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Geometry of spherical varieties and Newton–Okounkov polytopes

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Toric geometry and theory of Newton polytopes exhibited fruitful connections between algebraic geometry and convex geometry. After the Kouchnirenko and Bernstein–Khovanskii theorems were proved in the 1970-s (for a reminder see Section 1), Askold Khovanskii asked how to extend these results to the setting where a complex torus is replaced by an arbitrary connected reductive group. In particular, he advertised widely the problem of finding the right analogs of Newton polytopes for spherical varieties. The latter are natural generalizations of toric varieties and include classical examples such as Grassmannians, flag varieties and complete conics (see Section 2 for a reminder). Notion of Newton polytopes was extended to spherical varieties by Andrei Okounkov in the 1990-s [O97, O98]. Later, his construction was developed systematically in [KaKh, LM], and the resulting theory of Newton–Okounkov convex bodies is now an active field of algebraic geometry.

While Newton–Okounkov convex bodies can be defined for line bundles on arbitrary varieties (without a group action), they are easier to deal with in the case of spherical varieties because of connections with representation theory. For instance, Gelfand–Zetlin (GZ) polytopes and Feigin–Fourier–Littelmann–Vinberg (FFLV) polytopes arise naturally as Newton–Okounkov polytopes of flag varieties. My research focuses on explicit description of geometric and topological invariants of spherical varieties in terms of geometric and combinatorial invariants of their Newton–Okounkov polytopes. The goal is to extend the toric picture to the more general setting of varieties with a reductive group action. Section 3 is a survey of my results in this direction. Section 4 contains precise formulations of main results.

1. Newton–Okounkov convex bodies

In this section, we recall construction of Newton–Okounkov convex bodies for the general mathematical audience. Let us start from the definition of Newton polytopes.

Definition 1.1. Let $f = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha x^\alpha$ be a Laurent polynomial in n variables (here the multiindex notation x^α for $x = (x_1, \dots, x_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ stands for $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$). The *Newton polytope* $\Delta_f \subset \mathbb{R}^n$ is the convex hull of all $\alpha \in \mathbb{Z}^n$ such that $c_\alpha \neq 0$.

By definition, Newton polytope is a lattice polytope, that is, its vertices lie in $\mathbb{Z}^n$.

Example 1.2. For $n = 2$ and $f = 1 + 2x_1 + x_2 + 3x_1x_2$, the Newton polytope Δ_f is the square with the vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$.

Note that Laurent polynomials with complex coefficients are well-defined functions at all points $(x_1, \dots, x_n) \in \mathbb{C}^n$ such that $x_1, \dots, x_n \neq 0$. They are regular functions on the complex torus $(\mathbb{C}^*)^n := \mathbb{C}^n \setminus \bigcup_{i=1}^n \{x_i = 0\}$.

Theorem 1.3. [Kou] *For a given lattice polytope $\Delta \subset \mathbb{R}^n$, let $f_1(x_1, \dots, x_n)$,..., $f_n(x_1, \dots, x_n)$ be a generic collection of Laurent polynomials with the Newton polytope Δ. Then the system $f_1 = \dots = f_n = 0$ has $n!\mathrm{Volume}(\Delta)$ solutions in the complex torus $(\mathbb{C}^*)^n$.*

The Kouchnirenko theorem can be viewed as a generalization of the classical Bezout theorem. The Newton polytope serves as a refinement of the degree of a polynomial. This makes the Kouchnirenko theorem applicable to collections of polynomials which are not generic among all polynomials of given degree but only among polynomials with given Newton polytope. For instance, the Kouchnirenko theorem applied to a pair of generic polynomials with Newton polytope as in Example 1.2 yields the correct answer 2 while Bezout theorem yields an incorrect answer 4 (because of two extraneous solutions at infinity). A more geometric viewpoint on the Bezout theorem and its extensions stems from enumerative geometry and will be discussed in the next section. The Koushnirenko theorem was extended to the systems of Laurent polynomials with distinct Newton polytopes by David Bernstein and Khovanskii using mixed volumes of polytopes [B75]. Further generalizations include explicit formulas for the genus and Euler characteristic of complete intersections $\{f_1 = 0\} \cap \ldots \cap \{f_m = 0\}$ in $(\mathbb{C}^*)^n$ for $m < n$ [Kh78].

We now consider a bit more general situation. Fix a finite-dimensional vector space $V \subset \mathbb{C}(x_1, \ldots, x_n)$ of rational functions on $\mathbb{C}^n$. Let $f_1, \ldots, f_n$ be a generic collection of functions from V, and $X_0 \subset \mathbb{C}^n$ an open dense subset obtained by removing poles of these functions. How many solutions does a system $f_1 = \ldots = f_n = 0$ have in X_0? For instance, if V is the space spanned by all Laurent polynomials with a given Newton polytope, and $X_0 = (\mathbb{C}^*)^n$, then the answer is given by the Kouchnirenko theorem. Here is a simple non-toric example from representation theory.

Example 1.4. Let $n = 3$. Consider the adjoint representation of $GL_3(\mathbb{C})$ on the space $\mathrm{End}(\mathbb{C}^3)$ of all linear operators on $\mathbb{C}^3$. That is, $g \in GL_3(\mathbb{C})$ acts on an operator $X \in \mathrm{End}(\mathbb{C}^3)$ as follows:

$$\mathrm{Ad}(g) : X \mapsto gXg^{-1}.$$

Let $U^- \subset GL_3(\mathbb{C})$ be the subgroup of lower triangular unipotent matrices:

$$U^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix} \mid (x_1, x_2, x_3) \in \mathbb{C}^3 \right\}.$$

To define a subspace $V \subset \mathbb{C}(x_1, x_2, x_3)$ we restrict functions from the dual space $\mathrm{End}^*(\mathbb{C}^3)$ to the U^--orbit $\mathrm{Ad}(U^-)E_{13}$ of the operator $E_{13} := e_1 \otimes e_3^* \in \mathrm{End}(\mathbb{C}^3)$ (here e_1, e_2, e_3 is the standard basis in $\mathbb{C}^3$). More precisely, a linear function $f \in \mathrm{End}^*(\mathbb{C}^3)$ yields the polynomial $\hat{f}(x_1, x_2, x_3)$ as follows:

$$\hat{f}(x_1, x_2, x_3) := f\left(\begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix}^{-1} \right)$$

It is easy to check that the space V is spanned by 8 polynomials: 1, x_1, x_2, x_3, $x_1x_2 - x_1^2x_3$, x_1x_3, x_2x_3, $x_2^2 - x_1x_2x_3$. It will be clear from the next section that the Kouchnirenko theorem does not apply to the space V, that is, the normalized

volume of the Newton polytope of a generic polynomial from V is bigger than the number of solutions of a generic system $f_1 = f_2 = f_3 = 0$ with $f_i \in V$.

