Two-dimensional link homotopy via chart descriptions

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Outline

1. Two-dimensional braids
2. Chart diagrams
3. Unknotting and link homotopy of surface links

Two-dimensional braids: Motion pictures

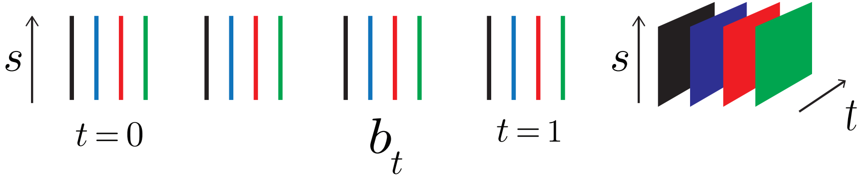
The trivial surface braid $\{x_1, \ldots, x_m\} \times I \times [0,1] \approx D^2 \times \{m \text{ points}\}$.

Two-dimensional braids: Motion pictures

Definition (Two-dimensional braid)

1. 1-parameter family $b_t, t \in [0,1]$ of geometric braids of degree m
2. b_0 and b_1 are the trivial braid $X_m \times I$, $X_m = \{x_1, \ldots, x_m\}$

$S = \cup_t (b_t \times \{t\})$ is a properly embedded surface in $(D^2 \times I) \times [0,1]$, called a surface braid of degree m.



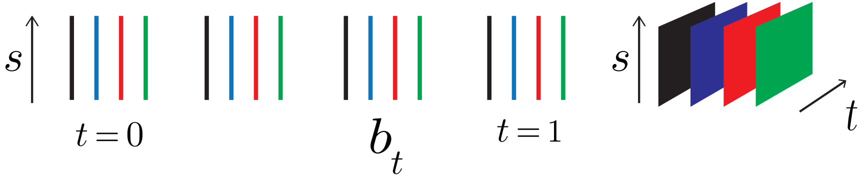
The trivial surface braid $\{x_1, \ldots, x_m\} \times I \times [0,1] \approx D^2 \times \{m \text{ points}\}$.

Two-dimensional braids: Motion pictures

Definition (Two-dimensional braid)

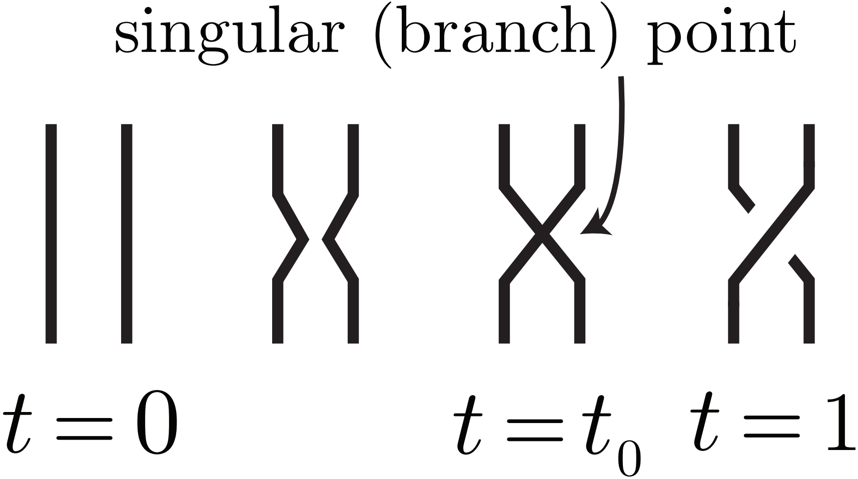
1. 1-parameter family $b_t, t \in [0,1]$ of geometric (possibly singular) braids of degree m
2. b_0 and b_1 are the trivial braid $X_m \times I$, $X_m = \{x_1, \ldots, x_m\}$

$S = \cup_t (b_t \times \{t\})$ is a properly embedded surface in $(D^2 \times I) \times [0,1]$, called a surface braid of degree m.

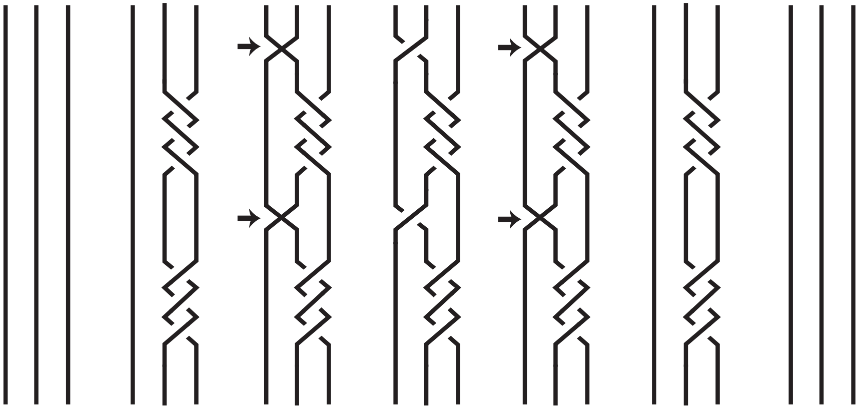


The trivial surface braid $\{x_1, \ldots, x_m\} \times I \times [0,1] \approx D^2 \times \{m \text{ points}\}$.

Two-dimensional braids: Singular points



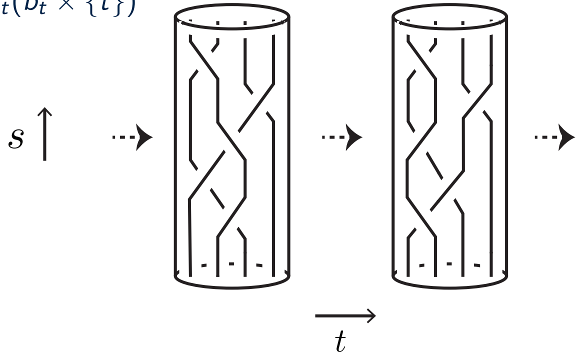
Two-dimensional braids: Example



$t = 0$ $t = t_1$ $t = t_2$ $t = 1$

Two-dimensional braids: Why singular points?

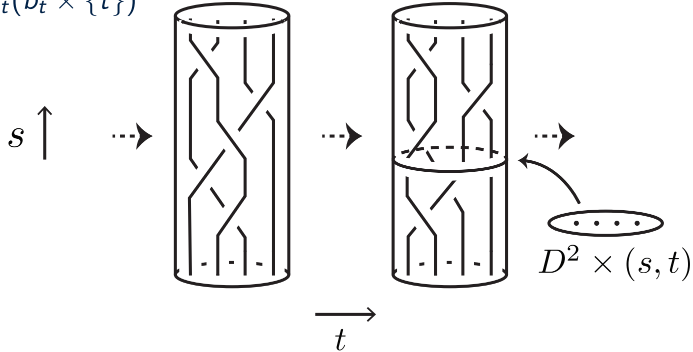
$S = \cup_t (b_t \times \{t\})$



Two-dimensional braids: Why singular points?

