ldots,\ J_1(k_m) = 0,\ m = 1, 2{,}3, \ldots.$

Therefore, the function $\alpha(t, r)$ acquires the form

$$\alpha = \frac{2r_0}{r} \sum_{m=1}^{\infty} \frac{A_m}{k_m} J_1(k_m r/r_0) \exp\left(-\nu k_m^2 t/r_0^2\right). \tag{50}$$

As is well known, the Bessel functions $J_\mu\left(k_m^{(\mu)} s\right)$, where $\mu > -1$ and $J_\mu\left(k_m^{(\mu)}\right) = 0$, $0 < k_1^{(\mu)} < k_2^{(\mu)} < \ldots < k_m^{(\mu)} < \ldots$, are an orthogonal system in the region $0 < s < 1$ and any continuous function $f(s)$ defined in this region can be expressed as the series[67]

$$f(s) = \sum_{m=1}^{\infty} f_m^{(\mu)} J_\mu(k_m^{(\mu)} s), \quad 0 < s < 1, \tag{51}$$

where

$$f_m^{(\mu)} = \frac{2}{J_{\mu+1}^2(k_m^{(\mu)})} \int_0^1 s f(s) J_\mu(k_m^{(\mu)} s) ds, \quad m = 1, 2, 3, \ldots. \tag{52}$$

That is why the coefficients A_m in (50) can be chosen so as to satisfy the initial condition at $t = 0$: $\alpha(0, r) = \alpha_0(r)$, where $\alpha_0(r)$ is an arbitrary continuous function.

Analogously, in order to satisfy an initial condition for the function γ at $t = 0$, we should choose the numbers θ_m in the form

$$\sqrt{(\theta_m/\nu)}r_0 = l_m, \quad m = 1, 2, 3, \ldots, \tag{53}$$

where l_m are different zeros of the Bessel function $J_0(x)$: $0 < l_1 < l_2 < l_3 < \ldots$, $J_0(l_m) = 0$, $m = 1, 2, 3, \ldots$.

From formulas (46) and (53), we obtain

$$\gamma = c(t) + 2r_0^2 \sum_{m=1}^{\infty} \frac{D_m}{l_m^2} \left[J_0\left(l_m r/r_0\right) - 1\right] \exp\left(-\nu l_m^2 t/r_0^2\right). \tag{54}$$

As follows from (48) and (54),

$$c(t) = 2r_0^2 \sum_{m=1}^{\infty} \frac{D_m}{l_m^2} \exp\left(-\nu l_m^2 t/r_0^2\right). \tag{55}$$

Thus, the obtained formulas (50), (54), and (55) describe solutions to the Navier-Stokes equations that satisfy the boundary conditions (48) in the considered case (44).

When $\nu \to 0$, from these formulas we find

$$\alpha = \frac{2r_0}{r} \sum_{m=1}^{\infty} \frac{A_m}{k_m} J_1\left(k_m r/r_0\right), \quad \gamma = 2r_0^2 \sum_{m=1}^{\infty} \frac{D_m}{l_m^2} J_0\left(l_m r/r_0\right), \quad \nu = 0. \tag{56}$$

When $\nu > 0$ formulas (50), (54), and (55) give that $\alpha(t) \to 0$ and $\gamma(t) \to 0$ as $t \to +\infty$.

C. A class of solutions depending on one function of *t* and *z*

Examine the case

$$a_0 = 0, \quad c_0 = c(t, z), \quad d_0 = D = \text{const}, \tag{57}$$

where $c(t, z)$ is some differentiable function. Then from formulas (25)–(27) we find

$$a_n = 0, \quad d_{n+1} = 0, \quad c_{n+2} = -\frac{c''_{n+1}}{4(n+2)^2}, \quad n \geq 0, \quad c_1 = -\frac{1}{2}\left(D + \frac{c''}{2}\right) \tag{58}$$

and hence

$$c_n = (-1)^n \frac{c_z^{(2n)}}{4^n (n!)^2}, \quad n \geq 2, \tag{59}$$

where $c_z^{(k)} \equiv \partial^k c/\partial z^k$.

