ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 15, No. 3, 2008, pp. 389–394. © Pleiades Publishing, Ltd., 2008.

Yang–Mills Fields of Nonstationary Spherical Objects with Big Charges

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Received September 2, 2007

Abstract. We study the nonlinear electrodynamics suggested in our paper in [Russ. J. Math. Phys. **12** (3), 379–385 (2005)] by using the Yang–Mills equations. This theory is intended to describe strong fields generated by objects with large electric charges. For nonstationary sources with spherical symmetry, we find formulas for the electric field strengths. This solution generalizes our previous result related to the stationary case. The corresponding formulas for nonlinear electric fields are used to explain some puzzling properties of the Earth atmosphere.

DOI: 10.1134/S1061920808030075

1. INTRODUCTION

Let us consider the nonlinear electrodynamics suggested in our paper [1] by using the Yang–Mills equations with SO(3) symmetry. This presents a reasonable generalization of the Maxwell theory, and it is intended to describe strong fields generated by macroscopic bodies with large electric charges whenever not only virtual photons but also virtual Z^0 and $W^{\pm}$ bosons can come into being.

The Yang–Mills equations can be represented as

$$D_\mu F^{k,\mu\nu} \equiv \partial_\mu F^{k,\mu\nu} + g\varepsilon_{klm} F^{l,\mu\nu} A^m_\mu = (4\pi/c) J^{k,\nu}, \tag{1}$$

$$F^{k,\mu\nu} = \partial^\mu A^{k,\nu} - \partial^\nu A^{k,\mu} - g\varepsilon_{klm} A^{l,\mu} A^{m,\nu}, \tag{2}$$

where $\mu,\nu = 0,1,2,3,\ k = 1,2,3,\ A^{k,\nu}$ and $F^{k,\mu\nu}$ are the potentials and strengths of a Yang–Mills field, D_μ is the Yang–Mills covariant derivative, ε_{klm} is the antisymmetric tensor, $\varepsilon_{123} = 1$, g is the coupling constant of electroweak interaction, and $\partial_\mu \equiv \partial/\partial x^\mu$, where x^μ are rectangular space-time coordinates of the Minkowski geometry.

As is well known, from the Yang–Mills equations (1)–(2), one can obtain [2, 3]

$$D_\nu D_\mu F^{k,\mu\nu} = 0, \tag{3}$$

$$D_\nu J^{k,\nu} \equiv \partial_\nu J^{k,\nu} + g\varepsilon_{klm} J^{l,\nu} A^m_\nu = 0. \tag{4}$$

Consider the following field sources:

$$J^{1,\nu} = J^\nu, \qquad J^{2,\nu} = J^{3,\nu} = 0, \tag{5}$$

where J^ν are the classical four-dimensional current densities.

Then equations (1)–(2) have a trivial particular solution in which $A^{2,\nu} = A^{3,\nu} = 0$ and the potentials $A^{1,\nu}$ and strengths $F^{1,\mu\nu}$ satisfy the Maxwell equations. Moreover, from (4) with $k = 1$ and from (5), we obtain the differential charge conservation equation [1]

$$\partial_\nu J^{1,\nu} = 0. \tag{6}$$

These properties prove that the nonlinear Yang–Mills equations under consideration with the classical sources of form (5) present a reasonable generalization of the linear Maxwell equations.

Since we examine the case $J^{2,\nu} = J^{3,\nu} = 0$, we choose the gauge

$$F^{2,\mu\nu} F^2_{\mu\nu} = F^{3,\mu\nu} F^3_{\mu\nu}, \tag{7}$$

which implies the equivalence of the second and third axes of the gauge space.

