

## ON INTERSECTION INDICES OF SUBVARIETIES IN REDUCTIVE GROUPS

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*To my Teacher Askold Khovanskii*

**ABSTRACT.** In this paper, I give an explicit formula for the intersection indices of the Chern classes (defined earlier by the author) of an arbitrary reductive group with hypersurfaces. This formula has the following applications. First, it allows to compute explicitly the Euler characteristic of complete intersections in reductive groups thus extending the beautiful result by D. Bernstein and Khovanskii, which holds for a complex torus. Second, for any regular compactification of a reductive group, it computes the intersection indices of the Chern classes of the compactification with hypersurfaces. The formula is similar to the Brion–Kazarnovskii formula for the intersection indices of hypersurfaces in reductive groups. The proof uses an algorithm of De Concini and Procesi for computing such intersection indices. In particular, it is shown that this algorithm produces the Brion–Kazarnovskii formula.

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### 1. INTRODUCTION

Let  $G$  be a connected complex reductive group of dimension  $n$ , and let  $\pi: G \rightarrow \mathrm{GL}(V)$  be a faithful representation of  $G$ . A generic *hyperplane section*  $H_\pi$  corresponding to  $\pi$  is the preimage  $\pi^{-1}(H)$  of the intersection of  $\pi(G)$  with a generic affine hyperplane  $H \subset \mathrm{End}(V)$ . There is a nice explicit formula for the self-intersection index  $H_\pi^n$  of  $H_\pi$  in  $G$ , and more generally, for the intersection index of  $n$  generic hyperplane sections corresponding to different representations (see Theorem 1.1 below) in terms of the weight polytopes of the representations [3], [9]. In this paper, I give a similar formula for the intersection indices of the *Chern classes* of  $G$  (defined in [11]) with generic hyperplane sections (see Theorem 1.2).

The *Chern classes* of  $G$  can be defined using the Chern classes of the *logarithmic* tangent bundle over a *regular* compactification of  $G$  (see Section 3 for a precise definition). They were introduced in [11] as main ingredients in a formula for the Euler characteristic of a generic hyperplane section and of complete intersections

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of several hyperplane sections. In the case where a reductive group is a complex torus  $(\mathbb{C}^*)^n$ , there are beautiful explicit formulas for the Euler characteristic due to D. Bernstein and A. Khovanskii [10]. The result of the present paper combined with [11] provides analogous formulas in the case of an arbitrary reductive group.

Denote by  $k$  the rank of  $G$ , i. e. the dimension of a maximal torus in  $G$ . Only the first  $(n-k)$  Chern classes are not trivial [11]. These Chern classes are elements of the *ring of conditions* of  $G$ , which was introduced by C. De Concini and C. Procesi [7] (see also Section 2.4 for a brief reminder). They can be represented by subvarieties  $S_1, \dots, S_{n-k} \subset G$ , where  $S_i$  has codimension  $i$ . All enumerative problems for  $G$ , such as the computation of the intersection index  $S_i H_\pi^{n-i}$ , make sense in the ring of conditions.

First, I recall the usual Brion–Kazarnovskii formula for the intersection indices of hyperplane sections. Choose a maximal torus  $T \subset G$ , and denote by  $L_T$  its character lattice. Choose also a Weyl chamber  $\mathcal{D} \subset L_T \otimes \mathbb{R}$ . Denote by  $R^+$  the set of all positive roots of  $G$  and denote by  $\rho$  the half of the sum of all positive roots of  $G$ . The inner product  $(\cdot, \cdot)$  on  $L_T \otimes \mathbb{R}$  is given by a nondegenerate symmetric bilinear form on the Lie algebra of  $G$  that is invariant under the adjoint action of  $G$  (such a form exists since  $G$  is reductive).

**Theorem 1.1** [3], [9]. *If  $H_\pi$  is a hyperplane section corresponding to a representation  $\pi$  with the weight polytope  $P_\pi \subset L_T \otimes \mathbb{R}$ , then the self-intersection index  $H_\pi^n$  of  $H_\pi$  is equal to*

$$n! \int_{P_\pi \cap \mathcal{D}} \prod_{\alpha \in R^+} \frac{(x, \alpha)^2}{(\rho, \alpha)^2} dx.$$

The measure  $dx$  on  $L_T \otimes \mathbb{R}$  is normalized so that the covolume of  $L_T$  is 1.

This theorem was first proved by B. Kazarnovskii [9]. Later, M. Brion proved an analogous formula for arbitrary spherical varieties using a different method [3].

The integrand in this formula has the following interpretation. The direct sum  $L_T \oplus L_T$  can be identified with the Picard group of the product  $G/B \times G/B$  of two flag varieties. Here  $B$  is a Borel subgroup of  $G$ . Hence, to each lattice point  $(\lambda_1, \lambda_2) \in L_T \oplus L_T$  one can assign the self-intersection index of the corresponding divisor in  $G/B \times G/B$ . The resulting function extends to the polynomial function  $(n-k)!F$  on  $(L_T \oplus L_T) \otimes \mathbb{R}$ , where

$$F(x, y) = \prod_{\alpha \in R^+} \frac{(x, \alpha)(y, \alpha)}{(\rho, \alpha)^2}.$$

Note that the integrand is the restriction of  $F$  onto the diagonal  $\{(x, x) : x \in L_T \otimes \mathbb{R}\}$ .

This interpretation leads to another proof of the Brion–Kazarnovskii formula (different from those of Kazarnovskii and Brion). Namely, take any regular compactification  $X$  of  $G$  that lies over the compactification  $X_\pi$  corresponding to the representation  $\pi$  (see Section 2.2). Then reduce the computation of  $H_\pi^n$  to the computation of the intersection indices of divisors in the closed orbits of  $X$  (see Section 4). All closed orbits are isomorphic to the product of two flag varieties. The precise algorithm for doing this was given by De Concini and Procesi [6] in the

case, where  $X$  is a wonderful compactification of a symmetric space. Then E. Bifet extended this algorithm to all regular compactifications of symmetric spaces [2]. I will show that in the case, where a symmetric space is a reductive group, this algorithm actually produces the Brion–Kazarnovskii formula if one uses the weight polytope of  $\pi$  to keep track of all transformations.

Moreover, the De Concini–Procesi algorithm works not only for divisors. It can also be carried over to the Chern classes of  $G$  (which are, in general, not linear combinations of complete intersections). In particular, there is the following explicit formula for the intersection indices of the Chern classes of  $G$  with hyperplane sections. Assign to each lattice point  $(\lambda_1, \lambda_2) \in L_T \oplus L_T$  the intersection index of the  $i$ -th Chern class of the tangent bundle over  $G/B \times G/B$  with the divisor  $D(\lambda_1, \lambda_2)$  corresponding to  $(\lambda_1, \lambda_2)$ , that is, the number  $c_i(G/B \times G/B)D^{n-k-i}(\lambda_1, \lambda_2)$ . Extend this function to the polynomial function on  $(L_T \oplus L_T) \otimes \mathbb{R}$ . Since the Chern classes of  $G/B$  are known the resulting function can be easily computed (see Section 4). The final formula is as follows.

Let  $\mathbb{D}$  be the differential operator (on functions on  $(L_T \oplus L_T) \otimes \mathbb{R}$ ) given by the formula

$$\mathbb{D} = \prod_{\alpha \in R^+} (1 + \partial_\alpha)(1 + \tilde{\partial}_\alpha),$$

where  $\partial_\alpha$  and  $\tilde{\partial}_\alpha$  are directional derivatives along the vectors  $(\alpha, 0)$  and  $(0, \alpha)$ , respectively. Denote by  $[\mathbb{D}]_i$  the  $i$ -th degree term in  $\mathbb{D}$ .

**Theorem 1.2.** *If  $H_\pi$  is a generic hyperplane section corresponding to a representation  $\pi$  with the weight polytope  $P_\pi \subset L_T \otimes \mathbb{R}$ , then the intersection index  $S_i H_\pi^{n-i}$  of the  $i$ -th Chern class of  $G$  with  $H_\pi^{n-i}$  is equal to*

$$(n - i)! \int_{P_\pi \cap \mathcal{D}} [\mathbb{D}]_i F(x, x) dx.$$

The measure  $dx$  on  $L_T \otimes \mathbb{R}$  is normalized so that the covolume of  $L_T$  is 1.