From formulas (19), (22), and (57)–(59), we obtain the following solution to Eqs. (15):

$$\alpha = 0, \quad \beta = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{c_z^{(2n+1)} r^{2n}}{4^n (n+1)! n!}, \quad \gamma = c - \frac{Dr^2}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{c_z^{(2n)} r^{2n}}{4^n (n!)^2}. \tag{60}$$

When $|c_z^{(n)}| \leq (K(t, z))^n$, $n \geq 0$, $-\infty < z < \infty$, where $K(t, z)$ is some positive function, the two series in (60) are absolutely convergent for any r and z.

Thus, formulas (5) and (60) give one more class of particular solutions to the Navier-Stokes equations.

It should be stressed that formulas (60) give expressions for the vector of velocity $\mathbf{v}$ independent of the kinematic viscosity ν. That is why these formulas can be applied to both the Navier-Stokes and Euler equations.

As can be easily verified, the functions β and γ determined by formulas (60) satisfy the equalities $\beta_{rr} + 3\beta_r/r + \beta_{zz} = 0$, $\gamma_{rr} + \gamma_r/r + \gamma_{zz} = -2D$. Therefore, from (16), we derive that in the considered case the functions Θ and q depend on the parameter ν when $D \neq 0$.

It should be noted that when the function $c(t,z)=\sum_{j=0}^{M} w_j(t)z^j$, where $w_j(t)$ are arbitrary functions and M is an arbitrary non-negative integer, from (60) we obtain finite sums for the functions β and γ.

The studied case (57) can be regarded as limiting when $\varepsilon \to 0$ in the following case:

$$a_0 = \varepsilon u_0(t,z), \quad c_0 = c(t,z), \quad d_0 = D = \text{const}, \tag{61}$$

where ε is a small parameter and $u_0(t,z)$ is some differentiable function.

Then, as follows from (26) and (27), expressions (60) for the functions β and γ are approximate and their deviations from the exact expressions are $O(\varepsilon^2)$.

Consider now the particular case

$$c(t,z) = \sum_{m=1}^{\infty} C_m(t)\sin(\lambda_m(t)z + \delta_m(t)), \tag{62}$$

where $C_m(t)$, $\lambda_m(t)$, and $\delta_m(t)$ are some differentiable functions of $t \geq 0$ and the infinite consequence $C_m(t)$ is absolutely summable for any $t \geq 0$.

Then from (60), we derive

$$\beta = -\frac{1}{2}\sum_{m=1}^{\infty} C_m(t)\cos(\lambda_m(t)z + \delta_m(t))\sum_{n=0}^{\infty}\frac{(\lambda_m(t))^{2n+1}\, r^{2n}}{4^n (n+1)!n!}, \tag{63}$$

$$\gamma = -\frac{1}{2}Dr^2 + \sum_{m=1}^{\infty} C_m(t)\sin(\lambda_m(t)z + \delta_m(t))\sum_{n=0}^{\infty}\frac{(\lambda_m(t))^{2n}\, r^{2n}}{4^n (n!)^2}. \tag{64}$$

Using the modified Bessel functions $I_0(x)$ and $I_1(x)$ of the first kind, these expressions can be represented as

$$\beta = -\frac{1}{r}\sum_{m=1}^{\infty} C_m(t)\cos(\lambda_m(t)z + \delta_m(t))\, I_1(\lambda_m(t)r), \tag{65}$$

$$\gamma = -\frac{1}{2}Dr^2 + \sum_{m=1}^{\infty} C_m(t)\sin(\lambda_m(t)z + \delta_m(t))\, I_0(\lambda_m(t)r). \tag{66}$$

It is well known that as $x \to +\infty$, the functions $I_0(x)$ and $I_1(x)$ are of the form $\frac{e^x}{\sqrt{2\pi x}}\left[1 + O(x^{-1})\right]$ and $I_0(-x) = I_0(x)$, $I_1(-x) = -I_1(x)$.

Therefore, formulas (65) and (66) can be applied to a finite spatial region. In particular, these formulas can be applied to a fluid occupying a cylindrical region with $0 \leq r \leq r_0$, where r_0 is some finite radius.

