National Research University Higher School of Economics

Asymptotics of tunneling for Schrödinger equation with hyperbolic frequency resonance

Vybornyi E.V.
Karasev M.V.

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Hamiltonian

We consider 2-dimensional Schrödinger operator $\hat{H} = H(\hat{z}^*|\hat{z})$ on the Heisenberg algebra

$$[\hat{z}_+, \hat{z}_+^*] = [\hat{z}_-, \hat{z}_-^*] = \hbar.$$

The Wick symbol $H(\bar{z}|z)$ of $\hat{H}$ has the form

$$H = \frac{1}{\hbar}\left(H_0 + H_1 + H_2 + \cdots\right), \qquad \deg H_j = j + 2, \tag{1}$$

where H_j is a homogeneous polynomial.

Here the main quadratic part of H is a hyperbolic oscillator, that is, a difference of two 1D harmonic oscillators,

$$H_0(\hat{z}^*|\hat{z}) = \omega_+ \hat{z}_+^* \hat{z}_+ - \omega_- \hat{z}_-^* \hat{z}_-.$$

We study the spectrum of $\hat{H}$ at a general frequency resonance

$$\omega_+ : \omega_- = k_+ : k_-, \qquad k_\pm \in \mathbb{N}. \tag{2}$$

The operator $\hat{H}$ arises in models of electromagnetic traps, such as hyperbolic Penning traps. In these traps, a charge is confined near a hyperbolic equilibrium of an electric field by an external magnetic field. The charge energy near the equilibrium takes the form of (1) after transforming to principal axes.

Rescaling of the parameters

We consider the spectrum of $\hat{H}$ in semiclassical approximation $\hbar \ll 1$.

Let the energy level of the harmonic part $H_0(\hat{z}^*|\hat{z})$ be equal to $\varepsilon = \hbar(\omega_+ n_+ - \omega_- n_-)$, where $n_\pm = 0, 1, 2, \dots$

We suppose that the energy ε is small, but it is much greater than $\hbar$ ($\varepsilon \gg \hbar$). Thus we consider the charge near the equilibrium, but in higher exited states.

For convenience, we introduce the rescaling $\hbar = h\varepsilon$, $\hat{a}_\pm = \hat{z}_\pm/\sqrt{\varepsilon}$. Therefore,

$$\hat{H} = \frac{1}{h}\left(H_0(\hat{a}^*|\hat{a}) + \sqrt{\varepsilon}H_1(\hat{a}^*|\hat{a}) + \varepsilon H_2(\hat{a}^*|\hat{a}) + \cdots\right), \tag{3}$$

$$[\hat{a}_j, \hat{a}_j^*] = h,$$

h is an effective parameter of the semiclassical approximation,
ε is a small parameter characterizing the magnitude of the anharmonic terms $H_1, H_2, \dots$,
the harmonic part H_0 in (3) is of oder of 1.

In the considered regime, the perturbation method ($\varepsilon \to 0$) and the semiclassical approximation ($h \to 0$) are simultaneously applicable for the operator $\hat{H}$.

Symmetry algebra and averaging

In the resonance case $\omega_+ : \omega_- = k_+ : k_-$, the eigenvalue ε of $\hat{H}_0$ has the infinite multiplicity.

The symmetry algebra, that is, an algebra of operators commuting with $\hat{H}_0$, can be defined by four generators:

the neutral action operators $\hat{S}_+ = \hat{a}_+^* \hat{a}_+$, and $\hat{S}_- = \hat{a}_-^* \hat{a}_-$;
the annihilation operator $\hat{B} = (\hat{a}_-)^{k_+}(\hat{a}_+)^{k_-}$, and the adjoint creation operator $\hat{B}^*$.

The generators of algebra have polynomial (non-Lie) commutation relations

$$[\hat{S}_\pm, \hat{B}^*] = h\, k_\mp \hat{B}^*, \qquad [\hat{B}, \hat{B}^*] = h\, r_h(\hat{S}_+, \hat{S}_-),$$

where r_h is a polynomial of degree $k_+ + k_- - 1$

$$r_h(s_+, s_-) = \frac{1}{h}\Big(g_h(s_+ + k_- h, s_- + k_+ h) - g_h(s_+, s_-)\Big), \quad g_h(s_+, s_-) = \prod_{j_1=0}^{k_- -1} (s_+ - j_1 h) \prod_{j_2=0}^{k_+ -1} (s_- - j_2 h).$$