Of course, this formula also allows one to compute the intersection index  $S_i \times H_{\pi_1} \dots H_{\pi_{n-i}}$  for any  $n - i$  generic hyperplane sections corresponding to different representations  $\pi_1, \dots, \pi_{n-i}$ .

Since, in general, the Chern classes of  $G$  are not complete intersections, this formula extends computation of the intersection indices to a bigger part of the ring of conditions of  $G$ . Theorem 1.2 also completes some results of [11]. Namely, the Chern classes  $S_1, \dots, S_{n-k}$  were used there in the following adjunction formula for the topological Euler characteristic of complete intersections of hyperplane sections in  $G$ .

**Theorem 1.3** [11]. *Let  $H_1, \dots, H_m$  be generic hyperplane sections corresponding to  $m$  (possibly different) representations of  $G$ . The Euler characteristic of the complete intersection  $H_1 \cap \dots \cap H_m$  is equal to the term of degree  $n$  in the expansion of the following product:*

$$(1 + S_1 + \dots + S_{n-k}) \cdot \prod_{i=1}^m H_i (1 + H_i)^{-1}.$$

The product in this formula is the intersection product in the ring of conditions.

Theorem 1.2 in the present paper allows to make this formula explicit, since it allows to compute all terms of the form  $S_i H_1^{k_1} \dots H_m^{k_m}$ , where  $k_1 + \dots + k_m = n - i$ . For example, if a complete intersection is just one hyperplane section  $H_\pi$ , then

$$\chi(H_\pi) = (-1)^{n-1} \int_{P_\pi \cap \mathcal{D}} (n! - (n-1)![\mathbb{D}]_1 + (n-2)![\mathbb{D}]_2 - \dots + k![\mathbb{D}]_{n-k}) F(x, x) dx.$$

There is also a formula for the Chern classes  $c_i(X)$  of the tangent bundle over any regular compactification  $X$  of  $G$  in terms of  $S_1, \dots, S_{n-k}$  (see Corollary 4.4 in [11]). Theorem 1.2 allows to compute explicitly the intersection index of  $c_i(X)$  with a complete intersection of complementary dimension in  $X$ .

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## 2. PRELIMINARIES

In this section, I recall some well-known facts which are used in the proof of Theorem 1.2. In Section 2.2, I define the regular compactification  $X$  of  $G$  associated with a representation  $\pi$  and describe the orbit structure of  $X$  in terms of the weight polytope of the representation. In Section 2.3, the Picard group of  $X$  is related to the space of virtual polytopes analogous to the weight polytope of  $\pi$ . The notion of analogous polytopes is discussed in Section 2.1. In Section 2.4, I recall the definition of the ring of conditions of  $G$ . Section 2.5 contains a formula for the integral of a polynomial function over a simplex, which is used to interpret the computation of intersection indices in terms of integrals over the weight polytope.

**2.1. Polytopes.** Let  $P \subset \mathbb{R}^k$  be a convex polytope. Define the *normal fan*  $P^*$  of  $P$ . This is a fan in the dual space  $(\mathbb{R}^k)^*$ . To each face  $F^i \subset P$  of dimension  $i$  there corresponds a cone  $F_i^*$  of dimension  $(n - i)$  in  $P^*$  defined as follows. The cone  $F_i^*$  consists of all linear functionals in  $(\mathbb{R}^k)^*$  whose maximum value on  $P$  is attained on the interior of the face  $F^i$ . In particular, to each facet of  $P$  there corresponds a one-dimensional cone, i. e. a ray, in  $P^*$ . If the dual space  $(\mathbb{R}^k)^*$  is identified with  $\mathbb{R}^k$  by means of the Euclidean inner product, the ray corresponding to a facet is spanned by a normal vector to the facet.

Two convex polytopes are called *analogous* if they have the same normal fan. All polytopes analogous to a given polytope  $P$  form a semigroup  $S_P$  with respect to Minkowski sum. This semigroup is also endowed with the action of the multiplicative group  $\mathbb{R}^{>0}$  (polytopes can be dilated). Hence,  $S_P$  can be regarded as a cone in the vector space  $V_P$ , where  $V_P$  is the minimal group containing  $S_P$  (i. e. the Grothendieck group of  $S_P$ ). The elements of  $V_P$  are called *virtual polytopes* analogous to  $P$ .

We now introduce special coordinates in the vector space  $V_P$  in the case where  $P$  is *simple*. A polytope in  $\mathbb{R}^k$  is called *simple* if it is generic with respect to parallel translations of its facets. Namely, exactly  $k$  facets must meet at each vertex. This implies that any other face is also the transverse intersection of those facets that contain it. Let  $\Gamma_1, \dots, \Gamma_l$  be the facets of  $P$ , and let  $\Gamma_1^*, \dots, \Gamma_l^*$  be the corresponding rays in  $P^*$ . Choose a non-zero functional  $h_i \in \Gamma_i^*$  in each ray. Call  $h_i$

a *support function* corresponding to the facet  $\Gamma_i$ . For any polytope  $Q$  analogous to  $P$ , denote by  $h_i(Q)$  the maximal value of  $h_i$  on the polytope  $Q$ . For instance, if  $h_i$  is normalized so that its value on the external unit normal to the facet  $\Gamma_i$  is 1, then  $h_i(P)$  is up to a sign the distance from the origin to the hyperplane that contains the facet  $\Gamma_i$  (the sign is positive if the origin and the polytope  $P$  are to the same side of this hyperplane, and negative otherwise). The numbers  $h_1(Q), \dots, h_l(Q)$  are called the *support numbers* of  $Q$ . Clearly, the polytope  $Q$  is uniquely defined by its support numbers. The coordinates  $h_1(Q), \dots, h_l(Q)$  can be extended to the space  $V_P$ , providing the isomorphism between  $V_P$  and the coordinate space  $\mathbb{R}^l$ .

In what follows, we will deal with *integer polytopes*, i. e. polytopes whose vertices belong to a given lattice  $\mathbb{Z}^k \subset \mathbb{R}^k$ . For such polytopes, the natural way to normalize the support functions is to require that  $h_i(P)$  be equal to the *integral distance* from the origin to the hyperplane that contains the facet  $\Gamma_i$ . Suppose that a hyperplane  $H$  not passing through the origin is spanned by lattice vectors. Then the *integral distance* from the origin to the hyperplane  $H$  is the index in  $\mathbb{Z}^k$  of the subgroup spanned by  $H \cap \mathbb{Z}^k$ . To compute the integral distance one can apply a unimodular (with respect to the lattice  $\mathbb{Z}^k$ ) linear transformation of  $\mathbb{R}^k$  so that  $H$  becomes parallel to a coordinate hyperplane. Then the integral distance is the usual Euclidean distance from the origin to this coordinate hyperplane.

**2.2. Regular compactifications of reductive groups.** With any representation  $\pi: G \rightarrow \text{GL}(V)$  one can associate the following compactification of  $\pi(G)$ . Take the projectivization  $\mathbb{P}(\pi(G))$  of  $\pi(G)$  (i. e. the set of all lines in  $\text{End}(V)$  passing through a point of  $\pi(G)$  and the origin), and then take its closure in  $\mathbb{P}(\text{End}(V))$ . We obtain a projective variety  $X_\pi \subset \mathbb{P}(\text{End}(V))$  with a natural action of  $G \times G$  coming from the left and right action of  $\pi(G) \times \pi(G)$  on  $\text{End}(V)$ . For example, when  $G = (\mathbb{C}^*)^n$  is a complex torus, all projective toric varieties can be constructed in this way.

Assume that  $\mathbb{P}(\pi(G))$  is isomorphic to  $G$ . Consider all weights of the representation  $\pi$ , i. e. all characters of the maximal torus  $T$  occurring in  $\pi$ . Take their convex hull  $P_\pi$  in  $L_T \otimes \mathbb{R}$ . Then it is easy to see that  $P_\pi$  is a polytope invariant under the action of the Weyl group of  $G$ . It is called the *weight polytope* of the representation  $\pi$ . The polytope  $P_\pi$  contains information about the compactification  $X_\pi$ .

