A Way for Graph Reduction for the Independent Set Problem and Its Application

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Definition

An independent set of a graph is a subset of vertices of the graph, no two of which are adjacent.

Definition

A maximum independent set of a graph G is an independent set of G with the maximum number of vertices. Its size is called the independence number of G and denoted as $\alpha(G)$.

Definition

The independent set problem (IS problem, for short), for a given graph G and natural number k, is a problem of verification, whether the inequality $\alpha(G) \geq k$ holds or not.

A way of graph reduction

Definition

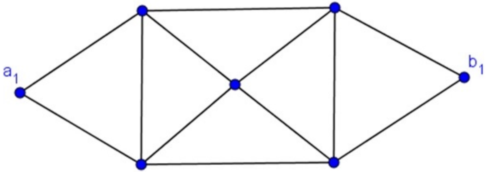
Let H_1 and H_2 be graphs, $A \subseteq V(H_1) \cap V(H_2)$. We call the graphs H_1 and H_2 α–similar with respect to A, if an injection $f : A \mapsto V(H_2)$ and a constant c exist, such that for any $X \subseteq A$ the equality $\alpha(H_1 \setminus X) = \alpha(H_2 \setminus X) + c$ holds.

Definition

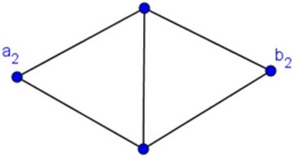
The function f is called α-similarity between H_1 and H_2 with respect to A.

Example

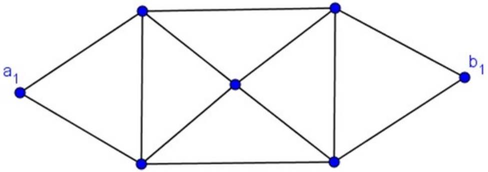
Let $A = \{a_1, b_1\}$ and $f : \{a_1, b_1\} \mapsto \{a_2, b_2\}$ with $f(a_1) = a_2$, $f(b_1) = b_2$. The graphs H_1 and H_2 are α-similar with respect to A.



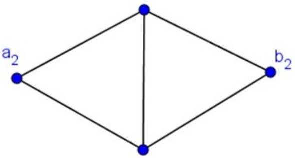
Graph H_1



Graph H_2



Graph H_1



Graph H_2

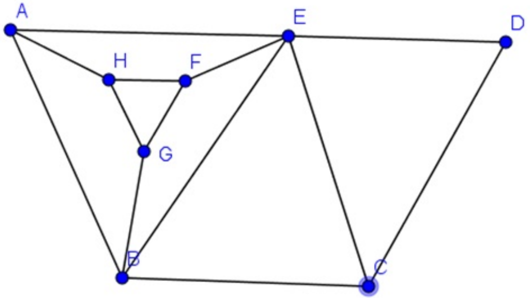
$\alpha(H_1) = 3$	$\alpha(H_2) = 2$	$\alpha(H_1) - \alpha(H_2) = 1$
$\alpha(H_1 \setminus a_1) = 2$	$\alpha(H_2 \setminus b_1) = 1$	$\alpha(H_1 \setminus a_1) - \alpha(H_2 \setminus b_1) = 1$
$\alpha(H_1 \setminus a_2) = 2$	$\alpha(H_2 \setminus b_2) = 1$	$\alpha(H_1 \setminus a_2) - \alpha(H_2 \setminus b_2) = 1$
$\alpha(H_1 \setminus \{a_1, b_1\}) = 2$	$\alpha(H_2 \setminus \{a_1, b_1\}) = 1$	$\alpha(H_1 \setminus \{a_1, b_1\}) - \alpha(H_2 \setminus \{a_1, b_1\}) = 1$

Definition

Let G be some graph with an induced subgraph H. We call a set of vertices $A \subseteq V(H)$ separating for the graph H, if no vertex from the graph $H \setminus A$ is adjacent to any vertex from the graph $G \setminus H$.

