A parameter space of cubic Newton maps with parabolics

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Main Object

Cubic Newton maps with a parabolic fixed point at infinity.
The Newton map of an entire function $g : \mathbb{C} \to \mathbb{C}$ is the meromorphic function $N_g : \mathbb{C} \to \hat{\mathbb{C}}$ defined by

$$N_g(z) := z - g(z)/g'(z).$$

The Newton maps for the family of entire functions of the form $g(z) = (z^2 + a)e^z$, parametrized by a single complex number $a \neq 0$, is the family given by the following cubic rational maps

$$f_a(z) = z - \frac{z^2 + a}{z^2 + 2z + a}. \tag{1}$$

Denote $f^{\circ n}$ the n-th iterate of f.
For a periodic point $f^{\circ n}(z) = z$ of period $n \geq 1$ the number $\lambda = (f^{\circ n})'(z)$ is called the *multiplier* of the orbit $\{z, f(z), \ldots, f^{\circ n-1}(z)\}$.

Definition (Classification of fixed points)

A periodic point $f^{\circ n}(z) = z$ is called

attracting if $|\lambda| < 1$, in particular,

superattracting if $\lambda = 0$

repelling if $|\lambda| > 1$

indifferent if $|\lambda| = 1$, in this case let $\lambda = e^{2\pi i\theta}$

- *rationally indifferent* (also called parabolic) if $\theta \in \mathbb{Q}$
- *irrationally indifferent* if $\theta \notin \mathbb{Q}$

The Basin of Attraction

Definition (The Basin of Attraction)

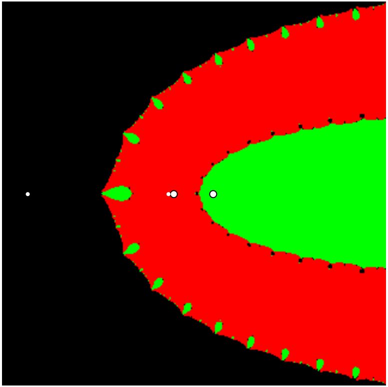
Let ξ be an attracting or parabolic fixed point of a rational map f. The *basin* $\mathcal{A}(\xi)$ of ξ is

$$\text{int}\{z \in \hat{\mathbb{C}} : \lim_{n\to\infty} f^{\circ n}(z) = \xi\},$$

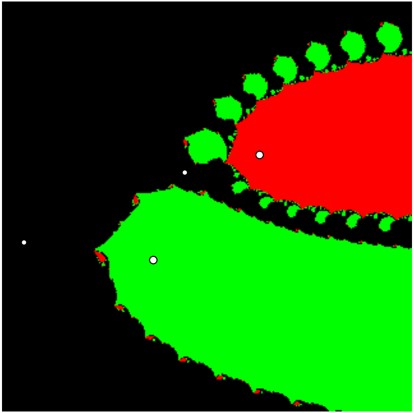
the interior of the set of starting points z which eventually converge to ξ under iteration.
The *immediate basin* $\mathcal{A}^{\circ}(\xi)$ of ξ is the forward invariant connected component of the basin.

Since Julia sets are connected for Newton maps, the basin components are simply connected.

The Julia set I



The Julia set II



Cubic Newton maps

The fixed points of f_a are:

1. the roots of $z^2 + a = 0$, which are superattracting, and
2. a point at infinity, which is a parabolic of multiplier $+1$ with one parabolic attracting petal.

The critical points of f_a are:

1. the roots of $z^2 + a = 0$, and
2. the roots of $z^2 + 4z + a + 2 = 0$
3. (Special case) If $a = 1$ then $z = -1$ is the only pole which is also a critical point

Motivation

1. Every cubic rational map has four fixed points counted with multiplicities.
2. If the two of the fixed points are attracting and the other two coincide creating a fixed point of multiplicity 2 then this must be a parabolic fixed point with the multiplier $+1$.
3. Sending this point to infinity by a Möbius map, conformal change of variable, we obtain a map which will be conjugate, via a quasiconformal map, to one of the cubic Newton maps of the family (1), not on the whole of $\hat{\mathbb{C}}$ but on some neighborhoods of their Julia sets. This can be done by a standard holomorphic surgery to make the two attracting fixed points superattracting.