This algebra has two central elements

$$\hat{H}_0 = \varepsilon\left(\omega_+ \hat{S}_+ - \omega_- \hat{S}_-\right), \qquad \hat{K} = \hat{B}^* \hat{B} - g_h(\hat{S}_+, \hat{S}_-) = 0.$$

Symmetry algebra and averaging

Operator averaging methods allow one to reduce the anharmonic terms of H to the operator $\hat{L}_\varepsilon$ that commutes with $\hat{H}_0$,

$$\hat{H} = \frac{1}{h}\hat{U}_\varepsilon^* \left(\hat{H}_0 + \sqrt{\varepsilon}\,\hat{L}_\varepsilon\right)\hat{U}_\varepsilon \qquad \text{mod } O(\varepsilon^{+\infty}), \tag{4}$$

$$[\hat{H}_0, \hat{L}_\varepsilon] = 0.$$

Operator $\hat{L}_\varepsilon$ can be expressed as a function on the generators of the symmetry algebra $\hat{S}_\pm$, $\hat{B}$, and $\hat{B}^*$.

At the lower resonances $1:(-1)$ and $2:(-1)$, we get

$$\hat{L}_\varepsilon = \mu\hat{B} + \bar{\mu}\hat{B}^* + O(\sqrt{\varepsilon}).$$

If $\mu \neq 0$, the operator $\mu\hat{B} + \bar{\mu}\hat{B}^*$ has a continuous spectrum. The charge is unstable in this case.

At the higher resonances $k_+ + k_- > 3$, we get

$$\hat{L}_\varepsilon = \frac{\mu}{2}\hat{B} + \frac{\bar{\mu}}{2}\hat{B}^* + p(\hat{S}_+, \hat{S}_-) + O(\sqrt{\varepsilon}), \tag{5}$$

where p is a quadratic polynomial.

Irreducible representations

The eigenspaces of the operator $\hat{H}_0$ are invariant with respect to the operator $\hat{L}_\varepsilon$. Conciser the operator $\hat{L}_\varepsilon$ on a fixed eigenspace of the operator $\hat{H}_0$, that is, on an irreducible representation of an algebra of symmetries.

Assume that $\omega_+\hat{S}_+ - \omega_-\hat{S}_- = 1$ ($\hat{H}_0 = \varepsilon$). Then we can consider the algebra with only one neutral operator $\hat{A} = \hat{S}_-/k_+ - a_0$ such that

$$[\hat{A}, \hat{B}^*] = h\hat{B}^*, \quad [\hat{B}, \hat{B}^*] = h\rho_h(\hat{A}),$$

and the spectrum of $\hat{A}$ consists of integers multiplied by h

$$\text{Spectrum of } \hat{A} = \{0, h, 2h, \ldots\}.$$

The central element of this algebra has the form

$$\hat{K} = \hat{B}^*\hat{B} - \varphi_h(\hat{A}) = 0.$$

Here the polynomials ρ and φ can be easily found from explicit forms of r and g.

The problem is reduced to the study of the semiclassical spectrum of the effective Hamiltonian

$$\hat{E} = \frac{\mu}{2}(\hat{B} + \hat{B}^*) + \alpha\hat{A}^2 + \beta\hat{A},$$

where $\mu > 0$, α and β are real coefficients.

Effective Hamiltonian

Operator averaging allows us to reduce the operator $\hat{H}$, defined on the Heisenberg algebra, to the operator $\hat{E}$, defined on the irreducible representations of the symmetry algebra of the operator $\hat{H}_0$.

The effective operator $\hat{E}$ defines the dynamics on the subspaces of constant energy $\hat{H}_0$, caused by the anharmonic terms of $\hat{H}$.

The corresponding classical mechanical system with Hamiltonian $E = \mu Y_1 + \alpha A^2 + \beta A$ is defined on the symplectic leaf $K = 0$ of the Poisson algebra

$$\{Y_2, Y_1\} = \rho_0(A)/2, \quad \{A, Y_1\} = Y_2, \quad \{A, Y_2\} = -Y_1,$$

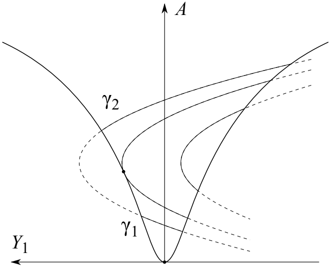
where $B = Y_1 + iY_2$.