To assign the *Newton–Okounkov convex body* to V we need an extra ingredient. Choose a translation-invariant total order on the lattice $\mathbb{Z}^n$ (e.g., we can take the lexicographic order). Consider a map

$$v : \mathbb{C}(x_1, \ldots, x_n) \setminus \{0\} \to \mathbb{Z}^n,$$

that behaves like the lowest order term of a polynomial, namely: $v(f + g) \geq \min\{v(f), v(g)\}$ and $v(fg) = v(f) + v(g)$ for all nonzero f, g. Recall that maps with such properties are called *valuations*. A straightforward construction of valuations is shown in Example 1.7 below.

Definition 1.5. The *Newton–Okounkov convex body* $\Delta_v(V)$ is the closure of the convex hull of the set

$$\bigcup_{k=1}^{\infty} \left\{ \frac{v(f)}{k} \mid f \in V^k \right\} \subset \mathbb{R}^n.$$

By V^k we denote the subspace spanned by the k-th powers of the functions from V.

Different valuations might yield different Newton–Okounkov convex bodies. An important application of Newton–Okounkov bodies is the following analog of Kouchnirenko theorem. Recall that by $X_0 \subset \mathbb{C}^n$ we denoted an open dense subset where all functions from V are regular (that is, do not have poles).

Theorem 1.6. [KaKh, LM] *If V is sufficiently big, then a generic system $f_1 = \ldots = f_n = 0$ with $f_i \in V$ has $n!\mathrm{Volume}(\Delta_v(V))$ solutions in X_0.*

In particular, it follows that all Newton–Okounkov convex bodies for V have the same volume. For more details (in particular, for the precise meaning of "sufficiently big") we refer the reader to [KaKh, Theorem 4.9].

Example 1.7. Let V be the space from Example 1.4. Define a valuation v by assigning to a polynomial $f \in \mathbb{C}[x_1, x_2, x_3]$ its lowest order term with respect to the lexicographic ordering of monomials. More precisely, we say that $x_1^{k_1} x_2^{k_2} x_3^{k_3} \succ x_1^{l_1} x_2^{l_2} x_3^{l_3}$ iff there exists $j \leq 3$ such that $k_i = l_i$ for $i < j$ and $k_j > l_j$. It is easy to check that $v(V)$ consists of 8 lattice points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(0,2,0)$. Their convex hull is depicted on Figure 1. This is the FFLV polytope $FFLV(1,1)$ for the adjoint representation of GL_3 (in this case, it happens to be unimodularly equivalent to the GZ polytope). In particular, $FFLV(1,1) \subset \Delta_v(V)$.

2. Spherical varieties

In this section, we give a brief introduction to geometry of spherical varieties for the general mathematical audience. Spherical varieties arise naturally in enumerative geometry. Recall two classical problems of enumerative geometry from the 19-th century.

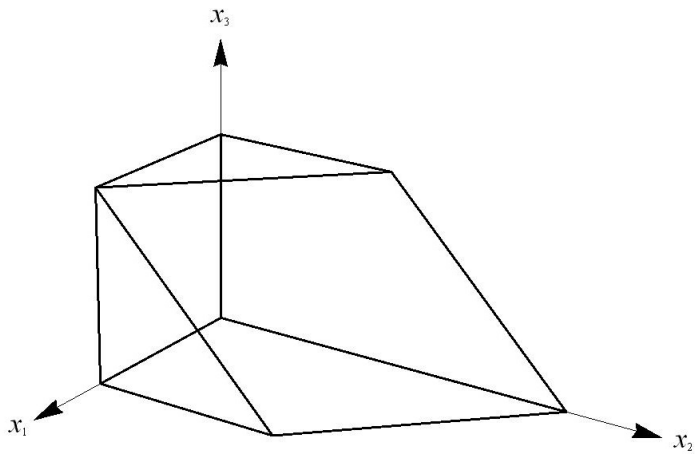

FIGURE 1

Problem 2.1 (Schubert). How many lines in a 3-space intersect four given lines in general position?

We can identify lines in $\mathbb{CP}^3$ with vector planes in $\mathbb{C}^4$, that is, a line can be viewed as a point on the Grassmannian $G(2,4)$. The condition that a line $l \in G(2,4)$ intersects a fixed line l_1 defines a hypersurface $H_1 \subset G(2,4)$. Hence, the problem reduces to computing the number of intersection points of four hypersurfaces in $G(2,4)$. It is not hard to check that the hypersurface H_1 is just a hyperplane section of the Grassmannian under the Plücker embedding $G(2,4) \hookrightarrow \mathbb{P}(\Lambda^2\mathbb{C}^4) \simeq \mathbb{CP}^5$. The image of the Grassmannian is a quadric in $\mathbb{CP}^5$. The number of intersection points of a quadric in $\mathbb{CP}^5$ with four hyperplanes in general position is equal to 2 by the Bezout theorem. Hence, the answer to the Schubert problem is 2.

Problem 2.2 (Steiner). How many smooth conics are tangent to five given conics?