Components of the parameter space

- Every parabolic fixed point attracts a critical point, then
- We have only one critical point that has a “free” dynamics, call this a free critical point.

Definition (Stable components)

The components of the parameter space of the cubic Newton maps of the family (1) for which the free critical point belongs to attracting basins and the basin of the parabolic fixed point at ∞ are called stable components.

We study topological properties of these components.

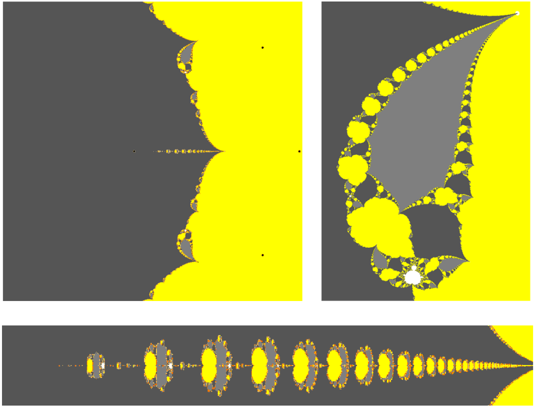


Figure: The a-parameter plane of cubic Newton maps f_a with two zoom-ins. Yellow – the free critical point belongs to the basin of ∞. Grey – the free critical point belongs to the basins of the two superattracting fixed points. Some centers (pcm) and half-centers (pcnm) are shown.

Main Theorem

Theorem

1. *Every stable component is a topological open disk containing a unique center, which is a cubic postcritically minimal Newton map.*
2. *The main parabolic component: the free critical point belongs to the immediate basin of ∞, the quasiconformal conjugacy classes of maps are of the following three types: type-I -homeomorphic to an open vertical strip, type-II -an analytic arc, type-III -a point, which is the center or a cubic postcritically non-minimal Newton map.*
3. *The boundary of every such a type-I class in H consists of type-II arcs meeting at type-III points.*

Let c_1 and c_2 be critical points of a rational function f. We say that c_1 and c_2 are in a critical orbit relation if $f^{\circ m}(c_1) = f^{\circ n}(c_2)$ for some non-negative integers m and n, if $c_1 = c_2$ we require $m \neq n$.

Definition (Minimal and non-minimal critical orbit relations)

Let f_a be a cubic Newton map of the form (1) and let $c_1 \in U_1$ and $c_2 \in U_2$ be its critical points with U_1 and U_2 the connected components of the basin of the parabolic fixed point at ∞. Assume $f_a^{\circ m}(U_1) = U_2$ with minimal such $m \geq 0$. We say that c_1 and c_2 are in a *minimal critical orbit relation* if $f_a^{\circ m}(c_1) = c_2$ with the same m. If $f_a^{\circ(m+n)}(c_1) = f_a^{\circ k}(c_2)$ with $n > 0$ and $k \geq 0$ then we say that c_1 and c_2 are in *non-minimal critical orbit relations*.

Post-Critically Minimal Newton map

Definition (Postcritically minimal and Postcritically non-minimal cubic Newton maps)

A cubic Newton map f_a of the form (1) is called postcritically minimal -pcm (postcritically non-minimal -npcm)

1. if its Fatou set consists of superattracting basins and the parabolic basin of ∞ and
2. if its free critical point is in a minimal critical orbit relation (a non-minimal critical orbit relation) with the other critical point of f_a in the parabolic basin of ∞.

As the critical orbit relations are given by a system of algebraic equations for the parameter a, these pcm and npcm Newton maps form a discrete set in stable components. Thus a small perturbation destroys the relation, thus these type of maps are rigid.