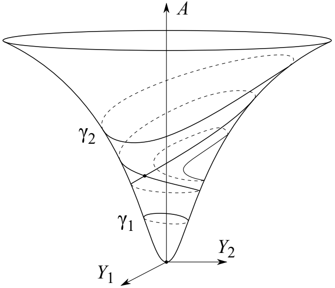
The symplectic leaf $K = 0$ is the algebraic surface of revolution

$$Y_1^2 + Y_2^2 = \varphi_0(A), \quad \varphi_0(A) = (k_- A + 1/\omega_+)^{k_-} (k_+ A)^{k_+}.$$

Classical trajectories

The classical trajectories can be obtained as intersections of surfaces of constant energy $E = \mathcal{E}$ with the symplectic leaf $K = 0$:





Of particular interest is such a regime, where a single energy $E = \mathcal{E}$ corresponds to a pair of periodic classical trajectories. We say that in this case the double-well regime is realized.

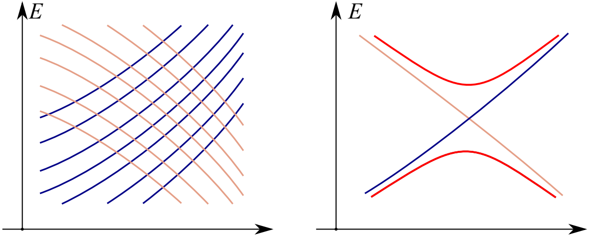
Proposition

Let $\alpha\beta < 0$. Then there exists μ_0 such that the double-well regime is realized for any $\mu \in (0, \mu_0)$. There are two periodic classical trajectories for any energy $\mathcal{E}$ in some interval.

Tunneling effects

In the double well regime, the stationary state of $\hat{E}$ can be localized near both of the two periodic trajectories as $h \to 0$. The bilocalized state corresponds to the avoided crossing of the energy levels of $\hat{E}$, that is, an implementation of the asymmetric quantum tunneling.

The main quantity characterizing the scale of these tunneling effects is the minimal value of the energy splitting $\Delta\mathcal{E}$ in the avoided crossing of a pair of energy levels.



The tunneling splitting $\Delta\mathcal{E}$ is exponentially small as $h \to 0$

$$\Delta\mathcal{E} = \exp\left\{-\frac{\mathcal{S}}{h}\big(1 + o(1)\big)\right\}, \quad (h \to 0),$$

where $\mathcal{S} > 0$ is a tunneling action.

Tunneling action and periodic instantons

To obtain a geometric interpretation of tunnel asymptotics, we consider the complexification of the classical mechanical system, that is, the complexification of the Poisson algebra, the Hamiltonian E and the symplectic leaf $K = 0$.

We say that the periodic trajectory γ of the complexified system is an instanton if it corresponds to the pure imaginary time $t = -i\tau$, and it crosses a pair of real trajectories γ_1 and γ_2 for a given energy $\mathcal{E}$.

Theorem

The tunneling splitting $\Delta\mathcal{E}$ that appears in the avoided crossing of energy levels in the double well regime has an asymptotic

$$\Delta\mathcal{E} = \exp\left\{-\frac{\mathcal{S}}{h}\big(1 + o(1)\big)\right\}, \quad (h \to 0).$$

The tunneling action has the form

$$\mathcal{S} = \frac{1}{2i}\int_\Sigma \omega > 0,$$

where the surface Σ is spanned by an instanton γ ($\partial\Sigma = \gamma$), and ω is the symplectic 2-form on the complexified leaf $K = 0$ of the Poisson algebra.

References

The further details can be found in

[1] Karasev M., Novikova E., Vybornyi E. Instantons via breaking geometric symmetry in hyperbolic traps // Mathematical notes. 2017. Vol. 102. No. 5-6. P. 776-786.

Similar methods were used for detailed analysis of Penning resonant traps

[2] Karasev M., Novikova E., Vybornyi E. Bi-states and 2-level systems in rectangular Penning traps // Russian Journal of Mathematical Physics. 2017. Vol. 24. No. 4. P. 454-464.

[3] Karasev M., Novikova E., Vybornyi E. Non-Lie Top tunneling and Quantum bilocalization in Planar Penning Trap // Mathematical notes. 2016. Vol. 100. No. 6. P. 807-819.

Thanks for your attention!
Questions?