As is shown in [1], the Yang–Mills equations (1)–(2) with the classical sources of form (5) give no possibility to uniquely determine the strengths $F^{1,\mu\nu}$. This happens because the Yang–Mills equations under consideration are not independent, and there is a differential relation for these equations (which follows from (3)–(6)):

$$D_\nu[D_\mu F^{k,\mu\nu} - (4\pi/c)J^{k,\nu}] \equiv 0 \quad \text{for} \quad k = 1. \tag{8}$$

For this reason, we posed the problem [1] of finding a new equation in addition to the Yang–Mills equations (1)–(2), and presented the equation

$$\sum_{k=1}^{3} \partial_\alpha F^{k,\alpha\nu} \partial^\beta F^k_{\beta\nu} = (4\pi/c)^2 \sum_{k=1}^{3} J^{k,\nu} J^k_\nu. \tag{9}$$

In [1], this equation was chosen due to its Lorentz covariance and the covariance under the gauge rotations of $J^{k,\nu}$ and $F^{k,\mu\nu}$ by constant angles, and also because, in the trivial case $A^{2,\nu} = A^{3,\nu} = J^{2,\nu} = J^{3,\nu} = 0$, equation (9) agrees with the Maxwell equations for the potentials $A^{1,\nu}$. Let us now give a physical interpretation of the additional equation (9).

First, we represent the Yang–Mills equations (1) in the form

$$\partial_\mu F^{k,\mu\nu} = (4\pi/c) I^{k,\nu}, \tag{10}$$

$$I^{k,\nu} = J^{k,\nu} - gc(4\pi)^{-1} \varepsilon_{klm} F^{l,\mu\nu} A^m_\mu. \tag{11}$$

Since $F^{k,\mu\nu} = -F^{k,\nu\mu}$, and hence $\partial_\nu \partial_\mu F^{k,\mu\nu} = 0$, we see from (10) that the components $I^{k,\nu}$ satisfy the three differential equations of charge conservation,

$$\partial_\nu I^{k,\nu} = 0. \tag{12}$$

Therefore, the components $I^{k,\nu}$, as well as $J^{k,\nu}$, can be regarded as three four-dimensional current densities. By (11), the components $I^{k,\nu}$ contain not only the source current densities $J^{k,\nu}$ but also additional components defined by field potentials and strengths. This enables one to treat the components $I^{k,\nu}$ as densities of full currents which include not only the source current densities $J^{k,\nu}$ but also current densities of virtual field particles.

It follows from (10) that equation (9) can be represented as

$$I^{k,\nu} I_{k,\nu} = J^{k,\nu} J_{k,\nu}. \tag{13}$$

Consider a small part of a source of a Yang–Mills electric field. Let Δq^k be the intrinsic charges corresponding to the source current densities $J^{k,\nu}$ and $\Delta\overline{q}^k$ the full charges corresponding to the full current densities $I^{k,\nu}$. The full charges $\Delta\overline{q}^k$ also include charges of virtual particles created inside the part of the source in question. Choosing a local inertial frame comoving with a small part of the source, we can represent the correlation (13) as

$$\Delta\overline{q}^k \Delta\overline{q}_k = \Delta q^k \Delta q_k. \tag{14}$$

As is well known, the classical expression for the electrostatic energy of a homogeneous body with charge Δq is proportional to $(\Delta q)^2$ [4]. If a small part of the source has three charges $\Delta\overline{q}^1, \Delta\overline{q}^2, \Delta\overline{q}^3$, then its electrostatic energy should be proportional to $\Delta\overline{q}^k \Delta\overline{q}_k$. Therefore, the correlation (14) implies that the electrostatic energy of a small part of a source should be the same in both cases, namely, if $\Delta\overline{q}^k = \Delta q^k$ and if $\Delta\overline{q}^k \neq \Delta\overline{q}^k$. Thus, (13) (and hence (9), which was added to the Yang–Mills equations) can be treated as a differential condition of conservation for the electrostatic energy of the field source when charged virtual particles are created inside the source.