Similarly to the Schubert problem, we can identify conics with points in $\mathbb{CP}^5$, namely, the conic given by an equation $ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$ corresponds to the point $(a : b : c : d : e : f) \in \mathbb{CP}^5$. Smooth conics form an open subset $C \subset \mathbb{CP}^5$ (the complement $\mathbb{CP}^5 \setminus C$ is the zero set of the discriminant). The condition that a conic is tangent to a given conic defines a hypersurface in $\mathbb{CP}^5$ of degree 6. Using Bezout theorem in $\mathbb{CP}^5$ one might guess (as Jacob Steiner himself did) that the answer to the Steiner problem is 6^5. However, the correct answer is much smaller. This is similar to the difference between the Bezout and Kouchnirenko theorems: the former yields extraneous solutions that have no enumerative meaning. The correct answer was found by Michel Chasles who used (in modern terms) a *wonderful compactification* of C, namely, the space of complete conics.

Hermann Schubert developed a powerful general method (calculus of conditions) for solving problems of enumerative geometry such as Problems 2.1, 2.2. In a sense, his method was based on an informal version of intersection theory. The 15-th

Hilbert problem asked for a rigorous foundation of Schubert calculus[1]. In the first half of the 20-th century, these foundations were developed both in the topological (cohomology rings) and algebraic (Chow rings) settings. However, Schubert's version of intersection theory was formalized only in the 1980-s by Corrado De Concini and Claudio Procesi [CP85].

Let G be a connected reductive group, and H a *spherical* algebraic subgroup, that is, a Borel subgroup $B \subset G$ acts on G/H with an open dense orbit. For a spherical homogeneous space G/H (not necessarily compact), De Concini and Procesi constructed the *ring of conditions* of G/H that encodes simultaneously all enumerative problems on G/H. It is easy to check that complex torus $(\mathbb{C}^*)^n$, Grassmannian $G(2,4)$ and the space C of smooth conics considered above are spherical homogenous spaces under the reductive groups $(\mathbb{C}^*)^n$, $GL_4(\mathbb{C})$ and $GL_3(\mathbb{C})$, respectively. In particular, their rings of conditions are well-defined. Elements of the ring of conditions are classes of subvarieties of G/H under natural numerical equivalence relation. Namely, two subvarieties of the same dimension are equivalent if their intersection indices with any subvariety of complementary dimension are equal. A transitive action of G is used to overcome the usual difficulty of intersecting non-transverse subvarieties. The ring product corresponds to the intersection of subvarieties.

In particular, many problems of enumerative geometry (including Problems 2.1, 2.2) reduce to computation of the self-intersection index of a hypersurface in G/H. In the toric case, the Kouchnirenko theorem yields an explicit formula for the self-intersection index of a hypersurface $\{f = 0\}$ where f is a generic polynomial with a given Newton polytope. In the spherical case, explicit formulas were obtained by Boris Kazarnovskii (case of $(G \times G)/G^{\mathrm{diag}}$) and Michel Brion (general case) [Kaz, Br89]. Though the Brion–Kazarnovskii formula was originally stated in different terms, it can be reformulated using Newton–Okounkov polytopes [KaKh2].

Example 2.3. We now place Example 1.4 into the context of enumerative geometry and spherical homogeneous spaces. Let $X = \{(V^1 \subset V^2 \subset \mathbb{C}^3) \mid \dim V^i = i\}$ be the variety of complete flags in $\mathbb{C}^3$. This is a homogeneous space under the action of $GL_3(\mathbb{C})$, namely, $X = GL_3(\mathbb{C})/B$ where a Borel subgroup B is the subgroup of upper-triangular matrices. It is easy to check that B acts on X with an open dense orbit $U^-B/B \simeq U^-$, in particular, X is spherical.

We say that two flags $V^1 \subset V^2$ and $W^1 \subset W^2$ in $\mathbb{C}^3$ are not in general position if either $V^1 \subset W^2$ or $W^1 \subset V^2$. How many flags in $\mathbb{C}^3$ are not in general position with three given flags? On the one hand, it is easy to show that the answer is 6 using high school geometry. On the other hand, the same answer can be found using the simplest projective embedding of X:

$$p : X \hookrightarrow \mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\Lambda^2\mathbb{C}^3) \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}(\mathrm{End}(\mathbb{C}^3)); \quad p : (V^1, V^2) \mapsto V^1 \times V^2 \mapsto V^1 \otimes \Lambda^2 V^2,$$

[1]Das Problem besteht darin, *diejenigen geometrischen Anzahlen strenge und unter genauer Feststellung der Grenzen ihrer Gültigkeit zu beweisen, die insbesondere Schubert auf Grund des sogenannten Princips der speciellen Lage mittelst des von ihm ausgebildeten Abzählungskalküls bestimmt hat* (Hilbert).

and counting the number of intersection points of $p(X)$ with 3 generic hyperplanes in $\mathbb{CP}^8$ (that is, the *degree* of $p(X)$). Restricting the map p to the open dense B-orbit $U^- \subset X$ we get that the latter problem reduces to the problem from Example 1.4. In particular, we can show that the inclusion $FFLV(1,1) \subset \Delta_v(V)$ is an equality. Indeed, by Theorem 1.6 the volume of $\Delta_v(V)$ times 3! is equal to the degree of $p(X)$, that is, to 6. Hence, the volume of $\Delta_v(V)$ is equal to 1. Since the volume of $FFLV(1,1)$ is also equal to 1, the inclusion $FFLV(1,1) \subset \Delta_v(V)$ implies the exact equality.

3. Results and publications

This section contains a brief overview of the results of the habilitation thesis. The main purpose is to place these results into the general context (without going in too much detail) and provide references to more recent developments. The precise statements and all necessary definitions can be found in [1]–[10] (see the list of publications in the end of this section).