$S \mapsto \{(s,t) \mapsto b_t \cap (D^2 \times \{s\})\}$

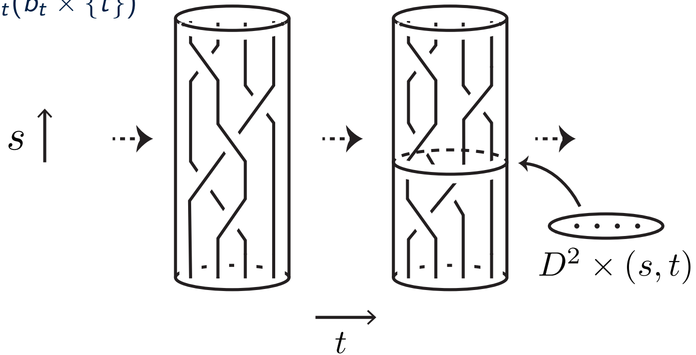
$S = \cup_t (b_t \times \{t\})$



Two-dimensional braids: Why singular points?

$$\overbrace{S \mapsto \{(s,t) \mapsto b_t \cap (D^2 \times \{s\})\}}^{(D^2,\partial D^2) \to \{m \text{ unordered distinct points in } D^2\} = C_m}$$

$S = \cup_t (b_t \times \{t\})$



Two-dimensional braids: Why singular points?

$$\overbrace{(D^2, \partial D^2) \to \{m \text{ unordered distinct points in } D^2\} = C_m}^{S \mapsto \{(s,t) \mapsto b_t \cap (D^2 \times \{s\})\}}$$

$\{\text{Surface braids}\} \to \pi_2(C_m, X_m)$, $X_m = \{x_1, \ldots, x_m\}$

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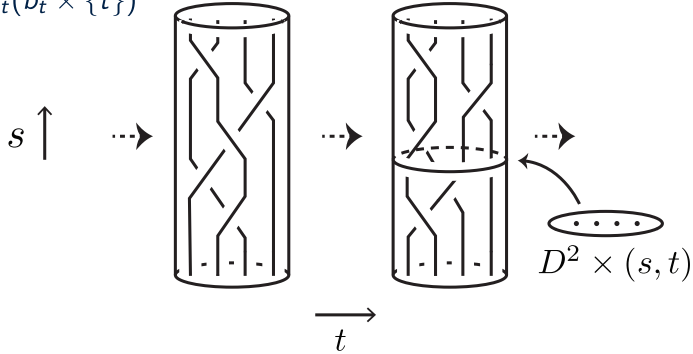


Two-dimensional braids: Why singular points?

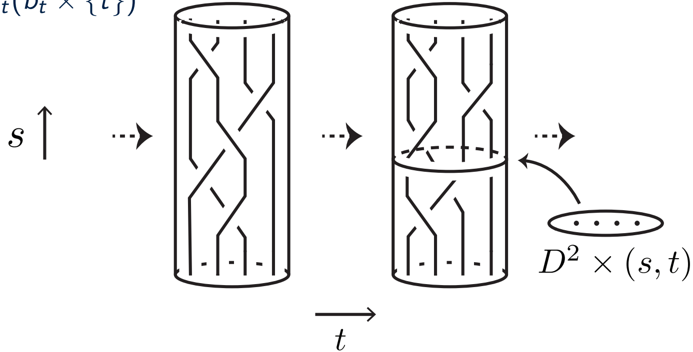
Surface braids without singular points are trivial

$$S \mapsto \{(s,t) \mapsto b_t \cap (D^2 \times \{s\})\}$$

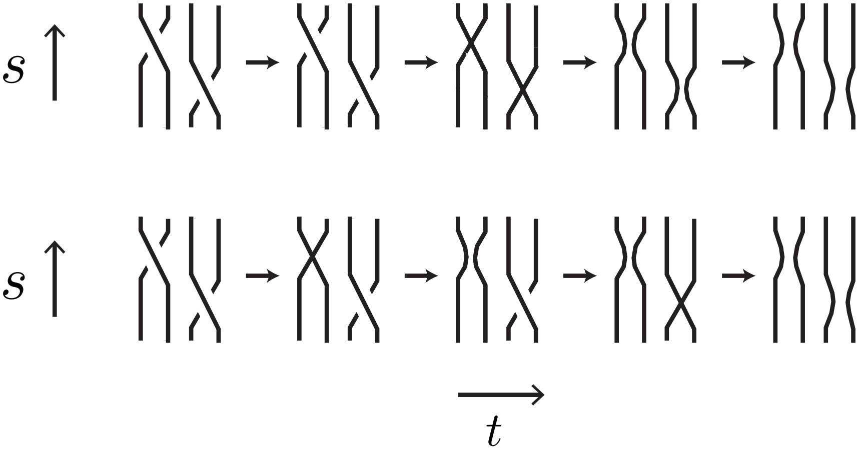
$$\overbrace{(D^2, \partial D^2) \to \{m \text{ unordered distinct points in } D^2\} = C_m}$$

$\{\text{Surface braids}\} \to \pi_2(C_m, X_m) = \{1\}$

$S = \cup_t (b_t \times \{t\})$



Two-dimensional braids: Equivalence



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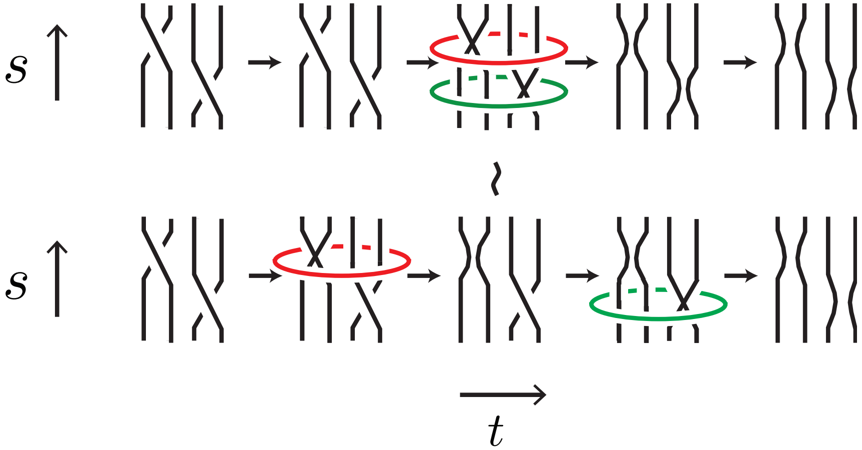

