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EXACT AXIALLY SYMMETRIC WAVE SOLUTIONS OF THE YANG–MILLS EQUATIONS

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We find a class of exact axially symmetric wave solutions of the Yang–Mills equations with $SU(2)$ symmetry. The solutions in this class describe running waves propagating at the speed of light in a vacuum and contain two arbitrary differentiable functions of their phase. We consider properties of field sources that can generate such running waves.

Keywords: Yang–Mills equations, Yang–Mills field potentials, $SU(2)$ symmetry, axially symmetric wave solutions

We consider the equations describing Yang–Mills fields with $SU(2)$ symmetry outside their sources [1]:

$$\partial_\mu F^{k,\mu\nu} + g\varepsilon_{klm} F^{l,\mu\nu} A^m_\mu = 0, \tag{1}$$

$$F^{k,\mu\nu} = \partial^\mu A^{k,\nu} - \partial^\nu A^{k,\mu} - g\varepsilon_{klm} A^{l,\mu} A^{m,\nu}, \tag{2}$$

where $\mu,\nu = 0,1,2,3$, $k,l,m = 1,2,3$, $A^{k,\nu}$ and $F^{k,\mu\nu}$ are the potentials and strengths of the Yang–Mills field, ε_{klm} is the antisymmetric tensor with $\varepsilon_{123} = 1$, g is the coupling constant of electroweak interactions, and $\partial_\mu \equiv \partial/\partial x^\mu$, where x^μ are orthogonal space–time coordinates of the Minkowski geometry. The Yang–Mills equations with $SU(2)$ symmetry play a major role in various models of electroweak interactions [1], [2], but the mathematical properties of their solutions are not well studied because they are nonlinear. In the spherically symmetric case, a number of exact solutions of the Yang–Mills equations have been obtained [3], but the problem of finding their solutions in more complicated cases is still relevant.

Our objective is to find new exact axially symmetric wave solutions of Yang–Mills equations (1), (2) describing running waves that propagate at the speed of light in a vacuum. We seek such solutions of Eqs. (1) and (2) in the form

$$\begin{gathered} A^{k,0} = \alpha^{k,0}, \qquad A^{k,1} = \frac{\alpha^{k,1}x + \alpha^{k,2}y}{\rho}, \\ A^{k,2} = \frac{\alpha^{k,1}y - \alpha^{k,2}x}{\rho}, \qquad A^{k,3} = \alpha^{k,3}, \\ \alpha^{k,\nu} = \alpha^{k,\nu}(\tau,\rho,z), \qquad \tau \equiv x^0, \qquad \rho \equiv (x^2+y^2)^{1/2}, \\ x \equiv x^1, \qquad y \equiv x^2, \qquad z \equiv x^3. \end{gathered} \tag{3}$$

It follows that each of the three two-dimensional vectors $(A^{k,1}, A^{k,2})$, $k = 1,2,3$, are sums of the mutually orthogonal vectors $(\alpha^{k,1}/\rho)\cdot(x,y)$ and $(\alpha^{k,2}/\rho)\cdot(y,-x)$. Hence, the considered potentials $A^{k,\nu}$ are covariant under spatial rotations of the coordinate system (x,y,z) about the z axis and are therefore axially symmetric.

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Substituting formulas (3) in expressions (2) for the field strengths $F^{k,\mu\nu}$, we obtain

$$
\begin{aligned}
&F^{k,01} = \frac{f^{k,1}x + f^{k,2}y}{\rho}, \qquad F^{k,12} = f^{k,4}, \qquad F^{k,13} = \frac{f^{k,5}x + f^{k,6}y}{\rho}, \\
&F^{k,02} = \frac{f^{k,1}y - f^{k,2}x}{\rho}, \qquad F^{k,23} = \frac{f^{k,5}y - f^{k,6}x}{\rho}, \\
&F^{k,03} = f^{k,3}, \qquad f^{k,q} = f^{k,q}(\tau,\rho,z), \quad q = 1,2,\dots,6,
\end{aligned}
\tag{4}
$$

where the functions $f^{k,q}(\tau,\rho,z)$ are given by

$$
\begin{aligned}
&f^{k,1} = \alpha_\tau^{k,1} + \alpha_\rho^{k,0} - g\varepsilon_{klm}\alpha^{l,0}\alpha^{m,1}, \qquad f^{k,4} = \alpha_\rho^{k,2} + \frac{\alpha^{k,2}}{\rho} + g\varepsilon_{klm}\alpha^{l,1}\alpha^{m,2}, \\
&f^{k,2} = \alpha_\tau^{k,2} - g\varepsilon_{klm}\alpha^{l,0}\alpha^{m,2}, \qquad f^{k,5} = \alpha_z^{k,1} - \alpha_\rho^{k,3} - g\varepsilon_{klm}\alpha^{l,1}\alpha^{m,3}, \\
&f^{k,3} = \alpha_\tau^{k,3} + \alpha_z^{k,0} - g\varepsilon_{klm}\alpha^{l,0}\alpha^{m,3}, \qquad f^{k,6} = \alpha_z^{k,2} - g\varepsilon_{klm}\alpha^{l,2}\alpha^{m,3}
\end{aligned}
\tag{5}
$$