**Theorem 2.1.** (1) ([14], Proposition 8) *The subvariety  $X_\pi$  consists of a finite number of  $G \times G$ -orbits. These orbits are in one-to-one correspondence with the orbits of the Weyl group acting on the faces of the polytope  $P_\pi$ . This correspondence preserves incidence relations. i. e. if  $F_1, F_2$  are faces such that  $F_1 \subset F_2$ , then the orbit corresponding to  $F_1$  is contained in the closure of the orbit corresponding to  $F_2$ .*

(2) *Let  $\sigma$  be another representation of  $G$ . The normalizations of subvarieties  $X_\pi$  and  $X_\sigma$  are isomorphic if and only if the normal fans corresponding to the polytopes  $P_\pi$  and  $P_\sigma$  coincide. If the first fan is a subdivision of the second, then there exists a  $G \times G$ -equivariant map from the normalization of  $X_\pi$  to  $X_\sigma$ , and vice versa.*

The second part of Theorem 2.1 follows from the general theory of spherical varieties (see [13], Theorem 5.1) combined with the description of compactifications  $X_\pi$  via colored fans (see [14], Sections 7, 8).

In what follows, we will only consider *regular* compactifications of  $G$ . The simplest example of a regular compactification is the *wonderful compactification* constructed by De Concini and Procesi. Suppose that the group  $G$  is of adjoint type, i. e. the center of  $G$  is trivial. Take any irreducible representation  $\pi$  with a strictly dominant highest weight. It is proved in [6] that the corresponding compactification  $X_\pi$  of the group  $G$  is always smooth and, hence, does not depend on the choice of a highest weight. Indeed, the normal fan of the weight polytope  $P_\pi$  coincides with the fan of the Weyl chambers and their faces, so the second part of Theorem 2.1 applies. This compactification is called the *wonderful compactification* and is denoted by  $X_{\text{can}}$ .

Other *regular* compactifications of  $G$  can be characterized as follows. The normalization  $X$  of  $X_\pi$  is *regular* if first, it is smooth, and second, there is a  $(G \times G)$ -equivariant map from  $X$  to  $X_{\text{can}}$ . These two conditions can be reformulated in terms of the weight polytope  $P_\pi$ . Namely, the first condition implies that  $P_\pi$  is *integrally* simple (see [14], Theorem 9), i. e. it is simple and the primitive vectors on the edges meeting at each vertex form a basis of  $L_T$ . The second condition implies that none of the vertices of  $P_\pi$  lies on the walls of the Weyl chambers, i. e. the normal fan of  $P_\pi$  subdivides the fan of the Weyl chambers and their faces.

A regular compactification  $X$  has the following nice properties (see [5] for details), which we will use in the sequel. The boundary divisor  $X \setminus G$  is a divisor with normal crossings. The  $G \times G$ -orbits of codimension  $s$  correspond to the faces of  $P_\pi$  of codimension  $s$  and have rank  $(k - s)$ . Recall that each face  $F \subset P_\pi$  is the transverse intersection of several facets of  $P_\pi$  (since  $P_\pi$  is simple). Then the closure of the orbit corresponding to  $F$  is the transverse intersection of the closures of the codimension one orbits that correspond to these facets. Each closed orbit of  $X$  (such orbits correspond to the vertices of  $P_\pi$ ) is isomorphic to the product of two flag varieties  $G/B \times G/B$ .

**2.3. Picard group of compactifications.** Let  $X$  be the normalization of the compactification  $X_\pi$  of  $G$ . We assume that  $X$  is regular, and hence smooth. Then the second cohomology group  $H^2(X)$  is isomorphic to the Picard group of  $X$  (see [2]). There is a description of the Picard group of a regular complete symmetric space due to Bifet (see [2], Theorem 2.4, see also [3], Proposition 3.2). In our case, this description can be reformulated as follows (such a reformulation is well-known in the toric case, and in the reductive case it was suggested by K. Kiumars). Denote by  $V(\pi)$  the group of all integer virtual polytopes analogous to the weight polytope  $P_\pi$  and invariant under the action of the Weyl group.

**Proposition 2.2.** *The Picard group  $\text{Pic}(X)$  of  $X$  is canonically isomorphic to the quotient group of  $V(\pi)$  modulo parallel translations. The isomorphism takes the hyperplane section corresponding to a representation  $\sigma$  to the weight polytope of  $\sigma$  and extends to the other divisors by linearity.*

In particular, if  $G$  is semisimple, then  $\text{Pic}(X) = V(\pi)$  (the only parallel translation taking a  $W$ -invariant polytope to a  $W$ -invariant polytope is the trivial one). Let us identify divisors in  $X$  with the corresponding polytopes using this isomorphism.

The variety  $X$  has  $l$  distinguished *boundary divisors*  $\bar{\mathcal{O}}_1, \dots, \bar{\mathcal{O}}_l$ , which are the closures of codimension one orbits. Let us describe the corresponding virtual polytopes. Choose  $l$  facets  $\Gamma_1, \dots, \Gamma_l$  of  $P_\pi$  so that each orbit of the Weyl group acting on the facets of  $P_\pi$  contains exactly one  $\Gamma_i$ . For example, one may take all facets that intersect the fundamental Weyl chamber. Choose the support functions  $h_1, \dots, h_l$  corresponding to these facets so that  $h_i(P_\pi)$  is equal to the integral distance (with respect to the weight lattice  $L_T$ ) from the origin to the facet  $\Gamma_i$ .

**Lemma 2.3.** *The closure  $\bar{\mathcal{O}}_i$  of codimension one orbit corresponds to the virtual polytope whose  $i$ -th support number is 1 and the other support numbers are 0.*

*Proof.* Let  $\sigma$  be any representation of  $G$  whose weight polytope  $P$  is analogous to  $P_\pi$ . Then  $X$  is isomorphic to the normalization of the compactification  $X_\sigma$ . Thus a generic linear functional  $f$  on  $X_\sigma$  can also be regarded as a rational function on  $X$ . Let us find the zero and the pole divisors of  $f$ . The zero divisor  $D$  is the divisor corresponding to the weight polytope of  $\sigma$ . The pole divisor is a linear combination of the divisors  $\bar{\mathcal{O}}_1, \dots, \bar{\mathcal{O}}_l$ . It is not hard to show that the coefficients are the support numbers  $h_1(P_\sigma), \dots, h_l(P_\sigma)$ , i.e. the integral distances from the origin to the facets of  $P_\sigma$  corresponding to  $\Gamma_1, \dots, \Gamma_l$ . Indeed, for toric varieties, this statement is well-known (see [8], Section 3.4). In particular, this holds for the closure  $\bar{T}$  in  $X$  of the maximal torus  $T \subset G$ . Note that the toric variety  $\bar{T}$  corresponds to the same polytope  $P_\sigma$  and the codimension one orbits of  $\bar{T}$  are the irreducible components of  $\mathcal{O}_i \cap \bar{T}$ . Hence, in order to have the right coefficients in the decomposition of  $D \cap \bar{T}$  along the hypersurfaces  $\bar{\mathcal{O}}_1 \cap \bar{T}, \dots, \bar{\mathcal{O}}_l \cap \bar{T}$  in  $\bar{T}$ , we must have

$$D = h_1(P_\sigma)\bar{\mathcal{O}}_1 + \dots + h_l(P_\sigma)\bar{\mathcal{O}}_l.$$

It follows that  $h_i(\bar{\mathcal{O}}_j) = 0$ , unless  $i = j$ . □

Another useful collection of divisors consists of the closures in  $X$  of codimension one Bruhat cells in  $G$ . Denote these divisors by  $D_1, \dots, D_k$ . They can also be described as follows. Denote by  $\omega_1, \dots, \omega_k$  the fundamental highest weights of  $G$ . Let  $X_i$  be the compactification corresponding to the irreducible representation  $\pi_i$  of  $G$  with the highest weights  $\omega_i$ . Then by Theorem 2.1 there is an equivariant map from the wonderful compactification  $X_{\text{can}}$  to  $X_i$ , and hence, there is also an equivariant map  $p: X \rightarrow X_i$ . The divisor  $D_i$  is the preimage under the map  $p$  of the hyperplane section (regarded as a divisor in  $X_i$ ) corresponding to the representation  $\pi_i$ .