Let us assume that the immediate basin $U = \mathcal{A}^\circ(\infty)$ contains the two critical points c_1 and c_2 of f_a and let $\psi_U : U \to \mathbb{D}$ be a Riemann map of U with $\psi_U(c_1) = 0$. Denote $w = \psi_U(c_2) \in \mathbb{D}$ the image of the second critical point. Then the conjugation $\psi_U \circ f_a \circ \psi_U^{-1}$ is a Blaschke product on the unit disk $\mathbb{D}$, denote it by $B(z)$. The Blaschke product $B(z)$ then has critical points at 0 and w and as a self map of $\mathbb{D}$ its degree is 3.

Denote by H the set of such an f_a in the parameter space. We want to show that it is a topological disk.

- It has a very rich structure because of interaction of the two critical points.
- The idea is to construct a map from H to some model space, which will be parametrized by a coordinate in the unit disk.
- As a model space we take the space of cubic Blaschke products with normalizations.

Automorphisms of the unit disk $\mathbb{D}$, Aut($\mathbb{D}$)

Every conformal automorphism $\tau \in$Aut($\mathbb{D}$) of the unit disk $\mathbb{D}$ is a fractional linear transformation of the form

$$\tau(z) = \gamma \cdot \frac{z-a}{1-\bar{a}z}, \tag{2}$$

for some constants $|a| < 1$ and $|\gamma| = 1$. For $\gamma = \frac{1-\bar{a}}{1-a}$, let us denote by

$$\beta_a(z) = \frac{1-\bar{a}}{1-a} \cdot \frac{z-a}{1-\bar{a}z}, \tag{3}$$

the unique automorphism of $\mathbb{D}$ sending a to the origin and normalized to fix $z = 1$.

For given $d+1$ complex numbers (constants) $a_1, a_2, \ldots, a_{d+1}$ in $\mathbb{D}$ and a constant $|\gamma| = 1$, we define a Blaschke product of degree $d+1$ by the following product

$$f(z) = \gamma \cdot \beta_{a_1}(z) \cdot \beta_{a_2}(z) \cdots \beta_{a_{d+1}}(z).$$

1. Every proper holomorphic map f of the unit disk $\mathbb{D}$ has a finite degree and f is the restriction of a Blaschke product on $\mathbb{D}$ of the same degree.
2. Blaschke products are determined uniquely by the constants

$$\gamma = f(1) \text{ and } \{a_1, a_2, \ldots, a_{d+1}\} = f^{-1}(\{0\}).$$

3. It is clear that the poles of f are at $1/\bar{a}_1, 1/\bar{a}_2, \ldots, 1/\bar{a}_{d+1}$.

We are only interested in Blaschke products f normalized to fix 1, then these maps are invariant under the conjugation by the inversion $z \mapsto 1/\bar{z}$.

This symmetry yields the following.

1. If ξ is a fixed point of f then $1/\bar{\xi}$ is also a fixed point.
2. If c is a critical point then $1/\bar{c}$ is also a critical point.

As a rational map of $\hat{\mathbb{C}}$ a Blaschke product f of degree $d+1$ has

1. $d+1$ roots,
2. $d+1$ poles, and
3. $2d$ critical points in $\hat{\mathbb{C}}$ counted with their multiplicities.
4. $d+1$ fixed points in $\hat{\mathbb{C}}$ counted with their multiplicities.

If there is a fixed point in $\mathbb{D}$ then it is simple and is the only fixed point in $\mathbb{D}$. In this case the fixed point is attracting and it attracts every point of $\mathbb{D}$ under the iterations of the map.
If there is no fixed point in $\mathbb{D}$ then there exists a fixed point in $\mathbb{S}^1$ that is attracting or parabolic, and every point in the complement of $\mathbb{S}^1$ is attracted to this fixed point.

The critical points $\iff$ their Blaschke products.