with $\alpha_\rho^{k,\nu} \equiv \partial\alpha^{k,\nu}/\partial\rho$, $\alpha_\tau^{k,\nu} \equiv \partial\alpha^{k,\nu}/\partial\tau$, and $\alpha_z^{k,\nu} \equiv \partial\alpha^{k,\nu}/\partial z$. We substitute expressions (3) and (4) for $A^{k,\nu}$ and $F^{k,\mu\nu}$ in the left-hand side of Yang–Mills field equation (1), denoted by $J^{k,\nu}$ in what follows. We then find

$$
\begin{aligned}
&J^{k,\nu} \equiv \partial_\mu F^{k,\mu\nu} + g\varepsilon_{klm}F^{l,\mu\nu}A_\mu^m, \\
&J^{k,0} = j^{k,0}, \qquad J^{k,1} = \frac{j^{k,1}x + j^{k,2}y}{\rho}, \\
&J^{k,2} = \frac{j^{k,1}y - j^{k,2}x}{\rho}, \qquad J^{k,3} = j^{k,3}, \qquad j^{k,\nu} = j^{k,\nu}(\tau,\rho,z),
\end{aligned}
\tag{6}
$$

where the functions $j^{k,\nu}$ are given by

$$
\begin{aligned}
&j^{k,0} = -f_\rho^{k,1} - \frac{f^{k,1}}{\rho} - f_z^{k,3} + g\varepsilon_{klm}(f^{l,1}\alpha^{m,1} + f^{l,2}\alpha^{m,2} + f^{l,3}\alpha^{m,3}), \\
&j^{k,1} = f_\tau^{k,1} - f_z^{k,5} + g\varepsilon_{klm}(f^{l,1}\alpha^{m,0} - f^{l,4}\alpha^{m,2} + f^{l,5}\alpha^{m,3}), \\
&j^{k,2} = f_\tau^{k,2} - f_\rho^{k,4} - f_z^{k,6} + g\varepsilon_{klm}(f^{l,2}\alpha^{m,0} + f^{l,4}\alpha^{m,1} + f^{l,6}\alpha^{m,3}), \\
&j^{k,3} = f_\tau^{k,3} + f_\rho^{k,5} + \frac{f^{k,5}}{\rho} - g\varepsilon_{klm}(f^{l,5}\alpha^{m,1} + f^{l,6}\alpha^{m,2} - f^{l,3}\alpha^{m,0}).
\end{aligned}
\tag{7}
$$

From Yang–Mills field equations (1) and formulas (6), we obtain the system of equations

$$
j^{k,\nu}(\tau,\rho,z) = 0, \quad k = 1,2,3, \quad \nu = 0,1,2,3,
\tag{8}
$$

where $j^{k,\nu}$ are determined by formulas (5) and (7). We seek the components $\alpha^{k,\nu}$ of field potentials that can satisfy Eqs. (8) in the form

$$
\begin{aligned}
&\alpha^{1,0} = 0, \qquad \alpha^{2,0} = P(\eta,\rho), \qquad \alpha^{3,0} = Q(\eta,\rho), \qquad \eta = \tau - z, \\
&\alpha^{1,2} = \frac{\varphi(\rho)}{g}, \qquad \alpha^{2,2} = \alpha^{3,2} = 0, \\
&\alpha^{k,1} = 0, \qquad \alpha^{k,3} = \alpha^{k,0}, \quad k = 1,2,3,
\end{aligned}
\tag{9}
$$

where $P(\eta,\rho)$, $Q(\eta,\rho)$, and $\varphi(\rho)$ are some differentiable functions. From (5), we then find

$$\begin{aligned}
&f^{1,1}=0, \qquad f^{2,1}=P_\rho, \qquad f^{3,1}=Q_\rho,\\
&f^{1,2}=0, \qquad f^{2,2}=-\varphi Q, \qquad f^{3,2}=\varphi P,\\
&f^{k,3}=0, \qquad f^{1,4}=\frac{\varphi'+\varphi/\rho}{g}, \qquad f^{2,4}=f^{3,4}=0,\\
&f^{k,5}=-f^{k,1}, \qquad f^{k,6}=-f^{k,2}, \quad k=1,2,3.
\end{aligned} \tag{10}$$

Here and hereafter, $P_\rho\equiv\partial P/\partial\rho$ and $P_{\rho\rho}\equiv\partial^2P/\partial\rho^2$ and similarly for Q.