To each dominant weight  $\lambda = m_1\omega_1 + \dots + m_k\omega_k$  there corresponds the *weight divisor*  $D(\lambda) = m_1D_1 + \dots + m_kD_k$ . The polytope of this divisor is the weight polytope  $P_\lambda$  of the irreducible representation with the highest weight  $\lambda$ . Note that  $\lambda$  is the only vertex of  $P_\lambda$  inside the fundamental Weyl chamber. Hence, it belongs to all facets of  $P_\lambda$  corresponding to  $\Gamma_1, \dots, \Gamma_l$  (e.g. some of the facets might degenerate to the vertex  $\lambda$ ). This implies the following lemma.

**Lemma 2.4.** *Let  $D(\lambda)$  be the weight divisor corresponding to a weight  $\lambda \in L_T$  and let  $P_\lambda$  be its polytope. Then  $h_i(P_\lambda) = h_i(\lambda)$  for any  $i = 1, \dots, l$ .*

Combination of these two lemmas leads to the following result.

**Corollary 2.5.** *Let  $D$  be the divisor on  $X$  corresponding to a polytope  $P$ . We assume that  $P$  is analogous to  $P_\pi$  and identify the respective facets. Then for any face  $F \subset P$  of codimension  $s$  that intersects the fundamental Weyl chamber  $\mathcal{D}$  and for any point  $\lambda \in F \cap \mathcal{D}$ , the divisor  $D$  can be written uniquely as a linear combination of  $D(\lambda)$  and of boundary divisors  $\bar{\mathcal{O}}_i$  such that the corresponding facets  $\Gamma_i$  do not contain  $F$ . Namely, if  $F = \Gamma_{i_1} \cap \dots \cap \Gamma_{i_s}$ , then*

$$D = D(\lambda) + \sum_{j \in \{1, \dots, l\} \setminus \{i_1, \dots, i_s\}} [h_j(P) - h_j(\lambda)] \bar{\mathcal{O}}_j.$$

**2.4. Ring of conditions.** Let  $Z_1$  and  $Z_2$  be two algebraic subvarieties in  $G$ . One can define their intersection index  $Z_1 Z_2$  as the number of points in the intersection  $gZ_1 \cap Z_2$  for a generic  $g \in G$ . Kleiman’s transversality theorem ensures that such an intersection index is well-defined, i. e. for a generic  $g \in G$  the intersection  $gZ_1 \cap Z_2$  is transverse and its cardinality does not depend on the choice of  $g$  [12]. We now consider the group  $C^*(G)$  of all formal linear combinations of algebraic subvarieties in  $G$  up to the following equivalence relation. Two subvarieties  $Z_1, Z_2$  of the same dimension are equivalent if and only if for any subvariety  $Y$  of complementary dimension the intersection indices  $Z_1 Y$  and  $Z_2 Y$  coincide.

For any two equivalence classes in  $C^*(G)$  represented by subvarieties  $Z_1$  and  $Z_2$  define their product as the class of the subvariety  $gZ_1 \cap Z_2$  for a generic  $g \in G$ . De Concini and Procesi showed that this product is well defined in  $C^*(G)$ , i. e. it does not depend on the choice of representatives [7]. The ring  $C^*(G)$  is called the *ring of conditions* of  $G$ .

In what follows, we will also use the relation between the ring of conditions and the (co)homology ring of a regular compactification. Namely, the subvarieties  $Z_1$  and  $Z_2$  represent the same class in the ring of conditions  $C^*(G)$  if there exists a regular compactification  $X$  of the group  $G$  such that the closures of  $Z_1$  and  $Z_2$  in  $X$  represent the same homology class in  $X$  and have proper intersections with all  $G \times G$ -orbits [6]. In particular, we will use regular compactifications to compute the intersection indices of subvarieties in  $G$ .

**2.5. Integration of polynomials.** Let  $f(x_1, \dots, x_k)$  be a homogeneous polynomial function of degree  $d$  defined on a real affine space  $\mathbb{R}^k$  with coordinates  $(x_1, \dots, x_k)$ . There is a useful formula expressing the integral of  $f$  over a simplex in  $\mathbb{R}^k$  in terms of the *polarization* of  $f$ . Recall that the *polarization* of  $f$  is the unique symmetric  $d$ -linear form  $f_{\text{pol}}$  on  $\mathbb{R}^k$  such that the restriction of  $f_{\text{pol}}$  to the diagonal coincides with  $f$ . One can define  $f_{\text{pol}}$  explicitly as follows:

$$f_{\text{pol}}(v_1, \dots, v_d) = \frac{1}{d!} \frac{\partial^d}{\partial v_1 \dots \partial v_d} f,$$

where  $\partial_{v_i}$  is the directional derivative along the vector  $v_i$ .

Let  $\Delta \subset \mathbb{R}^k$  be a  $k$ -dimensional simplex with vertices  $a_0, \dots, a_k$  and let  $dx = dx_1 \wedge dx_2 \wedge \dots \wedge dx_k$  be the standard measure on  $\mathbb{R}^k$ .

**Proposition 2.6** [3]. *Let  $f_{\text{pol}}$  be the polarization of  $f$ . It can be regarded as a linear function on the  $d$ -th symmetric power of  $V$ . Then the average value of  $f$  on the simplex  $\Delta$  coincides with the average value of  $f_{\text{pol}}$  on all symmetric products of  $d$  vectors from the set  $\{a_0, \dots, a_k\}$ :*

$$\frac{1}{\text{Vol}(\Delta)} \int_{\Delta} f(x) dx = \frac{1}{\binom{d+k}{k}} \sum_{i_0+\dots+i_k=d} f_{\text{pol}}(\underbrace{a_0, \dots, a_0}_{i_0}, \dots, \underbrace{a_k, \dots, a_k}_{i_k}).$$

### 3. CHERN CLASSES

In this section, I recall the definition of the Chern classes of spherical homogeneous spaces (see [11] for more details). In the sequel, only Chern classes of  $G \times G$ -orbits in regular compactifications of  $G$  will be used. For these Chern classes, I prove a vanishing result for their intersection indices with certain weight divisors in regular compactifications. This result will be important in Section 4 when applying the De Concini–Procesi algorithm to the Chern classes of  $G$ .

Let  $G/H$  be a spherical homogeneous space under  $G$  of dimension  $d$ . The  $i$ -th Chern class  $S_i(G/H)$  of  $G/H$  is the  $i$ -th degeneracy locus of  $n$  generic vector fields  $v_1, \dots, v_n$  coming from the action of  $G$ , that is,  $S_i(G/H) = \{x \in G/H : v_1(x), \dots, v_{d-i+1}(x) \text{ are linearly dependent}\}$ . In what follows we will use the following reformulation of this definition. Denote by  $\mathfrak{g}$  and  $\mathfrak{h}$  the Lie algebras of  $G$  and  $H$ , respectively, and denote by  $m$  the dimension of  $\mathfrak{h}$ . Define the *Demazure map*  $\varphi$  from  $G/H$  to the Grassmannian  $G(m, \mathfrak{g})$  of  $m$ -dimensional subspaces in  $\mathfrak{g}$  as follows:

$$\varphi: G/H \rightarrow G(m, \mathfrak{g}); \quad \varphi: gH \rightarrow g\mathfrak{h}g^{-1}.$$

Let  $C_i \subset G(m, \mathfrak{g})$  be the Schubert cycle corresponding to a generic subspace  $\Lambda_i \subset \mathfrak{g}$  of codimension  $m + i - 1$ , i.e.  $C_i = \{\Lambda \in G(m, \mathfrak{g}) : \dim(\Lambda \cap \Lambda_i) \geq 1\}$ . Then it is easy to see that the  $i$ -th Chern class  $S_i(G/H)$  of  $G/H$  is the preimage of  $C_i$  under the map  $\varphi$ :

$$S_i(G/H) = \varphi^{-1}(C_i).$$

The class of  $S_i(G/H)$  in the ring of conditions of  $G/H$  is the same for all generic  $C_i$  [11]. It is related to the Chern classes of the tangent bundles over regular compactifications of  $G$  [4], [11]. Namely, if  $X$  is a regular compactification of  $G/H$ , then the closure of  $S_i(G/H)$  in  $X$  is the  $i$ -th Chern class of the *logarithmic* tangent bundle over  $X$  that corresponds to the divisor  $X \setminus (G/H)$ . This vector bundle is generated by all vector fields on  $X$  that are tangent to  $G$ -orbits in  $X$ . In what follows, this bundle will be called the *Demazure bundle* of  $X$ .