In Section 2, we investigate the Yang–Mills equations (1)–(2) and the additional equations (7) and (9) for nonstationary sources with spherical symmetry. In Section 3, in the case in question, we solve the partial differential equations for field strengths and find formulas for the strengths. This solution generalizes our previous result in [1] related to the stationary case. In Section 4, we use the corresponding formulas for nonlinear electric fields to explain some puzzling properties of the Earth atmosphere. In Section 5, we discuss the results.

2. YANG–MILLS EQUATIONS IN THE CASE OF NONSTATIONARY SOURCES WITH SPHERICAL SYMMETRY

Now let us consider the following spherically symmetric sources $J^{k,\nu}$ of the form (5) which are covariant under spatial rotations about the point $x^1 = x^2 = x^3 = 0$:

$$(4\pi/c)J^{1,0} = j^0(t,r), \quad (4\pi/c)J^{1,l} = x^l j(t,r), \quad J^{2,\nu} = J^{3,\nu} = 0,$$
$$l = 1,2,3, \quad t = x^0/c, \quad r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, \tag{15}$$

where x^ν are rectangular space-time coordinates of the Minkowski geometry, $\nu = 0,1,2,3$, while t is time, r is the distance from the source center, and j^0, j are some functions of t and r.

The function $j^0(t,r)$ can be represented in the form

$$j^0(t,r) = 4\pi\theta(t,r), \tag{16}$$

where θ is the charge density of the source.

Let us seek spherically symmetric potentials $A^{k,\nu}$ satisfying the Yang–Mills equations (1)–(2) with sources (15) in the form

$$A^{1,0} = \alpha^0(t,r),\ A^{2,0} = \beta^0(t,r),\ A^{3,0} = \gamma^0(t,r),\ A^{1,l} = x^l\alpha(t,r),\ A^{2,l} = x^l\beta(t,r),\ A^{3,l} = x^l\gamma(t,r), \tag{17}$$

where $l = 1,2,3$ and $\alpha^0, \alpha, \beta^0, \beta, \gamma^0, \gamma$ are some functions of t and r.

Substituting expressions (17) for $A^{k,\nu}$ into formula (2) for $F^{k,\mu\nu}$, we can see that

$$F^{k,ml} = 0, \qquad k,m,l = 1,2,3, \tag{18}$$

$$F^{1,0l} = x^l u(t,r), \quad F^{2,0l} = x^l v(t,r), \quad F^{3,0l} = x^l w(t,r), \quad F^{k,l0} = -F^{k,0l}, \tag{19}$$

where

$$u = (1/c)\partial\alpha/\partial t + (1/r)\partial\alpha^0/\partial r + g(\gamma^0\beta - \beta^0\gamma), \tag{20}$$
$$v = (1/c)\partial\beta/\partial t + (1/r)\partial\beta^0/\partial r + g(\alpha^0\gamma - \gamma^0\alpha), \tag{21}$$
$$w = (1/c)\partial\gamma/\partial t + (1/r)\partial\gamma^0/\partial r + g(\beta^0\alpha - \alpha^0\beta). \tag{22}$$

Substituting formulas (15)–(19) into the Yang–Mills equations (1), we obtain the following system of equations:

$$r\partial u/\partial r + 3u + gr^2(w\beta - v\gamma) = -j^0, \tag{23}$$
$$r\partial v/\partial r + 3v + gr^2(u\gamma - w\alpha) = 0, \tag{24}$$
$$r\partial w/\partial r + 3w + gr^2(v\alpha - u\beta) = 0, \tag{25}$$
$$(1/c)\partial u/\partial t + g(v\gamma^0 - w\beta^0) = j, \tag{26}$$
$$(1/c)\partial v/\partial t + g(w\alpha^0 - u\gamma^0) = 0, \tag{27}$$
$$(1/c)\partial w/\partial t + g(u\beta^0 - v\alpha^0) = 0. \tag{28}$$

Let us turn now to (4). Substituting (15) and (17) into (4), we obtain

$$(1/c)\partial j^0/\partial t + r\partial j/\partial r + 3j = 0, \tag{29}$$
$$j^0\gamma^0 - r^2 j\gamma = 0, \quad j^0\beta^0 - r^2 j\beta = 0. \tag{30}$$