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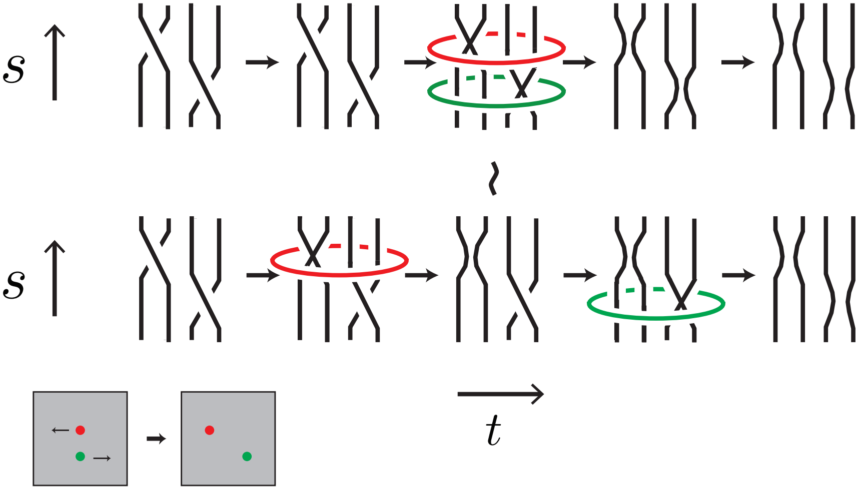
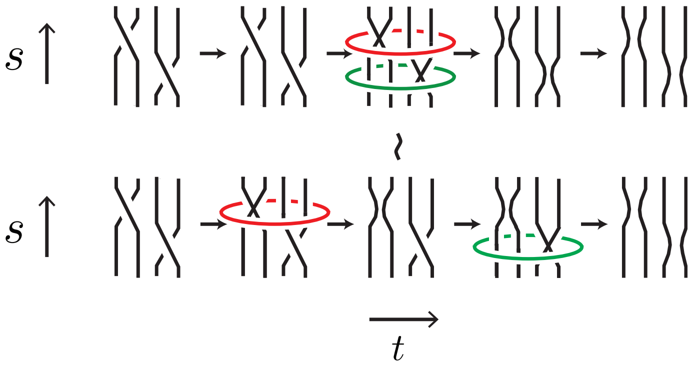


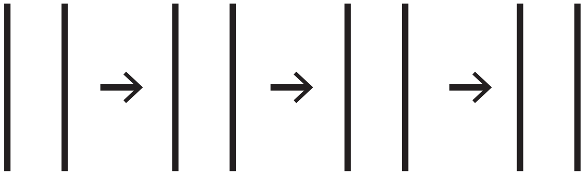
Two-dimensional braids: Equivalence

Two surface braids S and S' of degree m are equivalent if there is an ambient isotopy h_r, $r \in [0,1]$ of $D^2 \times I \times I$ such that $h_1(S) = S'$ and for each $r \in [0,1]$ we have

1. h_r maps (s,t)-fibers to (s,t)-fibers
2. $h_r|(D^2 \times \partial(I \times I))$ is the identity



Singular points as saddle bands



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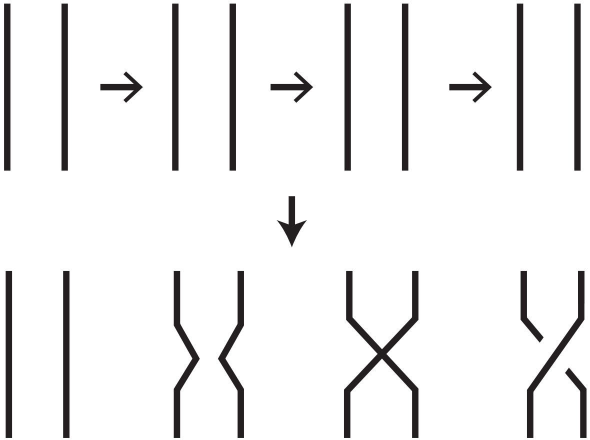

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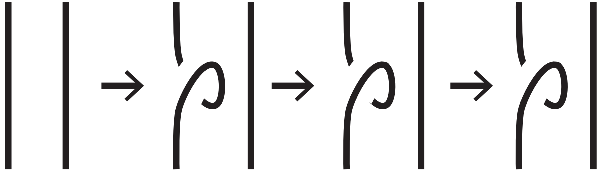

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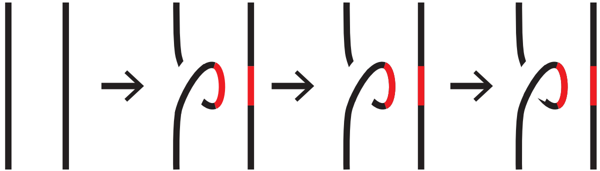

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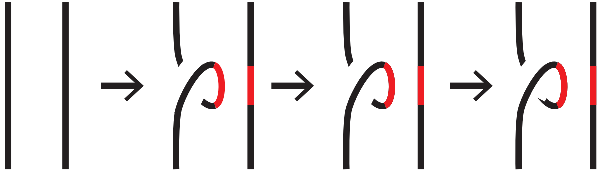

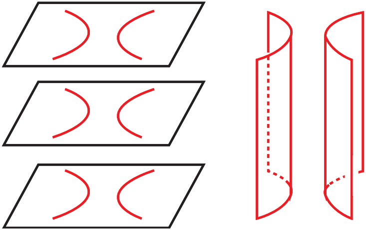

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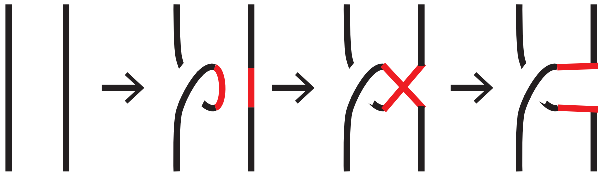

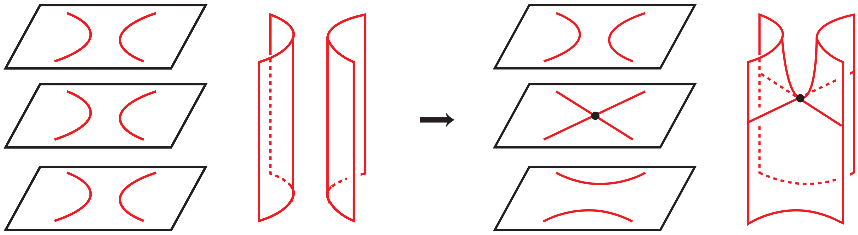

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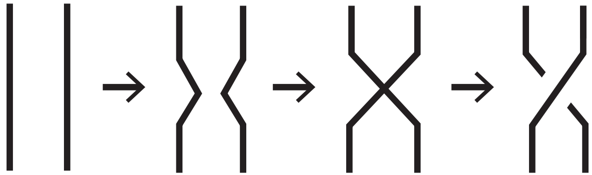

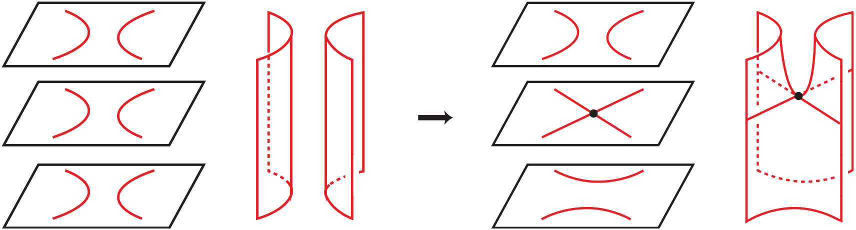

Surface braids and surface links

Definition

A surface link is an embedding of a oriented, closed surface into $\mathbb{R}^4$.

Surface braids and surface links

2D Alexander's theorem (Viro '90; S. Kamada '94)

Any surface link is equivalent to the closure of a (simple) surface braid of degree m for some m.