Theorem (Heins)

Let $c_1, c_2, \ldots, c_d$ be d (not necessarily distinct) points in $\mathbb{D}$. Then there is a unique Blaschke product f of degree $d+1$ with $f(0)=0$ and $f(1)=1$ and having $c_1, c_2, \ldots, c_d$ as its critical points. Moreover, if g is any other Blaschke product of degree $d+1$ with critical points $c_1, c_2, \ldots, c_d$, then there is a $\tau \in Aut(\mathbb{D})$ such that

$$\tau \circ g = f.$$

The cubic Blaschke products that we consider $f(1) = 1$, but $f(0) \neq 0$.
Additionally $f'(1) = 1$, and $f'(0) = 0$.
Moreover, we want that around $z = 1$ our map has the form $1 + (z-1) + A(z-1)^3 + o((z-1)^3)$, for a complex $A \neq 0$. It means that $f''(1) = 0$.

1. By theorem of Heins, every such a cubic Blaschke product is uniquely obtained by the position of its second critical point $w \in \mathbb{D}$.
2. $\forall w \in \mathbb{D}$ there exists a cubic Blaschke product normalized as above and such that 0 and w are its critical points in $\mathbb{D}$.

Let us call this the model space and denote by $\mathcal{B}_3$.

Let $\mathcal{M} = \mathcal{M}(\mathcal{B}_3)$ -the moduli space of $\mathcal{B}_3$, the conformal conjugacy classes of maps from $\mathcal{B}_3$.
Denote a class $[B]$ then $[B_1] = [B_2] \iff \exists\ \beta \in \mathrm{Aut}(\mathbb{D})$, $\forall z$

$$\beta \circ B_1(z) = B_2 \circ \beta(z). \tag{4}$$

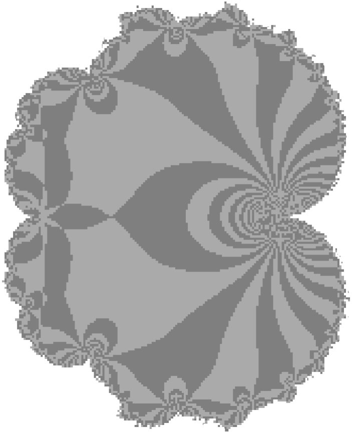
Then $\beta(1) = 1$, normalized.
Moreover, $\beta(\{0, w_1\}) = \{0, w_2\}$, the critical points of B_1 map to the critical points of B_2.
If $\beta(0) = 0$ then $\beta \equiv \mathrm{id}$ and $B_1 \equiv B_2$.
For the other case: $\beta(0) = w_2$ and $\beta(w_1) = 0$.
Then $\beta(z) = \beta_{w_1}(z) = \frac{1-\bar{w}_1}{1-w_1}\frac{z-w_1}{1-\bar{w}_1 z}$ and $\beta(0) = -w_1\frac{1-\bar{w}_1}{1-w_1} = w_2$.
The latter is a necessary and sufficient condition for B_1 and B_2 to be conformally conjugate.

The Equivalence

If w_1 is real then $w_2 = -w_1$. If $w_1 = vi$ is a pure imaginary for $v^2 < 1$ then $w_2 = \frac{2v^2}{1+v^2} - \frac{v(1-v^2)}{1+v^2} i$.
Let $z = (x, y)$ and $w = (u, v)$. Then $(x, y) \sim (u, v) \iff (u, v) = (-\frac{x(1-x)^2 - y^2(2-x)}{(1-x)^2+y^2}, -\frac{y(1-(x^2+y^2))}{(1-x)^2+y^2})$.
Thus $\mathcal{M} = \mathbb{D}/\sim$, the quotient of $\mathbb{D}$ by this equivalence relation.