3.1. **Euler characteristic of complete intersections in reductive groups.** In the torus case, almost all invariants of a complete intersection $Y = \{f_1 = 0\} \cap \ldots \cap \{f_m = 0\} \subset (\mathbb{C}^*)^n$ can be computed in terms of Newton polytopes $\Delta_{f_1}, \ldots, \Delta_{f_m}$. In the reductive case (for $(G \times G)/G^{\mathrm{diag}}$), the Brion–Kazarnovskii formula for $m = n$ (that is, for a zero-dimensional Y) was the only explicit formula for quite some time. Note that it can be interpreted as the formula for the (topological) Euler characteristic $\chi(Y)$. The main result of [1,2] is an explicit formula for $\chi(Y)$ for all $m \leq n$. The formula is obtained in two steps. First, (non-compact versions of) Chern classes of reductive groups are defined and studied as elements of the ring of conditions [1]. Second, an algorithm of De Concini–Procesi [CP83] is used to compute intersection indices of these Chern classes with complete intersections in order to use the adjunction formula [2].

It is proved in [2] that the De Concini–Procesi algorithm works for the Chern classes, which do not in general lie in the subring of conditions generated by complete intersections. It is also shown how to convert this algorithm to an explicit formula using the weight polytope of the representation associated with Y. In particular, this yields another proof of the Brion–Kazarnovskii formula. While formulas of [1,2] do not use Newton–Okounkov polytopes directly they have the same convex geometric flavor (see Section 4.1 for a formula in the case $m = 1$). Recently, more invariants (in particular, arithmetic genus) of complete intersections in spherical homogenous spaces were found in terms of Newton–Okounkov polytopes (see [KaKh3], which also contains a historical account of this problem).

3.2. **Convex geometric models for Schubert calculus.** The ring of conditions of a complex torus is generated by classes of hypersurfaces. The same is true for complete flag varieties G/B but not necessarily true for more general spherical homogeneous spaces. Thus complete flag varieties are first candidates for applying

convex geometric methods to computation of intersection products. Note that since G/B is compact its ring of conditions coincides with the Chow ring, and the latter has a natural basis of Schubert cycles given by the closures of B-orbits. In [3,5,8], we built convex geometric models for Schubert calculus on G/B and found convex geometric realizations of Schubert cycles.

In [3], the Chevalley-Pieri formula for G/B in type A is interpreted in terms of the GZ polytopes. In [5] (joint work with Evgeny Smirnov and Vladlen Timorin), we develop framework for realizing Schubert cycles by linear combinations of faces of a polytope so that the intersection of faces corresponds to the intersection product of Schubert cycles. In type A, we get explicit realizations for every Schubert cycle that allow us to represent the product of any two Schubert cycles by a nonnegative linear combination of faces of a GZ polytope. This result was inspired by the Ph.D. thesis of Mikhail Kogan who first associated faces of the GZ polytopes with Schubert varieties [Ko]. We also get formulas for the Demazure characters of Schubert varieties in terms of exponential sums over lattice points in Kogan faces of the GZ polytope.

In [8], a geometric algorithm is developed for realizing Schubert cycles by faces of polytopes in arbitrary type. In type A, this algorithm reduces to the Knutson–Miller mitosis on pipe dreams. In types B and C, it reduces to a new combinatorial algorithm that might yield explicit realizations of Schubert cycles in symplectic and orthogonal flag varieties by faces of symplectic GZ polytopes (see Section 4.2 for more details).

In [7] (joint work with Pavel Gusev and Vladlen Timorin), we study combinatorics of the GZ polytopes corresponding to different partial flag varieties (or in combinatorial terms, to partitions $1^{i_1}2^{i_2}\dots k^{i_k}$). Note that all GZ polytopes for a given partition have the same combinatorial type. We determine a recurrence relation for the number of vertices $V(1^{i_1}2^{i_2}\dots k^{i_k})$ and a PDE for the exponential generating function of the numbers $V(1^{i_1}2^{i_2}\dots k^{i_k})$. Recently, the recurrence relation was extended to f-vectors of the GZ polytopes in [ACK].

3.3. **Reincarnations of divided difference operators (DDO).** DDO and Demazure operators (also known as push-pull operators) are important tools in Schubert calculus and representation theory. They were used in [BGG] and [D] to express inductively Schubert cycles on complete flag varieties as polynomials in the Chern classes of line bundles (both in cohomology and K-theory). Another application is the Demazure formula for the Demazure characters of Schubert varieties [D]. In [4,6], we define and use analogs of DDO in (equivariant) Schubert calculus for algebraic cobordism. In [9], a convex geometric version of Demazure operators is defined and used to construct polytopes that capture Demazure characters. In [10], Newton–Okounkov polytopes of flag varieties for a geometric valuation are computed explicitly using a simple convex geometric construction, which is also motivated by DDO.

Note that the classical DDO and Demazure operators can be defined uniformly using the additive (Chow rings) and multiplicative (K-theory) formal group laws,

respectively. In [4] (joint work with Jens Hornbostel), we define DDO for the universal formal group law and apply them to study Schubert calculus in the algebraic cobordism ring (as defined by Levine–Morel and Levine–Pandharipande) of complete flag varieties. We were inspired by the results of Bressler–Evens on Schubert calculus in complex cobordism [BE], and part of our motivation was to transfer their results to the algebro-geometric setting. We also deduced a cobordism version of Chevalley–Pieri formula.

There are several major differences between Chow ring/cohomology and K-theory of flag varieties on the one hand, and algebraic/complex cobordism on the other hand. For instance, classes of Schubert varieties provide a natural basis in the former theories but not in the latter ones since Schubert varieties are in general not smooth. Instead, Bott–Samelson resolutions of Schubert varieties form a natural generating set (but not a basis) for the ring of algebraic/complex cobordism.

In [6] (joint work with Amalendu Krishna), we describe equivariant algebraic cobordism rings of flag varieties and wonderful compactifications of symmetric spaces of minimal rank. In particular, spaces $(G\times G)/G^{\mathrm{diag}}$ are symmetric of minimal rank. In the case of flag varieties, we used complex cobordism and topological arguments. Recently, a purely algebro-geometric approach was proposed in [CZZ]. In the case of wonderful compactifications, we used approach of Brion–Joshua who described equivariant Chow rings [BJ].