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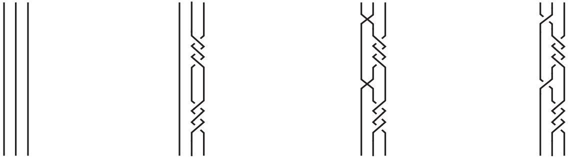
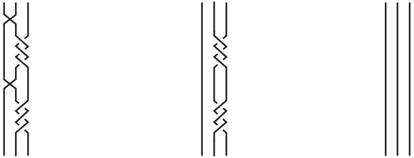


$t=0$



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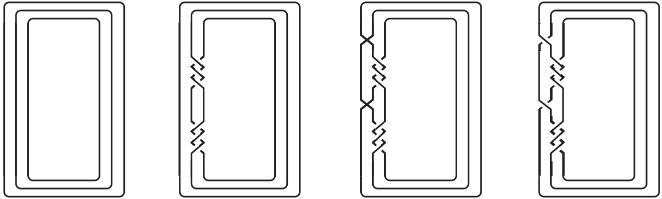
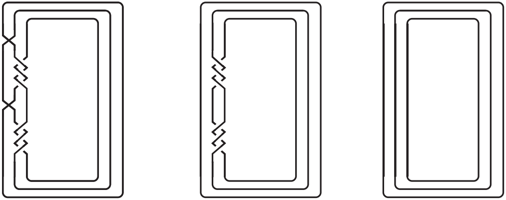


$t = 0$



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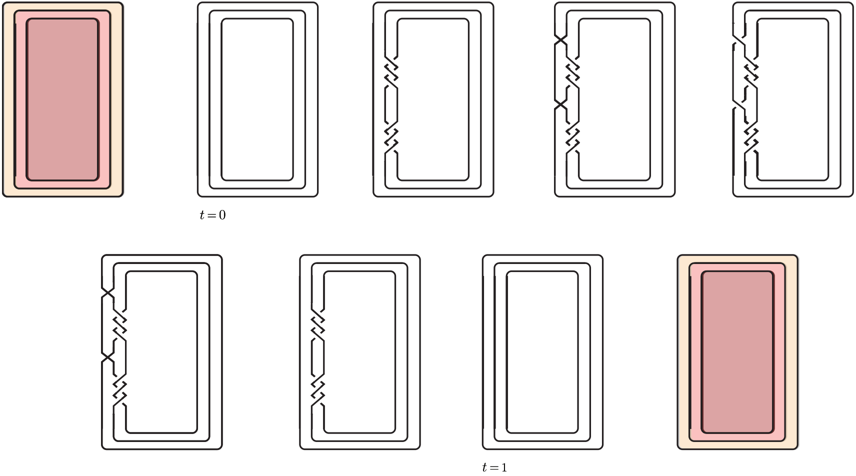

Immersed surface braids and surface links

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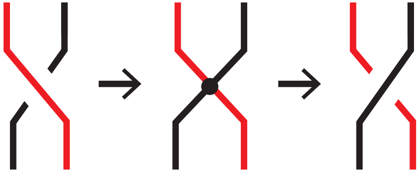
An <u>immersed</u> surface link is a generic immersion of an oriented, closed surface into $\mathbb{R}^4$.



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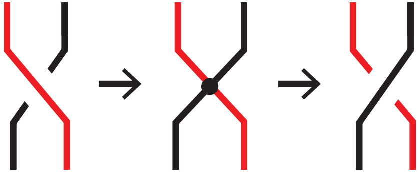

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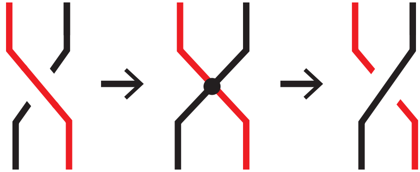

Immersed 2D Alexander's theorem (S. Kamada)

Any immersed surface link is equivalent to the closure of a immersed surface braid of degree m for some m.

Chart diagrams

Braid word chart

$\sigma_2\sigma_1^{-1}$

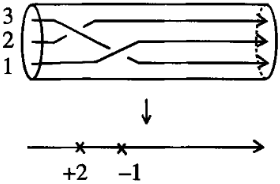

Chart diagrams

Braid word chart

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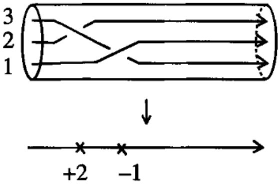
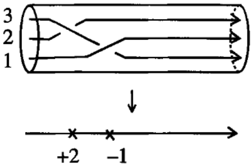


Braid word chart sequence



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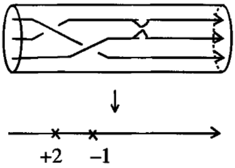
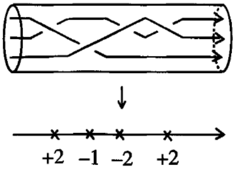
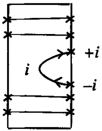
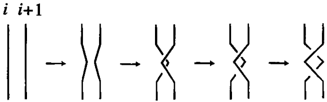




$\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_2$

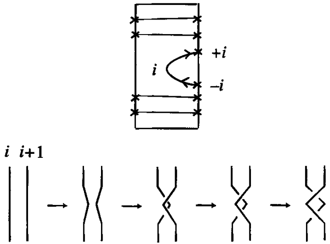
Chart diagrams for surface braids

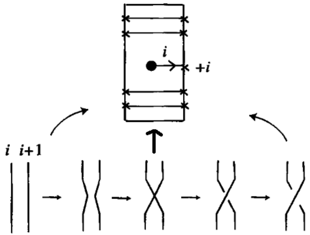




Illustrations taken from “Braid and Knot Theory in Dimension Four”, S. Kamada (2002)

Chart diagrams for surface braids





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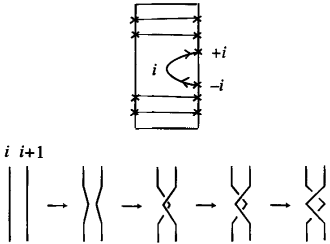

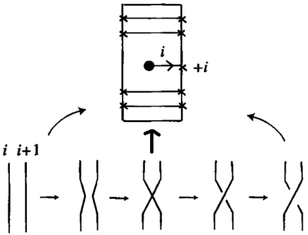

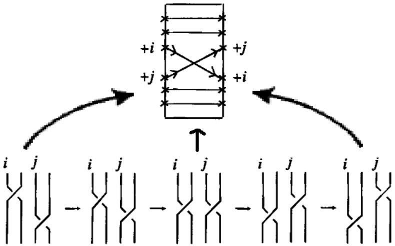

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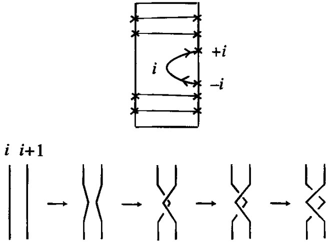