Substituting formulas (9) and (10) in expressions (7) for $j^{k,\nu}$, we obtain

$$\begin{aligned}
&j^{1,0}=0, \qquad j^{2,0}=-(P_{\rho\rho}+P_\rho/\rho-\varphi^2P), \qquad j^{3,0}=-(Q_{\rho\rho}+Q_\rho/\rho-\varphi^2Q),\\
&j^{1,2}=-\frac{(\varphi'+\varphi/\rho)'}{g}, \qquad j^{2,2}=j^{3,2}=0,\\
&j^{k,1}=0, \qquad j^{k,3}=j^{k,0}, \quad k=1,2,3.
\end{aligned} \tag{11}$$

From (8) and (11), we derive

$$P_{\rho\rho}+\frac{P_\rho}{\rho}-\varphi^2P=0, \qquad Q_{\rho\rho}+\frac{Q_\rho}{\rho}-\varphi^2Q=0, \tag{12}$$

$$\left(\varphi'+\frac{\varphi}{\rho}\right)'=0. \tag{13}$$

Equation (13) has the nonzero solution vanishing at infinity

$$\varphi=\frac{b}{\rho}, \quad b=\mathrm{const}\neq 0. \tag{14}$$

Substituting (14) in (12), we obtain two differential equations for the functions P and Q,

$$P_{\rho\rho}+\frac{P_\rho}{\rho}-\left(\frac{b}{\rho}\right)^2P=0, \qquad Q_{\rho\rho}+\frac{Q_\rho}{\rho}-\left(\frac{b}{\rho}\right)^2Q=0. \tag{15}$$

It is easy to verify that the functions ρ^b and ρ^{-b} satisfy Eqs. (15). Therefore, the solutions of Eqs. (15) that vanish as $\rho\to\infty$ have the form

$$P=\frac{G(\tau-z)}{\rho^{|b|}}, \qquad Q=\frac{H(\tau-z)}{\rho^{|b|}}, \quad b\neq 0, \tag{16}$$

where G and H are arbitrary differentiable functions.

From formulas (3), (9), (14), and (16), we express the field potentials $A^{k,\nu}$ as

$$\begin{aligned}
&A^{1,0}=0, \qquad A^{2,0}=\frac{G(\tau-z)}{\rho^{|b|}}, \qquad A^{3,0}=\frac{H(\tau-z)}{\rho^{|b|}},\\
&A^{1,1}=\frac{b}{g}\frac{y}{\rho^2}, \qquad A^{2,1}=A^{3,1}=0, \qquad A^{1,2}=-\frac{b}{g}\frac{x}{\rho^2}, \qquad A^{2,2}=A^{3,2}=0,\\
&A^{k,3}=A^{k,0}, \quad k=1,2,3.
\end{aligned} \tag{17}$$

Formulas (4), (10), (14), and (16) give the expressions for the field strengths $F^{k,\mu\nu}$:

$$
\begin{aligned}
&F^{1,01} = 0, \qquad F^{2,01} = -\frac{|b|G(\tau - z)x + bH(\tau - z)y}{\rho^{2+|b|}},\\
&F^{3,01} = \frac{bG(\tau - z)y - |b|H(\tau - z)x}{\rho^{2+|b|}},\\
&F^{1,02} = 0, \qquad F^{2,02} = -\frac{|b|G(\tau - z)y - bH(\tau - z)x}{\rho^{2+|b|}},\\
&F^{3,02} = -\frac{bG(\tau - z)x + |b|H(\tau - z)y}{\rho^{2+|b|}},\\
&F^{k,03} = F^{k,12} = 0, \qquad F^{k,13} = -F^{k,01}, \qquad F^{k,23} = -F^{k,02}, \quad k = 1, 2, 3.
\end{aligned}
\tag{18}
$$

Here, $\tau = x^0 = ct$, where t is time.

We have thus found a class of exact cylindrically symmetric solutions of the Yang–Mills equations. These solutions contain two arbitrary differentiable functions $G(\tau - z)$ and $H(\tau - z)$ and an arbitrary nonzero constant b and describe running waves propagating at the speed of light in the direction of the z axis. As follows from (18), the vectors $F^{k,01}$ and $F^{k,02}$ are orthogonal. Therefore, the obtained solutions of the Yang–Mills equations considered here are nontrivial.