In [9], analogs of Demazure operators on convex polytopes are defined. Such operators in general take a polytope P to a polytope or a convex chain P' of dimension one greater so that the exponential sums over the lattice points in P and P' are related by the classical Demazure operator. In particular, convex geometric Demazure operators can be applied inductively to construct the GZ polytopes in type A and Karshon–Grossberg cubes in any type from a single point. In type C_2, they were used to construct symplectic DDO polytopes that are combinatorially different both from symplectic GZ and FFLV polytopes. It turned out that these polytopes coincide with Newton–Okounkov polytopes of the symplectic flag variety for a natural geometric valuation [8]. Recently, Naoki Fujita conjectered that several classes of DDO polytopes coincide with Nakashima–Zelevinsky polyhedral realizations (for C_2 this follows from [FN, Example 5.10]).

In [10], Newton–Okounkov polytopes of complete flag varieties in type A are computed for a geometric valuation given by a flag of translated Schubert subvarieties that correspond to terminal subwords in the longest word decomposition $(s_1)(s_2s_1)(s_3s_2s_1)(\ldots)(s_{n-1}\ldots s_1)$ (Examples 1.4, 1.7, 2.3 illustrate this computation for $n=3$, see also Section 4.3). Surprisingly, the resulting polytopes turned out to coincide with the FFLV polytopes though the latter were originally constructed using a different approach. This coincidence stimulated further research (see [FaFL] for more details).

List of publications

[1] V. Kiritchenko, *Chern classes of reductive groups and an adjunction formula*, Ann. Inst. Fourier **56** (2006), no. 4, 1225–1256

[2] —, *On intersection indices of subvarieties in reductive groups*, Moscow Math. J., **7** (2007), no. 3 (issue dedicated to Askold Khovanskii), 489–505

[3] —, *Gelfand-Zetlin polytopes and flag varieties*, IMRN (2010), no. 13, 2512–2531

[4] J. Hornbostel, —, *Schubert calculus for algebraic cobordism*, J. reine angew. Math. (Crelles), 2011, no. 656, 59–85

[5] —, E. Smirnov, V. Timorin, *Schubert calculus and Gelfand–Zetlin polytopes*, Russian Math. Surveys **67** (2012), no. 4, 685–719

[6] —, A. Krishna, *Equivariant Cobordism of Flag Varieties and of Symmetric Varieties*, Transform. Groups, **18** (2013), no. 2, 391–413

[7] P. Gusev, —, V. Timorin, *Counting vertices in the Gelfand-Zetlin polytopes*, J. of Comb. Theory, Series A, **120** (2013), 960–969

[8] —, *Geometric mitosis*, Math. Res. Lett., **23** (2016), no. 4, 1069–1096

[9] —, *Divided difference operators on polytopes*, Adv. Studies in Pure Math. **71** (2016), 161–184

[10] —, *Newton–Okounkov polytopes of flag varieties*, Transform. Groups **22** (2017), no. 2, 387–402

4. Main results

This section includes more detailed statements of the main results of the habilitation thesis. We try to make formulations as self-contained as possible. However, we assume that the reader is familiar with representation theory and Schubert calculus.

4.1. **Euler characteristic of complete intersections in reductive groups.** Let G be a connected complex reductive group of dimension n and rank k, and let $\pi : G \to GL(V)$ be a faithful representation of G. A generic *hyperplane section* H_π corresponding to π is the preimage $\pi^{-1}(H)$ of the intersection of $\pi(G)$ with a generic affine hyperplane $H \subset \mathrm{End}(V)$. It is not hard to show that all generic hyperplane have the same (topological) Euler characteristic. Below we give an explicit formula for the Euler characteristic of H_π. It follows from [1, Theorem 1.1], [2, Theorem 1.3], which also imply an analogous formula for the Euler characteristic of complete intersections.

Choose a maximal torus $T \subset G$, and denote by L_T its character lattice. Choose also a Weyl chamber $\mathcal{D} \subset L_T \otimes \mathbb{R}$. Denote by R^+ the set of all positive roots of G and denote by ρ the half of the sum of all positive roots of G. The inner product $(\cdot,\cdot)$ on $L_T \otimes \mathbb{R}$ is given by a nondegenerate symmetric bilinear form on the Lie algebra of G that is invariant under the adjoint action of G (such a form exists since G is reductive). Let $P_\pi \subset L_T \otimes \mathbb{R}$ denote the weight polytopes of the representation π, that is, the convex hull of the weights of T that occur in π.

Define a polynomial function $F(x, y)$ on $(L_T \oplus L_T) \otimes \mathbb{R}$ by the formula:

$$F(x, y) = \prod_{\alpha \in R^+} \frac{(x, \alpha)(y, \alpha)}{(\rho, \alpha)^2}.$$

From a geometric viewpoint, this polynomial counts the self-intersection indices of divisors on the product $G/B \times G/B$ of two flag varieties (divisors correspond to the weights $(\lambda_1, \lambda_2) \in L_T \oplus L_T$).

Theorem 4.1. *[1,Theorem 1.1], [2,Theorem 1.2] Let $\mathbb{D}$ be the differential operator (on functions on $(L_T \oplus L_T) \otimes \mathbb{R}$) given by the formula*

$$\mathbb{D} = \prod_{\alpha \in R^+} (1 + \partial_\alpha)(1 + \widetilde{\partial}_\alpha),$$

where ∂_α and $\widetilde{\partial}_\alpha$ are directional derivatives along the vectors $(\alpha, 0)$ and $(0, \alpha)$, respectively. Denote by $[\mathbb{D}]_i$ the i-th degree term in $\mathbb{D}$ (if $\mathbb{D}$ is regarded as polynomial in ∂_α and $\widetilde{\partial}_\alpha$). Then

$$\chi(H_\pi) = (-1)^{n-1} \int\limits_{P_\pi \cap \mathcal{D}} \left(n! - (n-1)![\mathbb{D}]_1 + (n-2)![\mathbb{D}]_2 - \ldots + k![\mathbb{D}]_{n-k}\right) F(x,x)dx.$$

The volume form dx is normalized so that the covolume of the lattice L_T in $L_T \otimes \mathbb{R}$ is equal to 1.