Let $\lambda_m(t) = \Lambda(t)m$, where $\Lambda(t)$ is some differentiable function. Then formulas (65) and (66) correspond to certain boundary conditions at $r = r_0$ for the functions β and γ that are periodic in z.

V. CONCLUSION

We have considered the Navier-Stokes equations in the axially symmetric case. These equations were reduced to a nonlinear system of three partial differential equations. Their solutions were sought in the form of power series in r with coefficients depending on t and z. As a result, we came to recurrence relations for these coefficients which contain three arbitrary functions of t and z. Then we examined three particular cases in which analytical solutions for any t, r, and z to the Navier-Stokes equations were found. In the first of them a class of solutions of closed form was obtained. In the second case, we found a class of solutions independent of z. In the third case, we came to solutions depending on one function of t and z for which the vector of velocity $\mathbf{v}$ is independent of the kinematic viscosity ν.

[1] G. Gallavotti, *Foundations of Fluid Dynamics* (Springer-Verlag, Berlin, 2002).
[2] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis* (North-Holland Publishing Company, Amsterdam, 1977).
[3] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1987).
[4] A. Weinstein, Bull. Am. Math. Soc. **59**, 20 (1953).
[5] M. R. Ukhovskii and V. I. Iudovich, J. Appl. Math. Mech. **32**, 52 (1968).
[6] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow* (Gordon and Breach, New York, 1963).
[7] O. A. Ladyzhenskaya, Zap. Nauchn. Sem. LOMI **7**, 155 (1968).
[8] O. A. Ladyzhenskaya, Zap. Nauchn. Sem. LOMI **115**, 137 (1982).
[9] L. Caffarelli, R. Cohn, and L. Nirenberg, Commun. Pure Appl. Math. **35**, 771 (1982).
[10] C. Foias, C. Guillopé, and R. Temam, J. Differ. Equ. **57**, 440 (1985).
[11] M. E. Schonbek, Commun. Partial Differ. Equ. **11**, 733 (1986).
[12] P. Constantin, Commun. Math. Phys. **104**, 311 (1986).
[13] L. G. Loicansky, *Mechanics of Fluids and Gases* (Nauka, Moscow, 1987).
[14] P. Constantin and C. Foias, *Navier-Stokes Equations* (The University of Chicago Press, Chicago, London, 1988).
[15] V. I. Oseledets, Russ. Math. Surveys **44**, 210 (1989).
[16] C. Foias and R. Temam, J. Funct. Anal. **87**, 359 (1989).
[17] P. Constantin and C. Fefferman, Indiana Univ. Math. J. **42**, 775 (1993).
[18] G. Gallavotti, J. Math. Phys. **41**, 4061 (2000).
[19] H. Sohr, *The Navier-Stokes Equations* (Birkhäuser Verlag, Basel, 2001).
[20] Ya. G. Sinai, Russ. J. Math. Phys. **11**, 355 (2004).
[21] Ya. G. Sinai, J. Stat. Phys. **121**, 779 (2005).
[22] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge, 1981).
[23] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Dover, New York, 1981).
[24] P. Manneville, *Dissipative Structures and Weak Turbulence* (Academic Press, San Diego, 1990).
[25] P. Constantin and C. Fefferman, Nonlinearity **7**, 41 (1994).
[26] L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll, Science **261**, 578 (1993).
[27] U. Fisch, *Turbulence* (Cambridge University Press, Cambridge, 1995).
[28] P. Holmes, J. L. Lumley, and G. Berkooz, *Turbulence, Coherent Structures, Dynamical Systems and Symmetry* (Cambridge University Press, Cambridge, 1996).