Let  $X = G/H$  and  $Y = G/P$  be two spherical homogeneous spaces under  $G$ . Suppose that  $H$  is a subgroup of  $P$ . Consider the  $G$ -equivariant map

$$f: X \rightarrow Y; \quad f: gH \mapsto gP.$$

In general, it is not true that  $S_i(X)$  is the inverse image under the map  $f$  of a subset in  $Y$ . However, under the assumption that  $H$  contains a regular element of  $G$ , the intersection of  $S_i(X)$  (when it is nonempty) with a fiber of  $f$  has dimension at least  $\text{rk}(P) - \text{rk}(H)$ .

**Example.** In what follows, we will mostly deal with the case, where  $X$  and  $Y$  are spherical homogeneous spaces under the doubled group  $G \times G$ . Namely,  $X$  is a  $G \times G$ -orbit  $\mathcal{O}$  of a regular compactification of the group  $G$  and  $Y$  is a partial flag variety constructed as follows. Let  $H \subset G \times G$  be the stabilizer of a point in  $\mathcal{O}$ . Take the minimal parabolic subgroup  $P \subset G \times G$  that contains  $H$  and set  $Y = (G \times G)/P$ . It easily follows from an explicit description of the stabilizer  $H$  (see [14], Theorem 8) that  $H$  does contain a regular element of  $G \times G$ .

**Lemma 3.1.** *For a generic  $S_i(X)$ , there exists an open dense subset of  $S_i(X)$  such that for any element  $x$  of this subset the intersection of the fiber  $xP$  with  $S_i(X)$  has dimension greater than or equal to the  $\text{rk}(P) - \text{rk}(H)$ . In particular, the dimension of  $f(S_i(X))$  satisfies the inequality*

$$\dim f(S_i(X)) \leq \dim S_i(X) - (\text{rk}(P) - \text{rk}(H)).$$

*Proof.* Choose a generic vector space  $\Lambda \subset \mathfrak{g}$  of codimension  $\dim H + i - 1$ . Denote by  $\mathfrak{h}$  and  $\mathfrak{p}$  the Lie algebras of  $H$  and  $P$  respectively. Then by definition  $S_i(X)$  consists of all cosets  $gH$  such that  $g\mathfrak{h}g^{-1}$  has a nontrivial intersection with  $\Lambda$ , or equivalently  $\mathfrak{h} \cap g^{-1}\Lambda g$  is nontrivial.

Let  $gH$  be any element of  $S_i(X)$ . Estimate the dimension of the intersection of  $S_i(X)$  with the fiber  $gP$  of the map  $f$ . Because of the assumption on  $H$  stated above, for all  $g$  from a dense open subset of  $S_i(X)$ , the intersection  $\mathfrak{h} \cap g^{-1}\Lambda g$  contains an element  $v$  that is regular in  $\mathfrak{g}$ . Denote by  $C$  the centralizer in  $P$  of  $v \in \mathfrak{h} \subset \mathfrak{p}$ . Then  $\dim(C \cap H) = \text{rk}(H)$  while  $C$  has dimension at least  $\text{rk}(P)$ . Note that for any  $c \in C$  the coset  $gcH$  still belongs to  $S_i(X)$  since  $c^{-1}g^{-1}\Lambda gc$  contains  $c^{-1}vc = v$ . Hence,  $S_i(X) \cap gP$  contains a set  $gCH$  of dimension at least  $\text{rk}(P) - \text{rk}(H)$ . □

Lemma 3.1 is crucial for proving the following two vanishing results, which extend Proposition 9.1 from [6] and rely on the same ideas. Let  $X$  be a regular compactification of  $G$ , and let  $p: X \rightarrow X_{\text{can}}$  be its equivariant projection to the wonderful compactification. Denote by  $c_1, \dots, c_{n-k}$  the Chern classes of the Demazure vector bundle over  $X$ .

**Lemma 3.2.** *Let  $\mathcal{O}$  be a  $G \times G$ -orbit in  $X$  of codimension  $s < k$ , and  $\overline{\mathcal{O}} \subset X$  its closure. Suppose that the image  $p(\mathcal{O})$  under the map  $p: X \rightarrow X_{\text{can}}$  coincides with the closed orbit of  $X_{\text{can}}$ . In terms of polytopes, this means that the face corresponding to  $\mathcal{O}$  does not intersect the walls of the Weyl chambers.*

*Let  $\lambda$  be any weight of  $G$ , and  $D(\lambda)$  the corresponding weight divisor. Then the homology class  $c_i D^{n-i-s}(\lambda)$  vanishes on  $\overline{\mathcal{O}}$ , i. e. the following intersection index is zero:*

$$c_i D(\lambda)^{n-i-s} \overline{\mathcal{O}} = 0.$$

*Proof.* First of all, the intersection product  $c_i \cdot \overline{\mathcal{O}}$  is the  $i$ -th Chern class of the Demazure bundle over  $\overline{\mathcal{O}}$  (see [1], Proposition 2.4.2). Hence, it can be realized as the closure in  $\overline{\mathcal{O}}$  of the  $i$ -th Chern class  $S_i(\mathcal{O})$  of the spherical homogeneous space  $\mathcal{O}$ . The computation of the intersection index  $\overline{c_i \overline{\mathcal{O}}} D(\lambda)^{n-i-s}$  in  $X$  thus reduces to the computation of the intersection index  $\overline{S_i(\mathcal{O})} D(\lambda)^{n-i-s}$  in  $\overline{\mathcal{O}}$ . The latter is

equal to the intersection index  $S_i(\mathcal{O})D(\lambda)^{n-i-s}$  in the ring of conditions of  $\mathcal{O}$  since  $D(\lambda)$  and  $\overline{S_i(\mathcal{O})}$  have proper intersections with the boundary  $\overline{\mathcal{O}} \setminus \mathcal{O}$ .

To compute  $S_i(\mathcal{O})D(\lambda)^{n-i-s}$  we use the restriction of the map  $p: X \rightarrow X_{\text{can}}$  to  $\overline{\mathcal{O}}$ . By the hypothesis the image  $p(\overline{\mathcal{O}})$  is the closed orbit  $F$  in  $X_{\text{can}}$ , so it is isomorphic to the product  $G/B \times G/B$  of two flag varieties. Then the divisor  $D(\lambda)$  restricted to  $\overline{\mathcal{O}}$  is the inverse image under the map  $p$  of the divisor  $D(\lambda, \lambda)$  in  $F$ . Indeed,  $D(\lambda) = p^{-1}(\tilde{D}(\lambda))$ , where  $\tilde{D}(\lambda)$  is the weight divisor in  $X_{\text{can}}$  corresponding to  $\lambda$ . It is easy to check that  $\tilde{D}(\lambda) \cap F = D(\lambda, \lambda)$  (see Proposition 8.1 in [6]).

Hence, all the intersection points in  $S_i(\mathcal{O})D(\lambda)^{n-i-s}$  are contained in the preimage of  $p(S_i(\mathcal{O}))D(\lambda, \lambda)^{n-i-s}$ . But the latter is empty. Indeed, since  $\mathcal{O}$  has positive rank and  $F$  has zero rank, Lemma 3.1 implies that

$$\dim p(S_i(\mathcal{O})) < \dim S_i(\mathcal{O}) = n - i - s. \quad \square$$

It remains to deal with the orbits in  $X$  whose image under the map  $p$  is not the closed orbit in  $X_{\text{can}}$ . In this case, the face corresponding to such an orbit intersects the walls of the Weyl chambers, and hence, it is orthogonal to some of the fundamental weights  $\omega_1, \dots, \omega_k$ . Note that the codimension one orbits  $\mathcal{O}_1, \dots, \mathcal{O}_k$  in  $X_{\text{can}}$  are in one-to-one correspondence with the fundamental weights  $\omega_1, \dots, \omega_k$ . Namely, the facet corresponding to  $\mathcal{O}_i$  is orthogonal to  $\omega_i$ . Let  $\overline{\mathcal{O}}_1, \dots, \overline{\mathcal{O}}_k$  be the closures in  $X_{\text{can}}$  of  $\mathcal{O}_1, \dots, \mathcal{O}_k$ , respectively.