Since the sources of form (15) are invariant under their gauge rotations about the first axis, we can impose the following gauge condition on the functions α^0 and α:

$$j^0\alpha^0 - r^2 j\alpha = 0. \tag{31}$$

Let us multiply (23) by j and (26) by j^0 and add the products. Using (30), we see that

$$(1/c)j^0\partial u/\partial t + j(r\partial u/\partial r + 3u) = 0. \tag{32}$$

Multiplying (24) by j and (27) by j^0, adding the products, and using relations (30) and (31), we obtain

$$(1/c)j^0\partial v/\partial t + j(r\partial v/\partial r + 3v) = 0. \tag{33}$$

In the same way, from (25) and (28), using (30)–(31), we obtain

$$(1/c)j^0\partial w/\partial t + j(r\partial w/\partial r + 3w) = 0. \tag{34}$$

Moreover, multiplying (23) by u, (24) by v, and (25) by w and adding the products, we see that

$$u(r\partial u/\partial r + 3u) + v(r\partial v/\partial r + 3v) + w(r\partial w/\partial r + 3w) = -j^0 u. \tag{35}$$

Let us turn to (9). By (15), (18), and (19), it follows from (9) that

$$(r\partial u/\partial r + 3u)^2 + (r\partial v/\partial r + 3v)^2 + (r\partial w/\partial r + 3w)^2$$
$$- (r/c)^2[(\partial u/\partial t)^2 + (\partial v/\partial t)^2 + (\partial w/\partial t)^2] = (j^0)^2 - r^2 j^2. \tag{36}$$

Equations (32)–(36) for the unknown functions u, v, w are solved in the next section.

3. DETERMINATION OF YANG-MILLS ELECTRIC FIELDS

After multiplying (32) by r^3, we can represent the product as

$$(1/c)j^0\partial(r^3u)/\partial t + jr\partial(r^3u)/\partial r = 0. \tag{37}$$

To solve this equation, we introduce the function

$$q(t,r) = \int_0^r r^2 j^0(t,r)dr. \tag{38}$$

By (29), we have

$$(1/c)\partial q/\partial t = (1/c)\int_0^r r^2\partial j^0/\partial t\, dr = -\int_0^r r^2(r\partial j/\partial r + 3j)\, dr = -\int_0^r \partial(r^3 j)/\partial r\, dr = -r^3 j. \tag{39}$$

It follows from (38) and (39) that

$$\partial q/\partial r = r^2 j^0, \qquad (1/c)\partial q/\partial t = -r^3 j. \tag{40}$$

Formulas (40) give

$$(1/c)j^0\partial q/\partial t + jr\partial q/\partial r = 0. \tag{41}$$

Using relation (41), we find a solution of (37) of the form

$$r^3 u = P(q), \tag{42}$$

where P is an arbitrary differentiable function of q.

Actually, substituting (42) into (37) and using (41), we obtain

$$(1/c)j^0\partial(r^3u)/\partial t + jr\partial(r^3u)/\partial r = dP/dq\,[(1/c)j^0\partial q/\partial t + jr\partial q/\partial r] = 0. \tag{43}$$

Hence, formula (42) gives exact solutions to the first-order partial differential equation (37) and contains an arbitrary function P. As is well known [5], this means that formula (42) represents the general exact solution of (37) and, consequently, of (32), which is equivalent to (37).

Equations (33) and (34) for the functions v and w are similar to equation (32) for which we have found the exact solution (42). Therefore, using (42) and (38), we obtain the following exact solutions to equations (32), (33), and (34):

$$u = P(q)/r^3, \quad v = Q(q)/r^3, \quad w = S(q)/r^3, \quad q = \int_0^r r^2 j^0(t,r)dr, \tag{44}$$

where P, Q, and S are differentiable functions of q.