For instance, if $G = SL_3(\mathbb{C})$ and π is an irreducible representation with the highest weight $m\omega_1 + n\omega_2$ (by ω_1 and ω_2 we denote fundamental weights), then we get the following answer:

$$\begin{gathered}\chi(H_\pi) = -3(m^8 + 16m^7n + 112m^6n^2 + 448m^5n^3 + 700m^4n^4 + 448m^3n^5 + 112m^2n^6 + \\ 16mn^7 + n^8 + 18(m^6 + 12m^5n + 50m^4n^2 + 80m^3n^3 + 50m^2n^4 + 12mn^5 + n^6) + \\ +6(5m^4 + 40m^3n + 72m^2n^2 + 40mn^3 + 5n^4) + 6(m^2 + 4mn + n^2) - \\ -6(m+n)(m^6 + 13m^5n + 71m^4n^2 + 139m^3n^3 + 71m^2n^4 + 13mn^5 + n^6 + \\ +5(m^4 + 9m^3n + 19m^2n^2 + 9mn^3 + n^4) + 3(m^2 + 5mn + n^2))).\end{gathered}$$

4.2. **Convex geometric models for Schubert calculus.** In [3,5] Gelfand–Zetlin polytope is used to model Schubert calculus on the variety of complete flags in $\mathbb{C}^n$. Intersection product of cyles on the flag variety corresponds to the intersection of faces of the polytope. For an arbitrary reductive group G, we can also construct models for Schubert calculus on the flag variety G/B using polytopes. Recall that the Chow ring $CH^*(G/B)$ (regarded as a group under addition) is a free Abelian group with the basis od Schubert cycles $[X_w]$, which are labeled by elements $w \in W$ of the Weyl group of G. To construct a useful model we need to find out which linear combinations of faces of a polytope correspond to Schubert cycles. In the case of $G = GL_n(\mathbb{C})$, we achieved this goal using the combinatorial mitosis of Knutson–Miller. However, there were no suitable algorithms for the other groups. In [8,9], such algorithms are developed for an arbitrary G (in particular, for $G = GL_n(\mathbb{C})$ we get the Knutson–Miller mitosis).

Fix a reduced decomposition of the longest element $w_0 \in W$ (by s_i we denote simple reflections): $\overline{w_0} = s_{i_1} \ldots s_{i_d}$ (that is, $d = \dim G/B$). *Convex geometric*

Demazure operators D_i were constructed in [9] be elementary methods. Namely, with every simple reflection s_i we associate an operation on polytopes in $\mathbb{R}^d$ that raises dimension by one. In particular, using the decomposition $\overline{w_0}$ and a dominant weight λ we can construct inductively a polytope (possibly virtual) that encodes the Weyl character. First, we define a linear operator $p : \mathbb{R}^d \to \mathbb{R}^k$ that associates weights of G with the lattice points in $\mathbb{R}^d$ (recall that k denotes the rank of G).

Theorem 4.2. *[9,Theorem 3.6] For every dominant weight λ in the root lattice of G, and every point $a_\lambda \in \mathbb{Z}^d$ such that $p(a_\lambda) = w_0\lambda$ the convex chain*

$$P_\lambda := D_{i_1} D_{i_2} \dots D_{i_d}(a_\lambda)$$

yields the Weyl character $\chi(V_\lambda)$ of the irreducible G-module V_λ, that is,

$$\chi(V_\lambda) = \sum_{x \in P_\lambda \cap \mathbb{Z}^d} e^{p(x)}.$$

It is not hard to check that for every element $w \in W$ there exists a reduced decomposition $\overline{w} = s_{i_1} \dots s_{i_\ell}$ such that $\overline{w}$ is a subword of $\overline{w_0}$. In [9], for every simple reflection s_i we construct an operation M_i (geometric mitosis) on faces of the polytope P_λ. In particular, under additional assumptions we obtain by induction on ℓ a collection of faces that encode the Demazure character $\chi^w(\lambda)$ corresponding to the Schubert variety X_w and the weight λ:

Theorem 4.3. *[8, Corollary 3.6] Let $P_\lambda \subset \mathbb{R}^d$ be an admissible λ-balanced para-polytope, and $\mathfrak{S}_w \subset P_\lambda$ the union of all faces produced from the vertex $0 \in P_\lambda$ by applying successively the operations $M_{j_\ell}, \dots, M_{j_1}$. Suppose that for every $1 < k \le \ell$, the collection of faces $M_{j_k} \dots M_{j_\ell}(0)$ satisfies conditions (3) and (4) of [8, Theorem 3.4]. Then*

$$\chi^{w_0 w}(\lambda) = e^{w_0\lambda} \sum_{x \in \mathfrak{S}_w \cap \mathbb{Z}^d} e^{p(x)}.$$

For instance, for $G = Sp_4(\mathbb{C})$ and $\overline{w_0} = s_2 s_1 s_2 s_1$ we get a polytope $P_\lambda \subset \mathbb{R}^4$ that can also be obtained as the Newton–Okounkov polytope of the flag variety $X = Sp_4/B$ and the line bundle L_λ for the valuation associated with the flag $s_2 s_1 s_2 s_1 X_{\mathrm{id}} \subset s_2 s_1 s_2 X_{s_1} \subset s_2 s_1 X_{s_2 s_1} \subset s_2 X_{s_1 s_2 s_1} \subset X$ of translated Schubert subvarieties [8, Proposition 4.1]. Here s_1 and s_2 denote the reflections associated with the shorter and longer simple roots, respectively. One can show that this polytope is defined by 8 inequalities and moreover one can choose coordinates (y_1, y_2, y_3, y_4) in $\mathbb{R}^4$ so that precisely 4 inequalities become homogeneous, namely, $0 \le y_1$, $0 \le 2y_4 \le y_3 \le 2y_2$ [9, Example 3.4], [8, Example 2.9]. However, P_λ is combinatorially different from the string and FFLV polytopes [10, Section 3.4]. The faces of P_λ that contain 0 can be encoded by the following diagram:

$$\begin{array}{|c|c|}\hline & + \iff 0 = y_4 \\ \hline + \iff 0 = y_1 & + \iff y_4 = \frac{y_3}{2} \\ \hline & + \iff y_3 = 2y_2 \\ \hline \end{array}, \text{ e.g. } \{y_1 = 0,\ y_3 = 2y_2\} \text{ is encoded by } \begin{array}{|c|c|}\hline & \\ \hline + & \\ \hline & + \\ \hline \end{array}.$$