**Lemma 3.3.** *Let  $\mathcal{O}$  be a  $G \times G$ -orbit in  $X$  of codimension  $s < k$ . Suppose that the image  $p(\mathcal{O})$  under the map  $p: X \rightarrow X_{\text{can}}$  is not closed and lies in the intersection  $\overline{\mathcal{O}}_{i_1} \cap \dots \cap \overline{\mathcal{O}}_{i_s}$ . In terms of polytopes, this means that the face corresponding to  $\mathcal{O}$  is orthogonal to the weights  $\omega_{i_1}, \dots, \omega_{i_s}$ .*

*Let  $\lambda$  be any linear combination of the weights  $\omega_{i_1}, \dots, \omega_{i_s}$ . Then*

$$c_i D(\lambda)^{n-i-s} \overline{\mathcal{O}} = 0.$$

*Proof.* We use the  $G \times G$ -equivariant map  $r$  from  $\overline{\mathcal{O}}_{i_1} \cap \dots \cap \overline{\mathcal{O}}_{i_s}$  to a partial flag variety  $G/P \times G/P$  constructed in [6] (see [6], Lemma 5.1 for details). Consider the compactification  $X_{i_1, \dots, i_s}$  of  $G$  corresponding to the irreducible representation  $\pi_{i_1, \dots, i_s}$  whose highest weight lies strictly inside the cone spanned by  $\omega_{i_1}, \dots, \omega_{i_s}$ . This compactification has a unique closed orbit  $G/P \times G/P$ , where  $P \subset G$  is the stabilizer of the highest weight vector in the representation  $\pi_{i_1, \dots, i_s}$ . Clearly, the fan of the Weyl chambers and their faces subdivides the normal fan of the weight polytope of  $\pi_{i_1, \dots, i_s}$ . Hence, by Theorem 2.1 there is an equivariant map  $r: X_{\text{can}} \rightarrow X_{i_1, \dots, i_s}$ . This map takes  $\overline{\mathcal{O}}_{i_1} \cap \dots \cap \overline{\mathcal{O}}_{i_s}$  to the closed orbit  $G/P \times G/P$ .

The composition  $rp$  maps the orbit  $\mathcal{O}$  to the closed orbit  $G/P \times G/P$  of  $X_{i_1, \dots, i_s}$ . It is easy to show that the divisor  $D(\lambda)$  restricted to  $\mathcal{O}$  is the preimage of the divisor  $D(\lambda, \lambda) \subset G/P \times G/P$  under this map (see [6] Section 8.1). Now repeat the arguments of the proof of Lemma 3.2.  $\square$

These two lemmas imply the following vanishing result.

**Corollary 3.4.** *Let  $\mathcal{O}$  be any  $G \times G$ -orbit in  $X$  of codimension  $s < k$ , and let  $F$  be the face of the polytope of  $X$  that corresponds to  $\mathcal{O}$ . The intersection index*

$$c_i D(\lambda_1) \dots D(\lambda_{n-i-s}) \bar{\mathcal{O}}$$

*vanishes in the cohomology ring of  $X$  in the following two cases:*

- (1) *The face  $F$  does not intersect the walls of the Weyl chambers. Then weights  $\lambda_1, \dots, \lambda_{n-i-s}$  are any weights of  $G$ .*
- (2) *The face  $F$  intersects a wall of the Weyl chambers and weights  $\lambda_1, \dots, \lambda_{n-i-s}$  are orthogonal to  $F$  (with respect to the inner product  $(\cdot, \cdot)$  on  $L_T \otimes \mathbb{R}$  defined in the Introduction).*

#### 4. PROOF OF THEOREM 1.2

We use notation of Sections 2.2 and 2.3. Let  $X$  be any regular compactification lying over the compactification  $X_\pi$ . Then the closure  $\bar{H}_\pi$  of  $H_\pi$  in  $X$  has proper intersections with all  $G \times G$ -orbits in  $X$ , and thus  $S_i H_\pi^{n-i}$  coincides with the intersection index  $\bar{S}_i \bar{H}_\pi^{n-i}$  in the cohomology ring of  $X$ .

Assume that  $X$  corresponds to a representation of  $G$  with the weight polytope  $P_0$ . Let us compute  $\bar{S}_i D^{n-i}$  for a divisor  $D$  under the assumption that the polytope  $P$  corresponding to  $D$  is analogous to  $P_0$ . After we establish the formula of Theorem 1.2 for such divisors, it will automatically extend to the other divisors (in particular, for  $\bar{H}_\pi$ ) since any virtual polytope analogous to  $P_0$  is a linear combination of polytopes analogous to  $P_0$ . Since  $X$  is regular,  $P_0$  and hence  $P$  are simple.

All computations are carried in the cohomology ring of  $X$ . First, break  $D^{n-i}$  into monomials of the form  $\bar{\mathcal{O}}_{i_1} \dots \bar{\mathcal{O}}_{i_k} D(\lambda_1) \dots D(\lambda_{n-i-k})$ , where  $i_1, \dots, i_k$  are distinct integers from 1 to  $l$  and  $\lambda_1, \dots, \lambda_{n-i-k}$  are weights. Then every such monomial can be computed explicitly, since the intersection  $\bar{\mathcal{O}}_{i_1} \cap \dots \cap \bar{\mathcal{O}}_{i_k}$  is either empty or isomorphic to the product of two flag varieties.

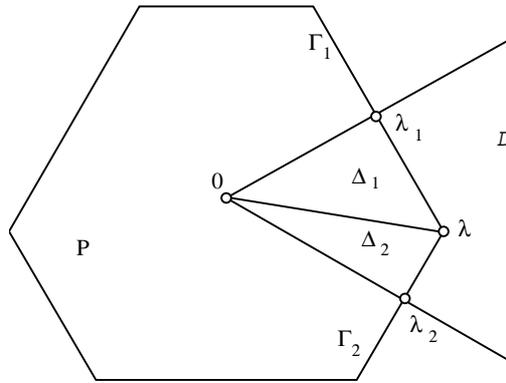
Since we are going to intersect  $D^{n-i}$  with  $\bar{S}_i$  we can ignore all monomials that are annihilated by  $\bar{S}_i$ . Recall that  $\bar{S}_i$  is the  $i$ -th Chern class of the Demazure bundle over  $X$ . In particular, Corollary 3.4 implies that  $\bar{S}_i$  annihilates the ideal  $I \subset H^*(X)$  generated by the monomials of the form  $D(\lambda_1) \dots D(\lambda_{n-i-s}) \bar{\mathcal{O}}$  such that either the face of  $P$  corresponding to the codimension  $s < k$  orbit  $\mathcal{O}$  does not intersect the walls of the Weyl chambers or, if it does, the weights  $\lambda_1, \dots, \lambda_{n-i-s}$  are orthogonal to this face.

To keep track of our calculations we use a subdivision of the polytope  $P \cap \mathcal{D}$  into simplices coming from the barycentric subdivision of  $P$  described below. For each face  $F \subset P$  choose a point  $\lambda_F \in F$  as follows. If  $F$  does not intersect the walls of the Weyl chamber  $\mathcal{D}$ , then  $\lambda_F$  is any point in the interior of the face. Otherwise, choose  $\lambda_F$  so that the corresponding vector is orthogonal to the face  $F$  (in particular,  $\lambda_F$  will belong to the intersection of the face with a wall of  $\mathcal{D}$ .) If  $F = P$  take  $\lambda_F = 0$ .

An  $s$ -flag  $\mathcal{F}$  is the collection  $\{F_1 \supset \dots \supset F_s\}$  of  $s \leq k$  nested faces of  $P$  such that each of them intersects  $\mathcal{D}$ , and  $F_i$  has codimension  $i$  in  $P$ . Denote by  $\bar{\mathcal{O}}_{\mathcal{F}}$  the closure in  $X$  of the orbit corresponding to the last face  $F_s$ , and by  $\Delta_{\mathcal{F}}$  the  $s$ -dimensional simplex with the vertices  $0, \lambda_{F_1}, \dots, \lambda_{F_s}$ . In particular, when  $s = k$ ,

the simplex  $\Delta_{\mathcal{F}}$  has full dimension and the orbit  $\mathcal{O}_{\mathcal{F}}$  is closed. The polytope  $\mathcal{D} \cap P$  is the union of simplices  $\Delta_{\mathcal{F}}$  over all possible  $k$ -flags  $\mathcal{F}$ .