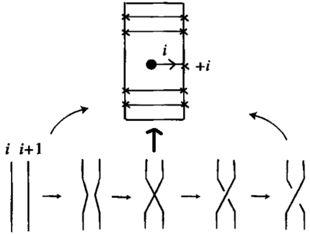

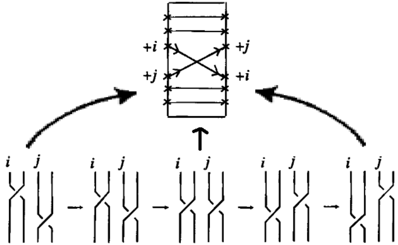

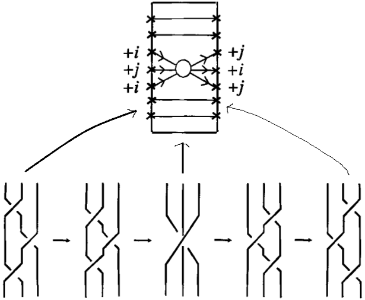

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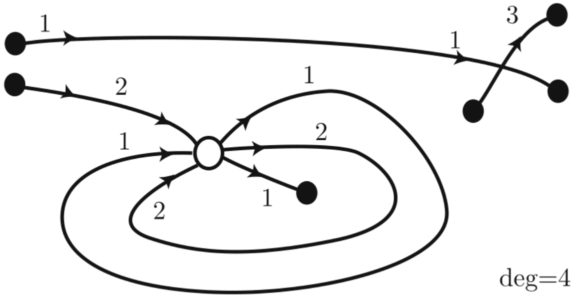
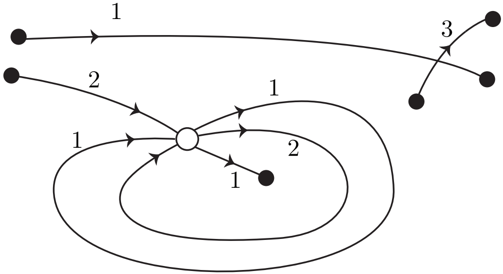
Chart diagrams for surface braids

Chart diagram degree m

A finite planar graph with edges oriented and labelled from $\{1, 2, \ldots, m-1\}$, and vertices with valence 1, 2 or 4.



Recovering a surface braid from a chart



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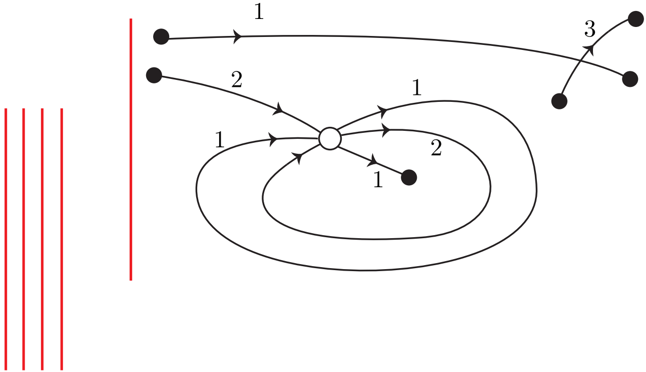

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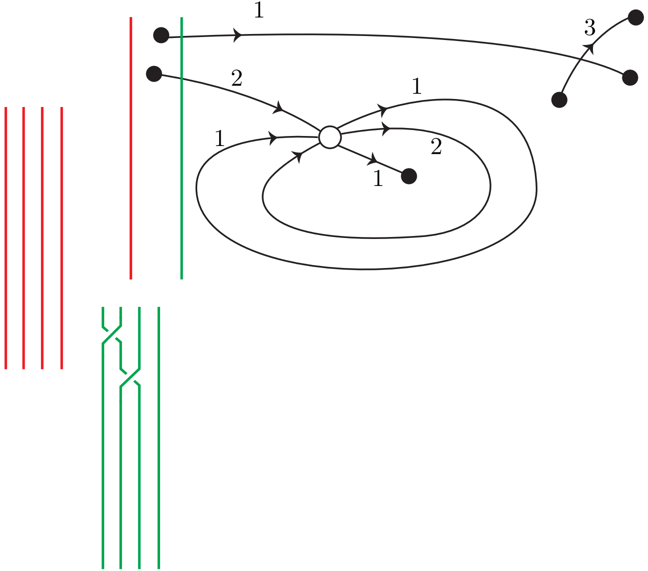

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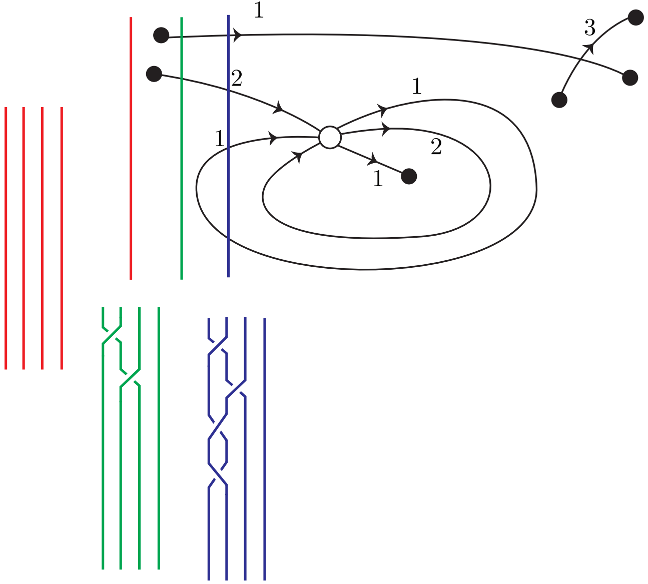

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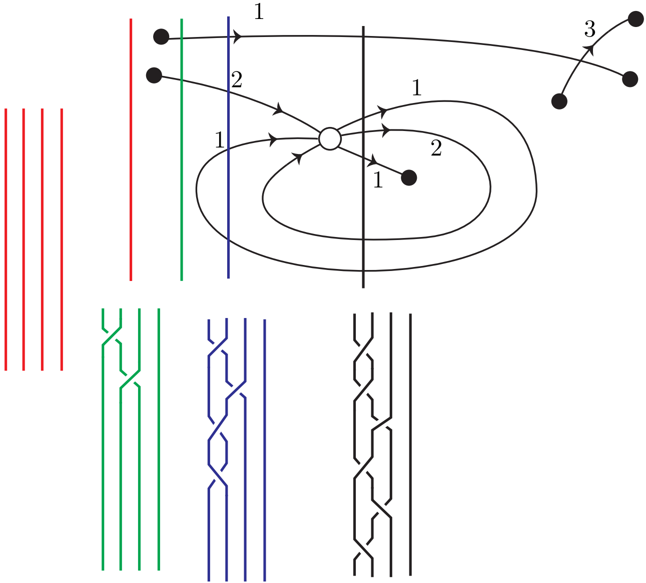
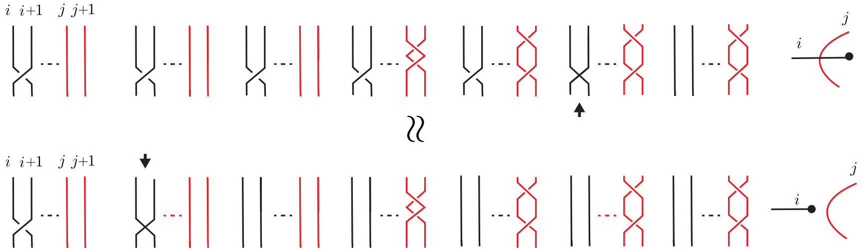


Surface braid equivalence via chart diagrams

Theorem (Kamada '92)

Two charts describe equivalent surface braids if and only if they related by certain chart moves.