Example

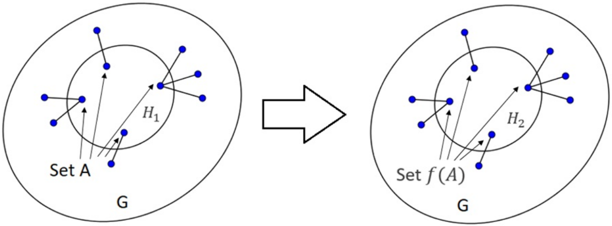
A separating set for the subgraph on vertices $\{A, B, E, F, H, G\}$ is $\{E, B\}$.

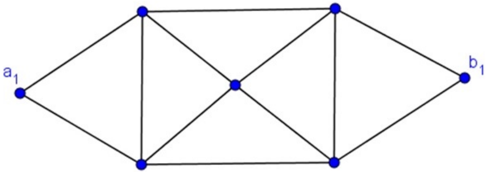


The notion of f-substitution

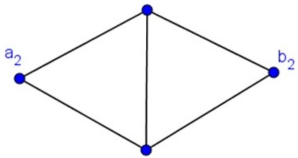
Definition

Let f be an α-similarity between graphs H_1 and H_1 regarding A. Let G be a graph with induced subgraph H_1 with H_1-separator A, while $V(G) \cap V(H_2) = \emptyset$. The operation of f-substitution of graph H_1 with graph H_1 in graph G is performed by removing from graph G all vertices of subgraph H_1, adding graph H_2 to $G \setminus H_1$ with all edges $(f(x), y)$, where $x \in A$, $y \in V(G) \setminus V(H_1)$, $xy \in E(G)$.

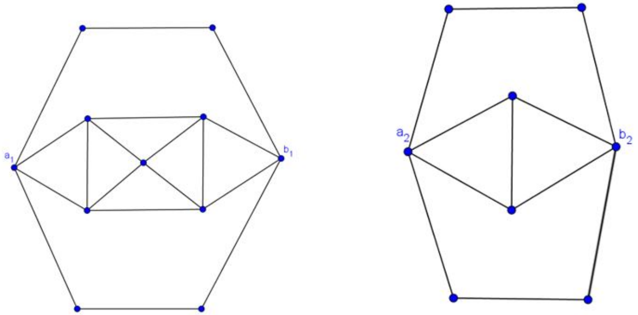




Graph H_1



Graph H_2



Graph before and after f-substitution of subgraph H_1 with subgraph H_2

Lemma

If graph G^* is a result of f-substitution of subgraph H_1 with subgraph H_2 in graph G, then f is an α-similarity between G^* and G with respect to A.

Corollary

If graph G^* is a result of f-substitution of subgraph H_1 with subgraph H_2 in graph G, then $\alpha(G^*) = \alpha(G) + \alpha(H_2) - \alpha(H_1)$.

Remark

Our substitutions is a particular case of the so-called **substitution schemes**, which has been proposed by Alekseev and Lozin in the paper «Local transformations of graphs, which keep the independence number» (Discrete Analysis and Operations Research, 1998)

Graph classes

Definition

A graph class is any set of simple graphs, which is closed under isomorphism.

Definition

We call a graph class IS-simple if IS problem for graphs in this class is solvable in polynomial time. A graph class, where IS problem is NP-complete, we call IS-hard.

Definition

A graph class is called hereditary if it is closed under vertex deletion.

Any hereditary class $\mathcal{X}$ can be defined by the set $\mathcal{S}$ of its minimal forbidden induced subgraphs.
It is denoted as $\mathcal{X} = Free(\mathcal{S})$.

Definition

A hereditary class is called finitely defined, if the set of its minimal forbidden induced subgraphs is finite.

Class $\mathcal{T}$

Definition

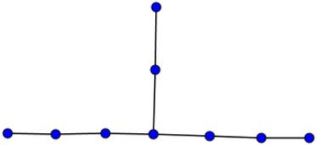
The triode $T_{i,j,k}$ is a tree, which can be created by coinciding end vertices of paths P_{i+1}, P_{j+1} and P_{k+1}.

Definition

The class $\mathcal{T}$ consists of all graphs, every connected component of which is a triode.