[29] *Hydrodynamics and Nonlinear Instabilities*, edited by C. Godrèche and P. Manneville (Cambridge University Press, Cambridge, 1998).
[30] Ya. G. Sinai, PhysicaA **263**, 565 (1999).
[31] G. I. Barenblatt, A. J. Chorin, and V. M. Prostokishin, J. Fluid Mech. **410**, 263 (2000).
[32] C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier-Stokes Equations and Turbulence* (Cambridge University Press, Cambridge, 2001).
[33] C. Foias, D. Holm, and E. Titi, Physica D **152–153**, 505 (2001).
[34] P. J. Schmid and D. S. Henningson, *Stability and Transition in Shear Flows* (Springer-Verlag, New York, 2001).
[35] A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics Vol. 26 (Cambridge University Press, Cambridge, 2002).
[36] S. Baroud, B. Plapp, Z.-S. She, and H. Swinney, Phys. Rev. Lett. **88**, 114501 (2002).
[37] C. Bardos, M. C. Lopes Filho, D. Niu, H. J. Nussenzveig Lopes, and E. S. Titi, SIAM J. Math. Anal. **45**, 1871 (2013).
[38] K. R. Rajagopal and A. S. Gupta, Int. J. Eng. Sci. **19**, 1009 (1981).
[39] E. D. Siggia, J. Fluid Mech. **107**, 375 (1981).
[40] W. H. Hui, J. Appl. Math. Phys. **38**, 689 (1987).
[41] C. Childress, G. R. Terley, E. A. Spiegel, and W. R. Young, J. Fluid Mech. **203**, 1 (1989).
[42] C. Y. Wang, Ann. Rev. Fluid Mech. **23**, 159 (1991).
[43] A. Vincent and M. Meneguzzi, J. Fluid Mech. **225**, 1 (1991).
[44] O. P. Chandna and E. O. Oku-Ukpong, Int. J. Math. Math. Sci. **17**, 155 (1994).
[45] F. Labropulu, Acta Mechanica **141**, 11 (2000).
[46] H. G. Choe and J. Lewis, J. Funct. Anal. **175**, 348 (2000).
[47] D. Chae and J. Lee, Nonlinear Anal. **46**, 727 (2001).
[48] M. Jamil, Int. J. Nonlin. Sci. **9**, 296 (2010).
[49] D. Li and Ya. G. Sinai, Adv. Math. **229**, 1976 (2012).
[50] A. S. Rabinowitch, Int. J. Adv. Math. Sci. **1**, 199 (2013).
[51] J. Kim and P. Moin, J. Comput. Phys. **59**, 308 (1985).
[52] J. B. Bell, P. Colella, and H. M. Glaz, J. Comput. Phys. **85**, 257 (1989).
[53] G. E. Kamiadakis, M. Israeli, and S. A. Orszag, J. Comput. Phys. **97**, 414 (1991).
[54] P. M. Gresho, Ann. Rev. Fluid Mech. **23**, 413 (1991).
[55] H. Chen, S. Chen, and W. H. Matthaeus, Phys. Rev. A **45**, R5339 (1992).
[56] S.-C. Chang, J. Comput. Phys. **119**, 295 (1995).
[57] L. Tobiska and R. Verfürth, SIAM J. Numer. Anal. **33**, 107 (1996).
[58] P. M. Gresho and R. L. Sani, *Incompressible Flow and the Finite Element Method* (John Wiley, Chichester, 1998).
[59] C. E. Baumann and J. T. Oden, Int. J. Numer. Methods Fluids **31**, 79 (1999).
[60] Kh. Khadra, Ph. Angot, S. Pameix, and J.-P. Caltagirone, Int. J. Numer. Methods Fluids **34**, 651 (2000).
[61] K. Ohkitani and J. D. Gibbon, Phys. Fluids **12**, 3181 (2000).
[62] D. L. Brown, R. Cortez, and M. L. Minion, J. Comput. Phys. **168**, 464 (2001).

[63] T. Chung, *Computational Fluid Dynamics* (Cambridge University Press, Cambridge, 2001).
[64] P. Wesseling, *Principles of Computational Fluid Dynamics* (Springer-Verlag, Berlin, Heidelberg, 2001).
[65] J. H. Ferziger and M. Perić, *Computational Method for Fluid Dynamics* (Springer-Verlag, Berlin, 2002).
[66] C. J. Clarke and R. F. Carswell, *Principles of Astrophysical Fluid Dynamics* (Cambridge University Press, Cambridge, 2007).
[67] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide* (Academic Press, Amsterdam, 2012).