For example:



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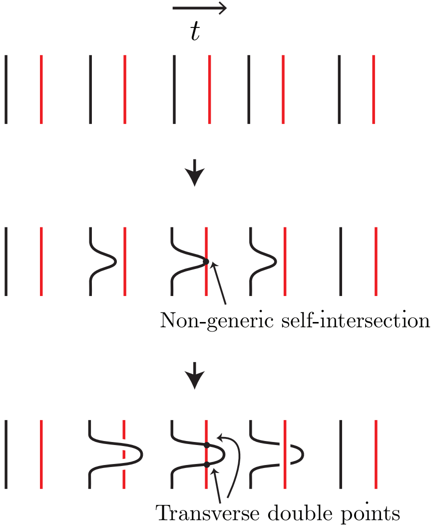
Finger move



So what problems can we solve?

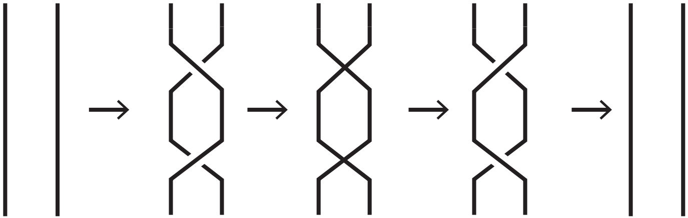
Theorem (Bartels-Teichner '99)

Let $f : S^2 \sqcup \ldots \sqcup S^2 \hookrightarrow S^4$ be an embedded link. Then by performing finger moves and their reverse (on the same component), f can be turned into the trivial link.

That is, f is link homotopic to the trivial link.

So what problems can we solve?

Finger moves in . . . surface braids



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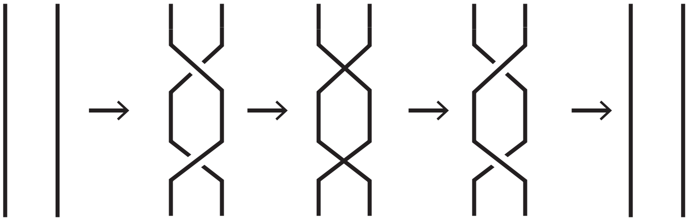
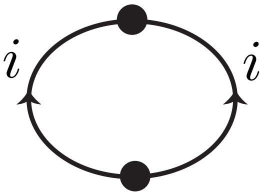
Finger moves in ... surface braids



Finger moves in ... charts



So what problems can we solve?

Trivial surface braids and their charts

$S^2 \sqcup S^2$

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Unknotted charts

a_1 $\quad a_2$ $\quad a_3$ $\quad \cdots\cdots \quad a_l$

Unknotting surface links via charts

Theorem (Iwakiri 2008)

Any chart can be transformed to an unknotted chart by inserting/deleting "quasi-hoops" and chart moves.

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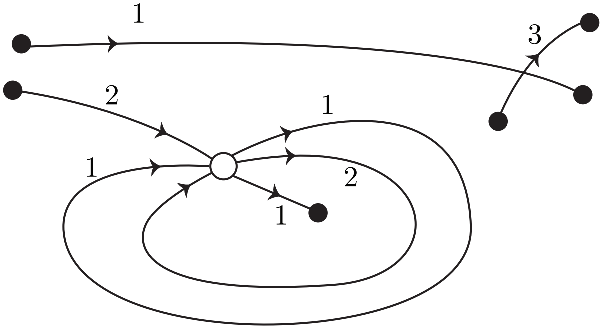

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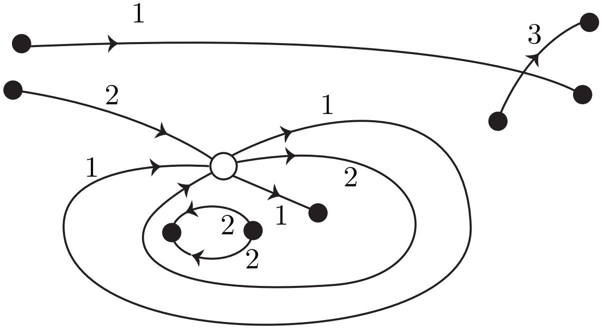

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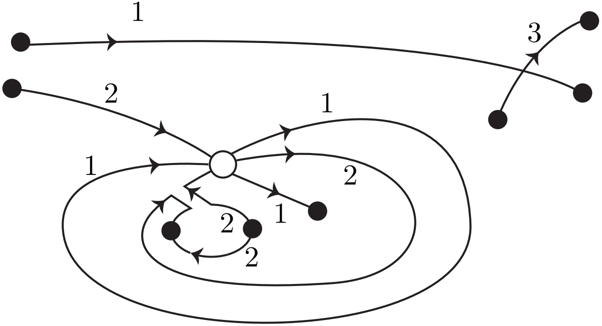

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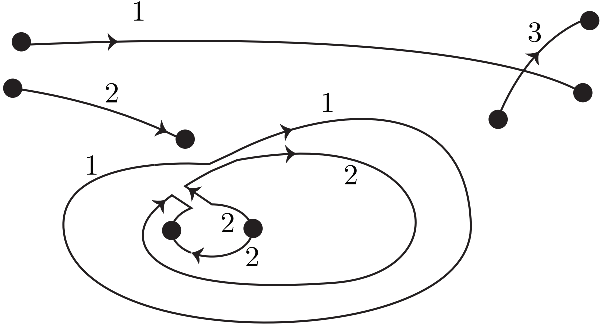

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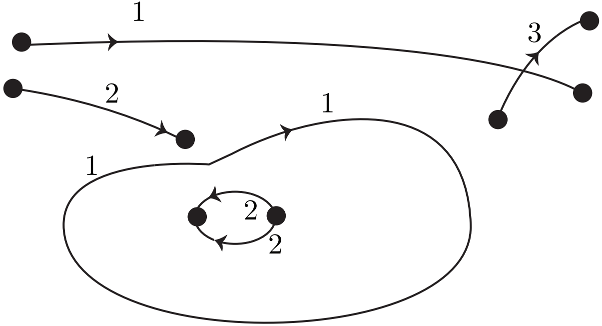

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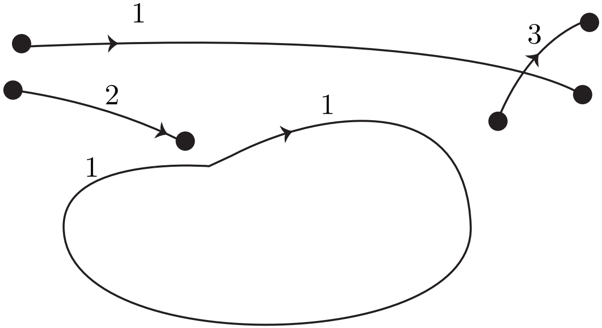