Geometric mitosis reduces to a simple combinatorial rule. According to this rule the Schubert cycles on Sp_4/B can be represented by the following unions of faces of P_λ.

$$\mathfrak{S}_{\mathrm{id}} = \{0\} = \begin{array}{|c|c|}\hline & + \\ \hline + & + \\ \hline & + \\ \hline\end{array}\,, \quad \mathfrak{S}_{s_1} = \begin{array}{|c|c|}\hline & + \\ \hline & + \\ \hline & + \\ \hline\end{array}\,, \quad \mathfrak{S}_{s_2} = \begin{array}{|c|c|}\hline & + \\ \hline + & + \\ \hline & \\ \hline\end{array}\,,$$

$$\mathfrak{S}_{s_1s_2} = \begin{array}{|c|c|}\hline & + \\ \hline & \\ \hline & + \\ \hline\end{array} \cup \begin{array}{|c|c|}\hline & + \\ \hline + & \\ \hline & \\ \hline\end{array}\,, \quad \mathfrak{S}_{s_2s_1} = \begin{array}{|c|c|}\hline & + \\ \hline & + \\ \hline & \\ \hline\end{array}\,, \quad \mathfrak{S}_{s_1s_2s_1} = \begin{array}{|c|c|}\hline & + \\ \hline & \\ \hline & \\ \hline\end{array}\,,$$

$$\mathfrak{S}_{s_2s_1s_2} = \begin{array}{|c|c|}\hline & \\ \hline & + \\ \hline & \\ \hline\end{array} \cup \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & + \\ \hline\end{array} \cup \begin{array}{|c|c|}\hline & \\ \hline + & \\ \hline & \\ \hline\end{array}\,, \quad \mathfrak{S}_{s_2s_1s_2s_1} = P_\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & \\ \hline\end{array}\,.$$

4.3. **Newton–Okounkov polytopes of flag varieties.** Fix the decomposition $\overline{w_0} = (s_1)(s_2s_1)(s_3s_2s_1)\dots(s_{n-1}\dots s_1)$ of the longest element $w_0 \in S_n$. Here $s_i := (i\ i+1)$ is the i-th elementary transposition (simple reflection in the case of the Weyl group S_n). Denote by $d := \binom{n}{2}$ the length of w_0.

Fix a complete flag of subspaces $F^\bullet := (F^1 \subset F^2 \subset \dots \subset F^{n-1} \subset \mathbb{C}^n)$ (this amounts to fixing a Borel subgroup $B \subset GL_n$). Also fix a basis $e_1,\dots, e_n$ in $\mathbb{C}^n$ compatible with $F^\bullet$ (or a maximal torus in B), that is, $F^i = \langle e_1, \dots, e_i\rangle$. In what follows, $\overline{w_\ell}$ for $\ell = 1,\dots, d$ denotes the subword of $\overline{w_0}$ obtained by deleting the first ℓ simple reflections in $\overline{w_0}$, and w_ℓ denotes the corresponding element of S_n. Consider the flag of translated Schubert subvarieties:

$$w_0X_{\mathrm{id}} \subset w_0w_{d-1}^{-1}X_{w_{d-1}} \subset w_0w_{d-2}^{-1}X_{w_{d-2}} \subset \dots \subset w_0w_1^{-1}X_{w_1} \subset GL_n/B, \qquad (*)$$

where Schubert subvarieties are taken with respect to the flag $F^\bullet$, i.e., $X_w = \overline{BwB/B}$. Recall that the open Schubert cell C with respect to $F^\bullet$ is defined as the set of all flags $M^\bullet$ that are in general position with the standard flag $F^\bullet$, i.e., all intersections $M^i \cap F^j$ are transverse. Let $y_1, \dots, y_d$ be coordinates on the open Schubert cell C (with respect to $F^\bullet$) that are compatible with $(*)$, i.e., $w_0w_\ell^{-1}X_{w_\ell} \cap C = \{y_1 = \dots = y_\ell = 0\}$.

For instance, we can identify the open Schubert cell C with an affine space $\mathbb{C}^d$ by choosing for every flag $M^\bullet$ a basis $v_1,\dots, v_n$ in $\mathbb{C}^n$ of the form:

$$v_1 = e_n + x_1^{n-1}e_{n-1} + \dots + x_1^1e_1,$$

$$v_2 = e_{n-1} + x_2^{n-2}e_{n-2} + \dots + x_2^1e_1, \quad \dots \quad , v_{n-1} = e_2 + x_{n-1}^1e_1, \quad v_n = e_n,$$

so that $M^i = \langle v_1, \dots, v_i\rangle$. Such a basis is unique, hence, the coefficients $(x_j^i)_{i+j<n}$ are coordinates on the open cell. It is not hard to check that the coordinates $(y_1, \dots, y_d) := (x_{n-1}^1; x_{n-2}^1, x_{n-2}^2; \dots; x_1^1, x_1^2, \dots, x_1^{n-1})$ are compatible with $(*)$. In

other words, every flag $M^\bullet \in C$ gets identified with a triangular matrix:

$$\begin{pmatrix} x_1^1 & x_2^1 & \dots & x_{n-1}^1 & 1 \\ x_1^2 & x_2^2 & \dots & 1 & 0 \\ \vdots & \vdots & & & 0 \\ x_1^{n-1} & 1 & \dots & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{pmatrix},$$

and we order the coefficients $(x_j^i)_{i+j<n}$ of this matrix by starting from column $(n-1)$ and going from top to bottom in every column and from right to left along columns.