Let us turn to (35). Using (44), we obtain

$$r\partial u/\partial r + 3u = j^0 dP/dq, \quad r\partial v/\partial r + 3v = j^0 dQ/dq, \quad r\partial w/\partial r + 3w = j^0 dS/dq. \tag{45}$$

Substituting (45) into (35), we have

$$(d/dq)[P^2(q) + Q^2(q) + S^2(q)] = -2P(q). \tag{46}$$

Writing

$$P = -R\cos\xi, \quad Q = -R\sin\xi\cos\eta, \quad S = -R\sin\xi\sin\eta, \quad R = R(q), \quad \xi = \xi(q), \tag{47}$$

we see from (46) that

$$dR/dq = \cos\xi. \tag{48}$$

Consider the gauge condition (7) by using (18)–(19), (44), and (47). To satisfy this condition, set

$$\eta = \pi/4. \tag{49}$$

Note that $P(0) = Q(0) = S(0) = 0$ by (44), and hence it follows from (48) that

$$R = \int_0^q \cos\xi(q)dq. \tag{50}$$

By (44) and (40), equation (36) yields

$$\partial u/\partial t = -cjdP/dq, \quad \partial v/\partial t = -cjdQ/dq, \quad \partial w/\partial t = -cjdS/dq. \tag{51}$$

Substituting (45) and (51) into (36), we obtain

$$[(j^0)^2 - r^2j^2][(dP/dq)^2 + (dQ/dq)^2 + (dS/dq)^2] = (j^0)^2 - r^2j^2. \tag{52}$$

Therefore,

$$(dP/dq)^2 + (dQ/dq)^2 + (dS/dq)^2 = 1. \tag{53}$$

Substituting formulas (47) into (53) and using (49) gives

$$(dR/dq)^2 + R^2(d\xi/dq)^2 = 1. \tag{54}$$

Equations (48) and (54) imply

$$dR/dq = \cos\xi, \qquad R^2(d\xi/dq)^2 = \sin^2\xi. \tag{55}$$

It is easy to show that equations (55) have the following solution:

$$\xi = q/K_0, \quad R = K_0\sin(q/K_0), \quad K_0 = \text{const}. \tag{56}$$

This solution was found in [1], where equations (55) were also studied.

Consider formulas (44), (47), (49), (56), and (16). Using (18)–(19), we can find the following expressions for the field strengths $F^{k,\mu\nu}$:

$$F^{1,l0} = K\sin(q/K)x^l/r^3, \quad K = K_0/2 = \text{const}, \quad F^{2,l0} = F^{3,l0} = 2^{-1/2}K[1-\cos(q/K)]x^l/r^3,$$
$$F^{k,ml} = 0, \quad k,m,l = 1,2,3, \quad q = q(t,r) = 4\pi\int_0^r r^2\theta(t,r)dr, \tag{57}$$

where θ is the charge density of a field source and q is the electric charge of the spherical domain of radius r, which depend on time t. It should be noted that the above solution (57) generalizes our previous result [1] for stationary sources with spherical symmetry.

It follows from (57) that

$$F^{1,l0} = q_{\text{eff}}x^l/r^3, \quad q_{\text{eff}} = K\sin(q/K). \tag{58}$$

In [1], using the value of the Earth's charge, the constant K was estimated as

$$K \sim 7\times 10^5 \text{ coulombs}. \tag{59}$$

Formula (58) presents a generalization of Coulomb's law, where q_{eff} and q can be called an effective charge and an actual charge, respectively. If the actual charge q is not large, $|q| \ll K$, then the effective charge q_{eff} and the actual charge q practically coincide, and formula (58) gives the classical Coulomb law. However, if $q \sim K$, then formula (58) substantially differs from the classical law.

As was shown in [1], formula (58) can be applied to unsolved problems concerning the origin of the Earth magnetic field and of the nature of ball lightning.