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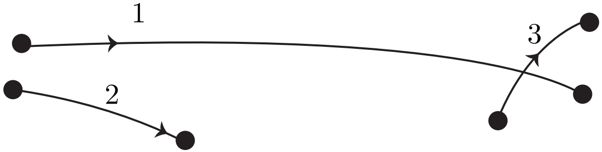

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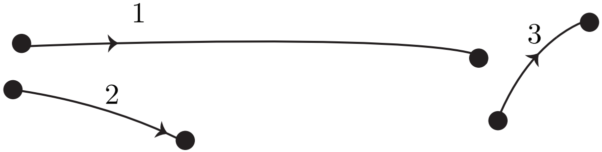

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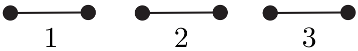
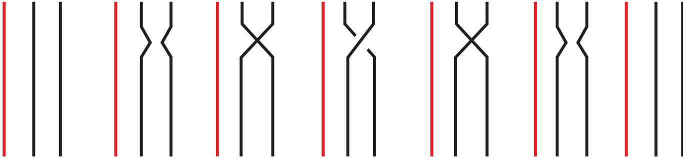
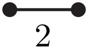


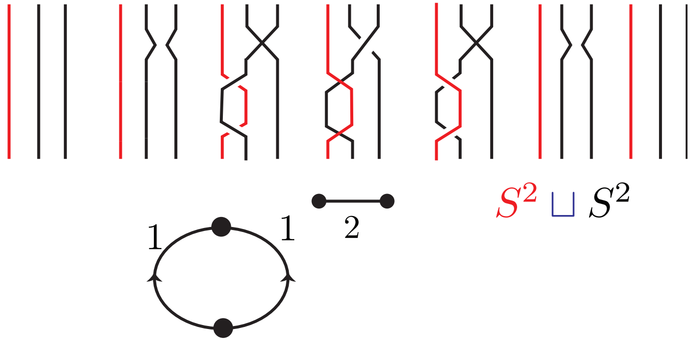
Unknotting via charts - link homotopy





$S^2 \sqcup S^2$

Unknotting via charts - link homotopy



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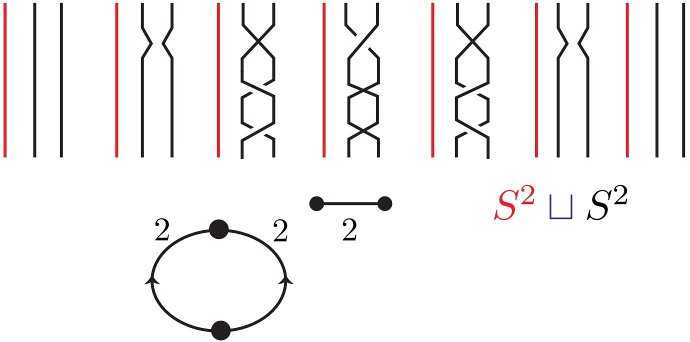

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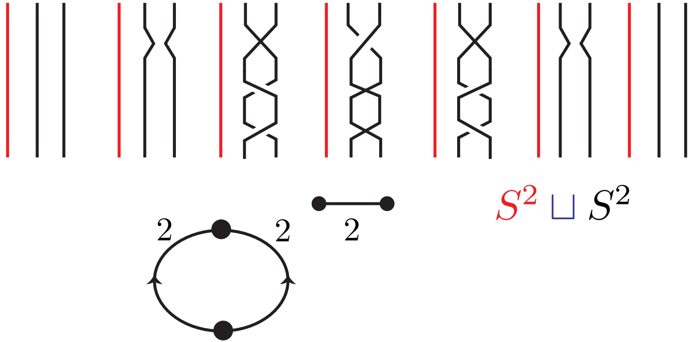
Unknotting via charts - link homotopy

Work in progress . . . to prove via chart diagrams:

If S be a surface braid of degree m representing an embedded link $S^2 \sqcup \ldots \sqcup S^2 \to \mathbb{R}^4$, then by performing finger moves and their reverse (<u>on the same component</u>), S can be turned into a trivial surface braid.



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Vassiliev Invariants

Vassiliev invariant order n

$\mathcal{K}$ — the set of isotopy classes of (classical) knots in 3-space

$v : \mathcal{K} \to A$ — an invariant of knots

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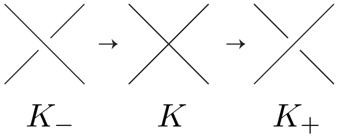
$$V(K) = V(K_+) - V(K_-).$$



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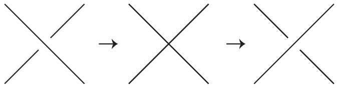
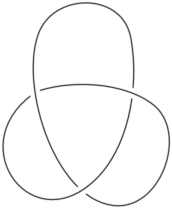


K_- $\quad$ K $\quad$ K_+

We say v is Vassiliev of order n if n is the smallest integer such that v vanishes on singular knots with more than n double points.

1D Vassiliev Invariants - definition 2

K



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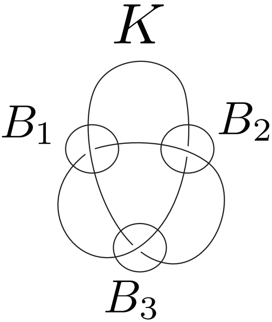

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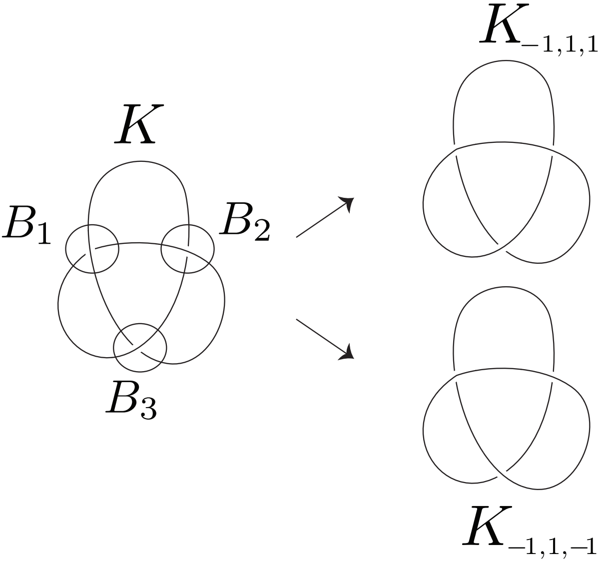

1D Vassiliev Invariants - definition 2

K — knot with n crossings

$K_{\varepsilon_1,\ldots,\varepsilon_n}$ — replace i^{th} crossing for each i such that $\varepsilon_i = -1$.