**Example.** Take  $G = \mathrm{PSL}_3(\mathbb{C})$ , and let  $X = X_{\mathrm{can}}$  be its wonderful compactification. Let divisor  $D$  be a hyperplane section corresponding to the irreducible representation with a strictly dominant highest weight  $\lambda$ . In this case,  $P$  is a hexagon symmetric under the action of the Weyl group, with two edges  $\Gamma_1$  and  $\Gamma_2$  intersecting  $D$ . Then  $\lambda_i = \lambda_{\Gamma_i} \in \Gamma_i$  is orthogonal to  $\Gamma_i$  for  $i = 1, 2$  and  $\Gamma_1 \cap \Gamma_2 = \lambda$ . The subdivision of  $P \cap D$  into simplices consists of two triangles  $\Delta_1$  and  $\Delta_2$  with the vertices  $0, \lambda_1, \lambda$  and  $0, \lambda_2, \lambda$ , respectively (see the figure).



**Lemma 4.1.** Denote by  $f_d(x_1, \dots, x_k)$  the sum of all monomials of degree  $d$  in  $k$  variables  $x_1, \dots, x_k$ . The following identity holds in the cohomology ring of  $X$  modulo the ideal  $I$ :

$$D^{n-i} \equiv k! \sum_{\mathcal{F}} \mathrm{Vol}(\Delta_{\mathcal{F}}) f_{n-k-i}(D, D(\lambda_{F_1}), \dots, D(\lambda_{F_{k-1}})) \mathcal{O}_{\mathcal{F}} \pmod{I},$$

where the sum is taken over all possible  $k$ -flags  $\mathcal{F} = \{F_1 \supset \dots \supset F_k\}$ . The volume form  $\mathrm{Vol}$  is normalized so that the covolume of  $L_T$  is equal to 1.

*Proof.* We will prove the following more general statement for  $s$ -flags. Denote by  $f_{d,s}(x_1, \dots, x_s)$  the sum of all monomials of degree  $d$  in  $s$  variables.

Recall that  $\Gamma_1, \dots, \Gamma_l$  denote the facets of  $P$  that intersect the Weyl chamber  $\mathcal{D}$ . An  $s$ -flag can be alternatively described by an ordered collection of facets  $\Gamma_{i_1}, \dots, \Gamma_{i_s}$  such that their intersection  $\Gamma_{i_1} \cap \dots \cap \Gamma_{i_s}$  has codimension  $s$ . Then  $F_j = \Gamma_{i_1} \cap \dots \cap \Gamma_{i_j}$ . This is a one-to-one correspondence, since the polytope  $P$  is simple. Assign to each  $s$ -flag  $\mathcal{F}$  the following number

$$c_{\mathcal{F}} = h_{i_1}(P) [h_{i_2}(P) - h_{i_2}(\lambda_{F_1})] \dots [h_{i_s}(P) - h_{i_s}(\lambda_{F_{s-1}})].$$

In particular, when  $s = k$ , i.e.  $F_s$  is just a vertex, the number  $c_{\mathcal{F}}$  coincides with the volume of  $\Delta_{\mathcal{F}}$  times  $k!$ . Indeed, by a unimodular linear transformation of  $L_T \otimes \mathbb{R}$  we can map the hyperplanes containing the facets  $\Gamma_{i_1}, \dots, \Gamma_{i_s}$  to the coordinate hyperplanes. Then  $[h_{i_j}(P) - h_{i_j}(\lambda_{F_{j-1}})]$  is just the Euclidean distance from the vertex  $\lambda_{F_{j-1}}$  of  $\Delta_{\mathcal{F}}$  to the hyperplane containing  $\Gamma_{i_j}$ . Note that to define

volumes we do not use the inner product  $(\cdot, \cdot)$  on the lattice  $L_T$ . We only use the lattice itself.

Then for any integer  $s$  such that  $1 \leq s \leq k$  the following is true:

$$D^{n-i} \equiv \sum_{\mathcal{F}} c_{\mathcal{F}} f_{n-s-i,s}(D, D(\lambda_{F_1}), \dots, D(\lambda_{F_{s-1}})) \bar{\mathcal{O}}_{\mathcal{F}} \pmod{I}, \tag{1}$$

where the sum is taken over all  $s$ -flags.

**Example.** If  $G$  is a complex torus, formula (1) is still meaningful but looks much simpler and reduces to

$$D^s = \sum_{\mathcal{F}} c_{\mathcal{F}} \bar{\mathcal{O}}_{\mathcal{F}}.$$

Prove formula (1) by induction on  $s$ . We use the notations of Section 2.3. For  $s = 1$ , the statement coincides with the decomposition  $D = h_1(P)\bar{\mathcal{O}}_1 + \dots + h_l(P)\bar{\mathcal{O}}_l$  from Lemma 2.3.

Assume that the formula is proved for some  $s < k$ . Prove it for  $s + 1$ . We now deal separately with each term on the right hand side of formula (1). First subtract the element  $f_{n-s-i,s}(D(\lambda_{F_s}), D(\lambda_{F_1}), \dots, D(\lambda_{F_{s-1}})) \bar{\mathcal{O}}_{\mathcal{F}} \in I$  from every term  $f_{n-s-i,s}(D, D(\lambda_{F_1}), \dots, D(\lambda_{F_{s-1}})) \bar{\mathcal{O}}_{\mathcal{F}}$ . This operation does not change the identity (1). A simple calculation shows that

$$\begin{aligned} f_{n-s-i,s}(x, x_1, \dots, x_{s-1}) - f_{n-s-i,s}(x_s, x_1, \dots, x_{s-1}) \\ = (x - x_s) f_{n-s-i-1,s+1}(x, x_1, \dots, x_{s-1}, x_s). \end{aligned}$$

Hence, after subtraction we can rewrite the difference as

$$(D - D(\lambda_{F_s})) f_{n-s-i-1,s+1}(D, D(\lambda_{F_1}), \dots, D(\lambda_{F_s})) \bar{\mathcal{O}}_{\mathcal{F}}.$$

Since  $\lambda_s$  lies in the intersection of  $s$  facets  $\Gamma_{i_1}, \dots, \Gamma_{i_s}$ , Corollary 2.5 implies that

$$(D - D(\lambda_{F_s})) \bar{\mathcal{O}}_{\mathcal{F}} = \sum_{j \neq i_1, \dots, i_s} [h_j(P) - h_j(\lambda_{F_s})] \bar{\mathcal{O}}_j \bar{\mathcal{O}}_{\mathcal{F}}.$$

Note that  $\bar{\mathcal{O}}_j \bar{\mathcal{O}}_{\mathcal{F}}$  is empty if and only if the intersection of  $\Gamma_j$  with  $\Gamma_{i_1} \cap \dots \cap \Gamma_{i_s}$  is empty. Hence,

$$(D - D(\lambda_{F_s})) \bar{\mathcal{O}}_{\mathcal{F}} = \sum_{\mathcal{F}'} [h_j(P) - h_j(\lambda_{F_s})] \bar{\mathcal{O}}_{\mathcal{F}'},$$

where the sum is taken over all  $(s + 1)$ -flags  $\mathcal{F}'$  that extend  $\mathcal{F}$ , i.e.  $\mathcal{F}' = \{F_1 \supset \dots \supset F_s \supset F_s \cap \Gamma_j\}$ . □

It remains to compute the term

$$\bar{S}_i \cdot f_{n-k-i}(D, D(\lambda_{F_1}), \dots, D(\lambda_{F_{k-1}})) \mathcal{O}_{\mathcal{F}} \tag{2}$$

for each  $k$ -flag  $\mathcal{F}$ . Suppose that the closed orbit  $\mathcal{O}_{\mathcal{F}}$  is the intersection of  $k$  hypersurfaces  $\bar{\mathcal{O}}_{i_1}, \dots, \bar{\mathcal{O}}_{i_k}$ . Then for any other codimension 1 orbit  $\mathcal{O}_j$  (such that  $j \neq i_1, \dots, i_k$ ), the intersection  $\mathcal{O}_{\mathcal{F}} \cap \bar{\mathcal{O}}_j$  is empty. Hence,  $D$  in (2) can be replaced by  $D(\lambda_{F_k})$  since

$$D = D(\lambda_{F_k}) + \sum_{j \neq i_1, \dots, i_k} (h_j(P) - h_j(\lambda_{F_k})) \bar{\mathcal{O}}_j.$$