For $\forall f_a \in H$ corresponds an element in $\mathcal{M} = \mathcal{M}(\mathcal{B}_3)$, and thus the stable component H can be identified with the moduli space $\mathcal{M}$.
Indeed, for f_a in H, let ϕ_1 be the Riemann map of the parabolic immediate basin $\mathcal{A}^\circ(\infty)$ such that $\phi_1(c_1) = 0$ and let $\phi_1(c_2) = w_1 \in \mathbb{D}$. Denote $B_1 = \phi_1 \circ f_a \circ \phi_1^{-1}$, which is a Blaschke product in $\mathcal{B}_3$.
Consider ϕ_2 the Riemann map of $\mathcal{A}^\circ(\infty)$ such that $\phi_2(c_2) = 0$ and let $\phi_2(c_1) = w_2 \in \mathbb{D}$.
Then the conjugacy $B_2 = \phi_2 \circ f_a \circ \phi_2^{-1}$ is also a Blaschke product.
B_1 and B_2 are conjugate to each other as we have
$f_a = \phi_1^{-1} \circ B_1 \circ \phi_1 = \phi_2^{-1} \circ B_2 \circ \phi_2$,
denote $\phi = \phi_2 \circ \phi_1^{-1}$
then $\phi \circ B_1 = B_2 \circ \phi$.

Main Parabolic Component H



Type-I and Type-II classes in $\mathcal{M} = H$

Demonstration of proof:

For $a \in H$, let P_a be the maximal attracting petal in $\mathcal{A}^\circ(\infty)$:

1. $\exists c_1 \in \partial P_a$, a critical point of f_a
2. P_a is open, forward invariant, simply connected
3. f_a is injective on P_a
4. $\exists$ a conformal map $\phi_a : P_a \to H_r$, the right half plane
5. $\phi_a(f_a(z)) = \phi_a(z) + 1$, the Abel equation.

Demonstration of Proof

Assume that and denote by $w' = f_a(c_2) \in P_a \setminus f_a(P_a)$. Then on H_r we have $0 < \operatorname{Re}\phi_a(w') \leq 1$ and $\operatorname{Im}\phi_a(w') \in \mathbb{R}$.
The idea is to change the position of the critical value on the vertical strip.
Let $x_0 + y_0 i = \phi(w')$, for the case $w' \notin \partial P$, it is easy to check that

$$\ell_{(h,t)} : (x,y) \mapsto \begin{cases} (\frac{x_0+h}{x_0}x, y + \frac{t}{x_0}x), & 0 \leq x \leq x_0, \\ ((1 - \frac{h}{1-x_0})x + \frac{h}{1-x_0}, y + \frac{t}{1-x_0}(1-x)), & x_0 \leq x \leq 1, \end{cases}$$

is a quasiconformal homeomorphism of the strip $0 \leq \operatorname{Re} z \leq 1$ parametrized by $z = x + iy$.
The real dilatation: $|\frac{\partial_{\bar{z}} l}{\partial_z l}(z)| = \frac{\sqrt{h^2+t^2}}{\sqrt{(2x_0+h)^2+t^2}}$ at $0 \leq \operatorname{Re} z < x_0$ and

$|\frac{\partial_{\bar{z}} l}{\partial_z l}(z)| = \frac{\sqrt{h^2+t^2}}{\sqrt{(2(1-x_0)-h)^2+t^2}}$ at $x_0 \leq \operatorname{Re} z \leq 1$.

1. Extend $\ell_{(h,t)}(x, y)$ continuously to the right half plane by the translation $z \mapsto z + 1$.
2. Pulling it back by the Fatou coordinate ϕ_a define an almost complex structure on the petal P_a
3. by the dynamics of f_a, pull it back to $\mathcal{A}(\infty)$
4. extend it by zero (the standard complex structure) to the rest of the plane, we obtain a Beltrami form on $\hat{\mathbb{C}}$ with the maximal dilatation equal to that of $\ell_{(h,t)}$.

By Measurable Riemann Mapping theorem there exists a quasiconformal map $\Phi_{(h,t)}$ on $\hat{\mathbb{C}}$ that solves the Beltrami equation. Then $\Phi_{(h,t)} \circ f_a \circ \Phi_{(h,t)}^{-1}$ is a Newton map in H and has required conditions.

Thank you.