In [10, Section 2.2] we give another example of coordinates compatible with $(*)$. The latter coordinates are more natural from the geometric viewpoint and are related to geometry of the Bott–Samelson variety associated with $\overline{w_0}$.

Fix the lexicographic ordering on monomials in coordinates $y_1, \dots, y_d$ so that $y_1 \succ y_2 \succ \dots \succ y_d$. Let v denote the lowest order term valuation on $\mathbb{C}(X_{\overline{w_0}}) = \mathbb{C}(GL_n/B)$ associated with these coordinates and ordering. Let L_λ be the line bundle on GL_n/B corresponding to a dominant weight $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ of GL_n. Denote by $\Delta_v(GL_n/B, L_\lambda) \subset \mathbb{R}^d$ the Newton–Okounkov convex body corresponding to GL_n/B, L_λ and v.

Theorem 4.4. *[10, Theorem 2.1] The Newton–Okounkov convex body $\Delta_v(GL_n/B, L_\lambda)$ coincides with the Feigin–Fourier–Littelmann–Vinberg polytope $FFLV(\lambda)$.*

We now recall the definition of $FFLV(\lambda)$. Label coordinates in $\mathbb{R}^d$ corresponding to $(y_1, \dots, y_d)$ by $(u_{n-1}^1; u_{n-2}^2, u_{n-2}^1; \dots; u_1^{n-1}, u_1^{n-2}, \dots, u_1^1)$. Arrange the coordinates into the table

$$\begin{array}{ccccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & & \dots & & \lambda_n \\ & u_1^1 & & u_2^1 & & \dots & & u_{n-1}^1 & \\ & & u_1^2 & & \dots & & u_{n-2}^2 & & \\ & & & \ddots & & \ddots & & & \\ & & & & u_1^{n-2} & & u_2^{n-2} & & \\ & & & & & u_1^{n-1} & & & \end{array} \qquad (FFLV)$$

The polytope $FFLV(\lambda)$ is defined by inequalities $u_m^l \geq 0$ and

$$\sum_{(l,m)\in D} u_m^l \leq \lambda_i - \lambda_j$$

for all Dyck paths going from λ_i to λ_j in table $(FFLV)$ where $1 \leq i < j \leq n$.

For instance, the computation of the polytope $\Delta_v(GL_n/B, L_\lambda)$ for $n = 3$ and $\lambda = (1, 0, -1)$ is illustrated in Examples 1.4, 1.7, 2.3.

References

[ACK] B. H. An, Yu. Cho, J. S. Kim, *On the f-vectors of Gelfand–Cetlin polytopes*, European J. of Comb., **67** (2018), 61–77

[B75] D.N. Bernstein, *The number of roots of a system of equations*, Funct. Anal. Appl., **9** (1975), 183–185

[BGG] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, *Schubert cells, and the cohomology of the spaces G/P*, Russian Math. Surveys **28** (1973), no. 3, 1–26.

[BE] P. Bressler, S. Evens, *Schubert calculus in complex cobordism*, Trans. Amer. Math. Soc., **331** (1992), no. 2, 799–813

[Br89] M. Brion, *Groupe de Picard et nombres caracteristiques des varietes spheriques*, Duke Math J. **58** (1989), no.2, 397–424

[BJ] M. Brion, R. Joshua, *Equivariant Chow ring and Chern classes of wonderful symmetric varieties of minimal rank*, Transform. Groups **13** (2008), no. 3-4, 471–493

[CP83] C. De Concini and C. Procesi, *Complete symmetric varieties I,* Lect. Notes in Math. **996**, Springer, 1983

[CP85] C. De Concini and C. Procesi, *Complete symmetric varieties II Intersection theory,* Advanced Studies in Pure Mathematics **6** (1985), Algebraic groups and related topics, 481–513

[CZZ] B. Calmès, K. Zainoulline, C. Zhong, *Equivariant oriented cohomology of flag varieties*, Documenta Math. Extra Volume: Alexander S. Merkurjev's Sixtieth Birthday (2015), 113–144

[D] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I. Ann. Sci. École Norm. Sup. 4 **7** (1974), 53–88

[FaFL] X. Fang, Gh. Fourier, P. Littelmann, *Essential bases and toric degenerations arising from generating sequences*, Adv. in Math. **312** (2017), 107–149

[FN] N. Fujita and S. Naito, *Newton-Okounkov convex bodies of Schubert varieties and polyhedral realizations of crystal bases*, Math. Z. **285** (2017), 325–352

[KaKh] K. Kaveh, A.G. Khovanskii, *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math.(2) **176** (2012), no.2, 925–978

[KaKh2] K. Kaveh, A.G. Khovanskii, *Convex bodies associated to actions of reductive groups*, Moscow Math. J. **12** (2012), no. 2, 369–396

[KaKh3] K. Kaveh, A.G. Khovanskii, *Complete intersections in spherical varieties*, Selecta Math. **22** (2016), no. 4, 2099–2141

[Kaz] B.Ya. Kazarnovskii, *Newton polyhedra and the Bezout formula for matrix-valued functions of finite-dimensional representations,* Functional Anal. Appl. **21** (1987), no. 4, 319–321

[Kh78] A.G. Khovanskii, *Newton polyhedra, and the genus of complete intersections,* Functional Anal. Appl. **12** (1978), no. 1, 38–46

[Ko] M. Kogan, *Schubert geometry of flag varieties and Gelfand–Cetlin theory*, Ph.D. thesis, Massachusetts Institute of Technology, Boston 2000

[Kou] A.G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), no.1, 1–31

[LM] R. Lazarsfeld, M. Mustata, *Convex Bodies Associated to Linear Series*, Annales Scientifiques de l'ENS **42** (2009), no. 5, 783–835

[O97] A. Okounkov, *Note on the Hilbert polynomial of a spherical variety*, Functional Anal. Appl. **31** (1997), no. 2, 138–140

[O98] A. Okounkov, *Multiplicities and Newton polytopes*, Kirillov's seminar on representation theory, Amer. Math. Soc. Transl. Ser. 2, **181** (1998), 231–244

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