Notations

1. $\mathcal{P}$ — class of planar graphs
2. $\mathcal{D}(d)$ — class of graphs with maximum degree of d
3. $\mathcal{P}(d)$ — class of planar graphs with maximum degree of d



Triod $T_{3,3,2}$

Theorem (Alekseev, 2003)

1. Every finitely defined class $\mathcal{X}$, which contains $\mathcal{T}$, is IS-hard.
2. The same is correct if instead $\mathcal{X}$ we consider the classes $\mathcal{P} \cap \mathcal{X}$, $\mathcal{D}(d) \cap \mathcal{X}$ or $\mathcal{P}(d) \cap \mathcal{X}$ with $d > 2$.

Assumption (Alekseev, 2003)

Every finitely defined class, which does not include $\mathcal{T}$, is IS-simple.

Equivalent assumption

For any graph $G \in \mathcal{T}$, the class $Free(\{G\})$ is IS-simple.

Remark

The same assumption can be made for finitely defined subclasses regarding the classes $\mathcal{P}$, $\mathcal{D}(d)$, $\mathcal{P}(d)$.

Known results

Theorem (Lozin, Milanic, 2007)

The class $\mathcal{D}(d) \cap Free(\{T_{1,i,i}\})$ is IS-simple, for any i and d.

Theorem (Alekseev et al., 2008)

The class $\mathcal{P} \cap Free(\{T_{1,i,i}\})$ is IS-simple, for any i.

Theorem (Malyshev, 2013)

The class $\mathcal{P}(3) \cap Free(\{T_{2,2,i}\})$ is IS-simple, for any i.

Theorem (Lozin et al., 2015)

The class $\mathcal{D}(3) \cap Free(\{T_{2,2,2}\})$ is IS-simple.

Main result

Theorem (Malyshev, Sirotkin, 2017)

The class $\mathcal{P}(3) \cap Free(\{T_{3,3,2}\})$ is IS-simple.

Proof sketch

The independent set problem for the class $\mathcal{P}(3) \cap Free(\{T_{3,3,2}\})$ can be reduced to the same problem for the class $\mathcal{P}(3) \cap Free(\{T_{2,2,10}\})$ in polynomial time. To this end, certain f-substitutions are used. The class $\mathcal{P}(3) \cap Free(\{T_{2,2,10}\})$ is IS-simple (Malyshev, 2013).
We use approximately 20 different f-substitutions.

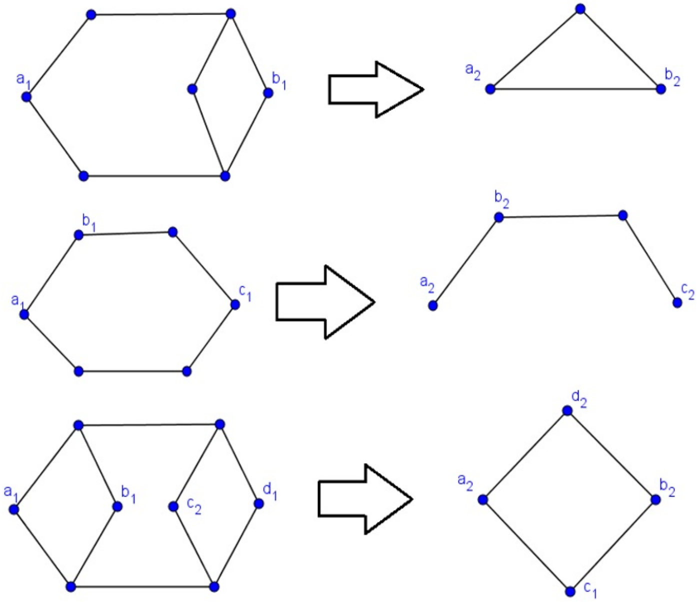
Main result

Theorem (Malyshev, Sirotkin, 2017)

The class $\mathcal{P}(3) \cap Free(\{T_{3,3,2}\})$ is IS-simple.

Proof sketch

The independent set problem for the class $\mathcal{P}(3) \cap Free(\{T_{3,3,2}\})$ can be reduced to the same problem for the class $\mathcal{P}(3) \cap Free(\{T_{2,2,10}\})$ in polynomial time. To this end, certain f-substitutions are used. The class $\mathcal{P}(3) \cap Free(\{T_{2,2,10}\})$ is IS-simple (Malyshev, 2013).
We use approximately 20 different f-substitutions.



Examples of used f-substitutions

Thank you for the attention!