We now consider expressions (6) and (11) for the components $J^{k,\nu}$ of field sources that can generate the running waves under consideration. They can be represented as

$$
\begin{aligned}
&J^{1,0} = 0, \qquad J^{2,0} = j^{2,0}(\tau - z, \rho), \qquad J^{3,0} = j^{3,0}(\tau - z, \rho),\\
&J^{1,1} = \frac{j^{1,2}(\rho)y}{\rho}, \qquad J^{2,1} = J^{3,1} = 0,\\
&J^{1,2} = -\frac{j^{1,2}(\rho)x}{\rho}, \qquad J^{2,2} = J^{3,2} = 0,\\
&J^{k,3} = J^{k,0}, \quad k = 1, 2, 3.
\end{aligned}
\tag{19}
$$

From (19), we easily find that the components of the field sources satisfy the differential charge-conservation equations

$$
\partial_\nu J^{k,\nu} = 0, \quad k = 1, 2, 3. \tag{20}
$$

As can be seen from (19), the source components $J^{1,\nu}$ correspond to a rotation of the current carriers about the z axis. The source components $J^{2,\nu}$ and $J^{3,\nu}$ satisfy the relativistically invariant correlations $J^{2,\nu}J^2{}_\nu = 0$ and $J^{3,\nu}J^3{}_\nu = 0$.

We now consider the properties of field sources located on the z axis that can generate axially symmetric wave solutions of form (17). We first consider the equation for $P(\tau - z, \rho)$ in (11), which becomes

$$
P_{\rho\rho} + \frac{P_\rho}{\rho} - \left(\frac{b}{\rho}\right)^2 P = -j^{2,0}, \tag{21}
$$

where we take (14) into account. Knowing two particular solutions ρ^b and ρ^{-b} of linear differential equation (21) with $j^{2,0} = 0$, we can easily find its solution for a nonzero function $j^{2,0}$ by the method of variation of constants. Applying this method, we obtain a solution of Eq. (21) that vanishes as $\rho \to \infty$:

$$
P = \frac{1}{2|b|}\left[\rho^{-|b|}\int_0^\rho \rho^{1+|b|} j^{2,0}\, d\rho + \rho^{|b|}\int_\rho^\infty \rho^{1-|b|} j^{2,0}\, d\rho\right]. \tag{22}
$$

Let $\delta(\rho)$ be the delta function that is rotation symmetric in the plane (x, y),

$$\delta(\rho) = 0, \quad \rho > 0, \qquad \delta(0) = \infty, \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\rho)\, dx\, dy = 2\pi \int_{0}^{\infty} \rho\delta(\rho)\, d\rho = 1. \tag{23}$$

From formula (22), we then find the function $j^{2,0}(\tau - z, \rho)$ that ensures the form of the function $P(\tau - z, \rho)$ in (16) as $\rho > 0$,

$$j^{2,0} = \frac{4\pi|b|G(\tau - z)\delta(\rho)}{\rho^{|b|}}. \tag{24}$$

Completely similarly, we find the function $j^{3,0}(\tau - z, \rho)$ that ensures the form the function $Q(\tau - z, \rho)$ in (16) as $\rho > 0$:

$$j^{3,0} = \frac{4\pi|b|H(\tau - z)\delta(\rho)}{\rho^{|b|}}. \tag{25}$$

We consider the equation in (11) for the function $\varphi(\rho)$ and write it as

$$\varphi'' + \frac{\varphi'}{\rho} - \frac{\varphi}{\rho^2} = -gj^{1,2}. \tag{26}$$

This equation is similar to Eq. (21) with $b = 1$. Therefore, we similarly find the function $j^{1,2}(\rho)$ that ensures the form of the function $\varphi(\rho)$ in (14) as $\rho > 0$:

$$j^{1,2} = \frac{4\pi b}{g}\frac{\delta(\rho)}{\rho}. \tag{27}$$

The sought field sources that are located on the z axis and generate the potentials $A^{k,\nu}$ and field strengths $F^{k,\mu\nu}$ of the respective forms (17) and (18) are determined by formulas (19), (23)–(25), and (27).

REFERENCES

1. L. H. Ryder, *Quantum Field Theory*, Cambridge Univ Press., Cambridge (1985); N. F. Nelipa, *Physics of Elementary Particles*: *Gauge Fields* [in Russian], Vysshaya Shkola, Moscow (1985).
2. A. A. Slavnov and L. D. Faddeev, *Introduction to the Quantum Theory of Gauge Fields* [in Russian], Nauka, Moscow (1988); English transl.: L. D. Faddeev and A. A. Slavnov, *Gauge Fields*: *Introduction to Quantum Theory*, Benjamin, London (1990); A. S. Rabinowitch, *Internat. J. Theoret. Phys.*, **39**, 2457 (2000).
3. G. 't Hooft, *Nucl. Phys. B*, **79**, 276 (1974); A. M. Polyakov, *JETP Letters*, **20**, 194 (1974); R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations*, Acad. Press, London (1982); A. S. Rabinowitch, *Russ. J. Math. Phys.*, **12**, 379 (2005).