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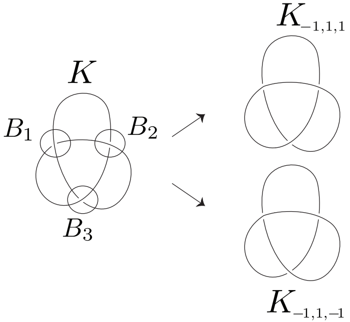

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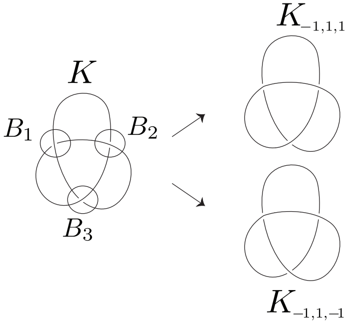
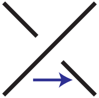
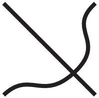
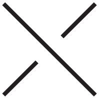


Vassiliev Invariants - 2-dimensional





Non-generic



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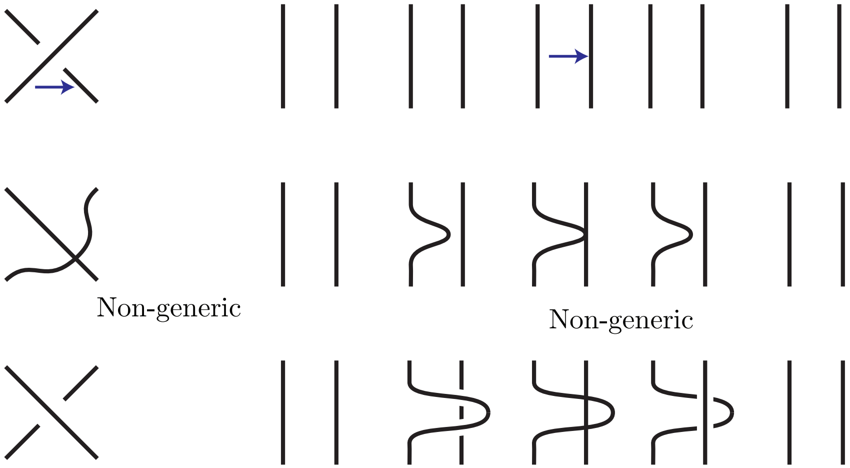

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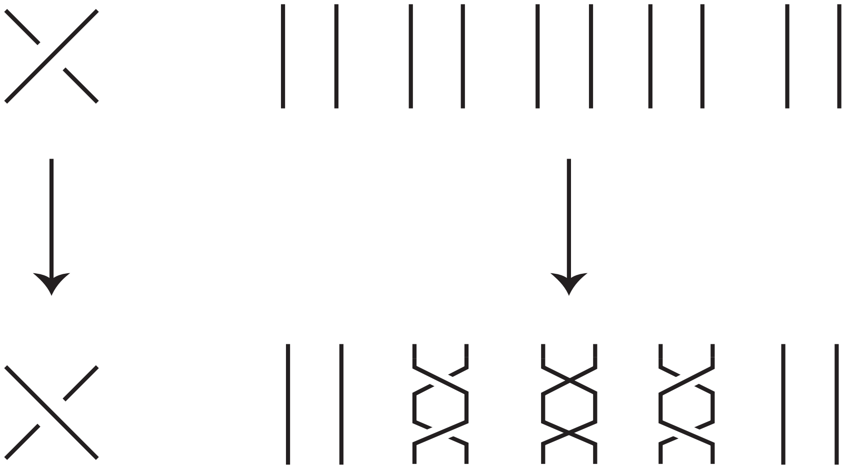

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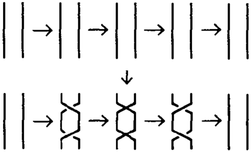

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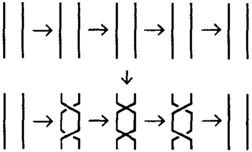

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Theorem (Iwakiri 2008)

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Vassiliev Invariants - 2-dimensional

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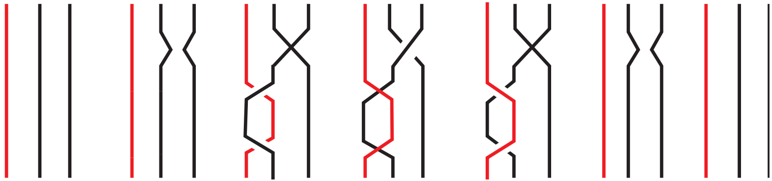
S, S' — immersed surface m-braids with l components.
Then $v(S) = v(S')$ if the l^{th} component of S and S' have the same genus and number of

(i) *sheets (equivalently, singular points),*

(ii) *positive double points, and*

(iii) *negative double points.*



$S^2 \sqcup S^2$

Kirk-Livingston (Vassiliev) Invariants

Kirk-Livingston order n

$\mathcal{L}_m$ — the set of isotopy classes of (classical) links $S^1 \sqcup S^1 \to S^3$ with linking number m

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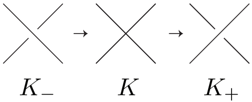

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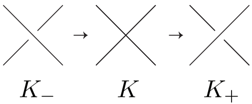

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Kirk-Livingston (Vassiliev) Invariants

Properties of KL invariants

- linking number is type 0

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- when $A = \mathbb{Z}$, KL '95 classified KL-type 1
- Open conjecture: the set of KL-type 2 invariants has infinite rank (for any linking number)

Linking number in $\mathbb{R}^3 \Leftrightarrow$ Kirk's invariant in $\mathbb{R}^4$

A link map is a map $f : S^p \sqcup S^q \to S^m$ with $f(S^p) \cap f(S^q) = \emptyset$.

A link homotopy is a homotopy through link maps.

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Theorem

The set of link maps $S^1 \sqcup S^1 \to S^3$ modulo link homotopy is classified by the linking number.

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Theorem (Schneiderman-Teichner, 2017)

The set of link maps $S^2 \sqcup S^2 \to S^4$ modulo link homotopy is classified by Kirk's σ invariant.

2D Vassiliev Invariants

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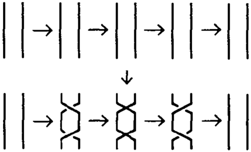

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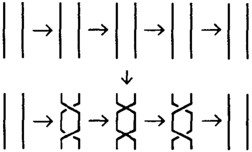

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for any $(S, B_1, \dots, B_n)$.

Kamada-Kirk-Livingston-Vassiliev Invariants

S — link map $S^2 \sqcup S^2 \to S^4$ with fixed σ value, with n "crossings" $\{B_i\}_i$ on the same component

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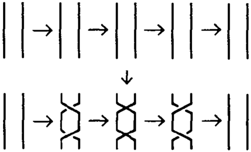

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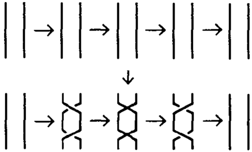

An invariant v of links maps $S^2 \sqcup S^2 \to S^4$ with fixed σ value is Kamada-Kirk-Livingston-Vassiliev of order n if n is the smallest integer such that

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for any $(S, B_1, \ldots, B_n)$.

Problems

Question

Are there interesting type >1 K-KL-V invariants? How are these related to σ, the signed number of double points and the "triple point number"?

Kirk-Livingston (Vassiliev) Invariants

Definition (example)

Vassiliev type n implies KL types $\leq n$. The linking number is type 0. $(-1)^{\mathrm{lk}(L)}$ is KL type 0 but not finite type. the linking number and the generalized Sato-Levine invariant are of KL-types 0 and 1, respectively, but of types exactly 1 and 3.

Definition (example)

Kirk and Livingston conjectured that the group of type 2 invariants in their sense has infinite rank for 2-component links with any linking number