Introduce the two relativistic invariants,

$$\Phi = \sum_{k=1}^{3} F^{k,\mu\nu}F^k_{\mu\nu}, \qquad \Phi^1 = F^{1,\mu\nu}F^1_{\mu\nu}. \tag{60}$$

Using (57), we come to the following relativistic-invariant correlation at infinity:

$$\Phi^1/\Phi = \cos^2 q(r_0)/(2K), \qquad r\to\infty, \tag{61}$$

where $q(r_0)$ is the charge of a field source occupying some domain $0 \leqslant r \leqslant r_0$.

The behavior of the strength tensor $F^{k,\mu\nu}$ at infinity should be independent of the shape and charge distribution of a finite field source. Therefore, the relativistic-invariant correlation (61) can be regarded as a boundary condition at infinity that should be added to the differential equations of the suggested nonlinear electrodynamics based on the Yang–Mills theory.

In the next section, we consider applications of the above results to unsolved problems of the Earth atmosphere.

4. APPLICATIONS TO PHYSICS OF EARTH ATMOSPHERE

Despite the fact that the Earth atmosphere has been explored for a very long time, there is a number of mysteries concerning its structure and its electric field. Let us discuss some of them.

1. The mystery of atmospheric structure. The protonosphere, which is the uppermost layer of the Earth atmosphere, admits very high temperatures. These range up to about 1300°C, and that is why the protonosphere should have rapidly dissipated [6]. However, nothing of this sort has happened to this layer.

The temperatures in the two neighboring layers, mesosphere and ionosphere, contrast sharply with each other. The temperatures in the ionosphere are very high and range up to about 1100°C, whereas the temperatures in the mesosphere are very low and their minimum value is about −130°C [6]. It is very surprising that there is no equalization of temperatures in these two atmospheric layers, and the nature of this phenomenon is not explained within the framework of classical physics.

2. The mystery of the atmospheric electric field. As is known, in fair weather, the electric field of the atmosphere at the Earth surface is 100 v/m, but at a height of 50 km above its surface, the electric field is close to zero [7]. Moreover, experiments carried out by means of rockets [8] show a repeated change in the sign of the atmospheric electric field at heights from 10 to 80 km. Within the framework of classical physics, this phenomenon can be caused only by the alternation of layers with positive and negative charges. However, neighboring layers should attract in this case, and such a structure cannot be stable. For this reason, explanations based on classical physics for the detected properties of the atmospheric electric field face serious obstacles.

In order to unravel these mysteries of Earth atmosphere, consider formula (58) of the suggested nonlinear electrodynamics. It follows from this formula that the atmosphere can contain stable spherical layers with the actual charge $q = \pm 2\pi K$. Indeed, consider a layer which consists of particles with positive (or negative) charges and has the actual charge $q = 2\pi K$ (or $-2\pi K$). Then it follows from (58) that the layer generates attractive forces in its upper part and repulsive forces in its lower part, which can ensure the stability of the layer. This property enables one to explain the separation of the atmosphere into stable layers and the repeated change in the sign of the atmospheric electric field.

5. CONCLUSION

We have considered the nonlinear electrodynamics suggested in [1] for nonstationary sources with spherical symmetry. This electrodynamics is a nonlinear generalization of the Maxwell theory in the case of sufficiently large electric charges. It is based on the Yang–Mills equations (1)–(2) and an additional equation (9), which enables one to uniquely determine the field strengths and can be interpreted as a differential condition of conservation of the electrostatic energy of a field source when charged virtual particles are created inside it.

For nonstationary sources with spherical symmetry of the form (15), we have obtained the system of partial differential equations (32)–(36) for the components u, v, w of the Yang–Mills field strengths $F^{k,l0}$. Solving these equations, we found the exact expressions (57) for the field strengths $F^{k,l0}$. The solution generalizes our previous result [1] obtained for stationary sources with spherical symmetry.

Using formula (58) for nonlinear electric fields generated by spherical domains with large charges, we gave explanations to several puzzling properties of atmospheric structure and its electric field.

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