Note also that the evaluation of (2) reduces to the computation of intersection indices in  $\mathcal{O}_{\mathcal{F}}$ , which is the product of two flag varieties. We have that  $\bar{S}_i \cdot \mathcal{O}_{\mathcal{F}} = c_i(\mathcal{O}_{\mathcal{F}})$  and  $D(\lambda) \cdot \mathcal{O}_{\mathcal{F}} = D(\lambda, \lambda)$ . Here  $c_i(\mathcal{O}_{\mathcal{F}})$  is the  $i$ -th Chern class of the tangent bundle of  $\mathcal{O}_{\mathcal{F}}$ , which coincides with the Demazure bundle over  $\mathcal{O}_{\mathcal{F}}$  since  $\mathcal{O}_{\mathcal{F}}$  is closed. Hence,

$$\begin{aligned} \bar{S}_i f_{n-k-i}(D(\lambda_{F_k}), D(\lambda_{F_1}), \dots, D(\lambda_{F_{k-1}}))\mathcal{O}_{\mathcal{F}} &= \\ &= c_i(G/B \times G/B) f_{n-k-i}(D(\lambda_{F_1}, \lambda_{F_1}), \dots, D(\lambda_{F_k}, \lambda_{F_k})). \end{aligned} \tag{3}$$

The intersection product in the right hand side of this formula is taken in  $G/B \times G/B$ .

The function  $F_i(\lambda) = c_i(G/B \times G/B)D(\lambda, \lambda)^{n-k-i}$  can be expressed explicitly in terms of the function  $F$  defined in the Introduction, since the  $i$ -th Chern class of  $G/B \times G/B$  is the term of degree  $i$  in the intersection product

$$\prod_{\alpha \in R^+} (1 + D(\alpha, 0))(1 + D(0, \alpha)).$$

One way to compute  $F_i$  is as follows. Let  $\mathbb{D}$  and  $[\mathbb{D}]_i$  be the differential operators defined in the Introduction. Then

$$F_i(x) = (n - k - i)! [\mathbb{D}]_i F(x, x).$$

This easily follows from the formula for the polarization mentioned in Section 2.5 and the fact that  $D^{n-k}(\lambda, \lambda) = (n - k)! F(\lambda, \lambda)$ .

We can now apply Proposition 2.6 to convert the sum (3) into the integral over the simplex  $\Delta_{\mathcal{F}}$ . Indeed, by definition of the function  $f_{n-k-i}$  we have that (3) can be rewritten as

$$\sum_{i_1 + \dots + i_k = n-k-i} (F_i)_{\text{pol}}(\underbrace{\lambda_{F_1}, \dots, \lambda_{F_1}}_{i_1}, \dots, \underbrace{\lambda_{F_k}, \dots, \lambda_{F_k}}_{i_k}).$$

This is equal to the integral

$$\binom{n-i}{k} \int_{\Delta_{\mathcal{F}}} F_i(x) dx / \text{Vol}(\Delta_{\mathcal{F}})$$

by Proposition 2.6 applied to the simplex  $\Delta_{\mathcal{F}}$  (with the vertices  $0, \lambda_{F_1}, \dots, \lambda_{F_k}$ ) and to the function  $F_i(x)$ . Combining this with Lemma 4.1 we get

$$\bar{S}_i D^{n-i} = \frac{(n-i)!}{(n-k-i)!} \sum_{\mathcal{F}} \int_{\Delta_{\mathcal{F}}} F_i(x) dx = (n-i)! \int_{P \cap \mathcal{D}} [\mathbb{D}]_i F(x, x) dx.$$

Note that when  $i = 0$ , we get the Brion–Kazarnovskii formula.

### 5. EXAMPLE

In this section, I give an example of computation of the Euler characteristic using Theorems 1.2 and 1.3. Namely, for the group  $G = \text{SL}_3(\mathbb{C})$ , I compute the Euler characteristic of a hyperplane section corresponding to an irreducible representation.

Let  $\pi$  be the irreducible representation of  $\mathrm{SL}_3(\mathbb{C})$  with the highest weight  $\lambda = m\omega_1 + n\omega_2$ , where  $\omega_1, \omega_2$  are the fundamental weights of  $\mathrm{SL}_3(\mathbb{C})$ , and  $m$  and  $n$  are nonnegative integers. Take a generic hyperplane section  $H_\pi$  corresponding to the representation  $\pi$ . We first find  $S_i H_\pi^{n-i}$ . The dimension of  $\mathrm{SL}_3(\mathbb{C})$  is 8, and the rank is 2, so there are 6 nontrivial Chern classes  $S_1, \dots, S_6$ .

Let us compute all ingredients of the formula of Theorem 1.2. The weight polygon  $P_\pi$  of  $\pi$  is depicted on the figure above. The domain  $P_\pi \cap \mathcal{D}$  is the union of two triangles  $\Delta_1$  and  $\Delta_2$ . The positive roots of  $\mathrm{SL}_3(\mathbb{C})$  are  $\alpha_1 = 2\omega_1 - \omega_2$ ,  $\alpha_2 = 2\omega_2 - \omega_1$  and  $\alpha_1 + \alpha_2$ . Write  $\mathbb{D}$  and  $F$  in the coordinates  $(s, t; \tilde{s}, \tilde{t})$  in  $(L_T \oplus L_T) \otimes \mathbb{R}$  associated with the basis  $\{\alpha_1, \alpha_2\}$  in  $L_T \otimes \mathbb{R}$ :

$$\mathbb{D} = (1 + \partial_s)(1 + \partial_t)(1 + \partial_s + \partial_t)(1 + \partial_{\tilde{s}})(1 + \partial_{\tilde{t}})(1 + \partial_{\tilde{s}} + \partial_{\tilde{t}}).$$

Since  $\rho = \alpha_1 + \alpha_2$  and  $(\alpha_1, \omega_1) = (\alpha_2, \omega_2) = 1$ ,  $(\alpha_1, \omega_2) = (\alpha_2, \omega_1) = 0$  we have that

$$F = \frac{1}{4}(2s - t)(2t - s)(t + s)(2\tilde{s} - \tilde{t})(2\tilde{t} - \tilde{s})(\tilde{t} + \tilde{s}).$$

If we plug  $\mathbb{D}$  and  $F$  in the formula of Theorem 1.2 and integrate, we get that

$$H_\pi^8 = 3(m^8 + 16m^7n + 112m^6n^2 + 448m^5n^3 + 700m^4n^4 + 448m^3n^5 + 112m^2n^6 + 16mn^7 + n^8);$$

$$S_1 H_\pi^7 = 18(m+n)(m^6 + 13m^5n + 71m^4n^2 + 139m^3n^3 + 71m^2n^4 + 13mn^5 + n^6);$$

$$S_2 H_\pi^6 = 54(m^6 + 12m^5n + 50m^4n^2 + 80m^3n^3 + 50m^2n^4 + 12mn^5 + n^6);$$

$$S_3 H_\pi^5 = 90(m+n)(m^4 + 9m^3n + 19m^2n^2 + 9mn^3 + n^4);$$

$$S_4 H_\pi^4 = 18(5m^4 + 40m^3n + 72m^2n^2 + 40mn^3 + 5n^4);$$

$$S_5 H_\pi^3 = 54(m+n)(m^2 + 5mn + n^2);$$

$$S_6 H_\pi^2 = 18(m^2 + 4mn + n^2).$$

We now apply Theorem 1.3 to obtain

$$\begin{aligned} \chi(H_\pi) = & -3(m^8 + 16m^7n + 112m^6n^2 + 448m^5n^3 + 700m^4n^4 + 448m^3n^5 + \\ & 112m^2n^6 + 16mn^7 + n^8) + 18(m^6 + 12m^5n + 50m^4n^2 + 80m^3n^3 + 50m^2n^4 + 12mn^5 + n^6) + \\ & 6(5m^4 + 40m^3n + 72m^2n^2 + 40mn^3 + 5n^4) + 6(m^2 + 4mn + n^2) - 6(m+n)(m^6 + 13m^5n + \\ & 71m^4n^2 + 139m^3n^3 + 71m^2n^4 + 13mn^5 + n^6) + 5(m^4 + 9m^3n + 19m^2n^2 + 9mn^3 + n^4) + \\ & 3(m^2 + 5mn + n^2)). \end{aligned}$$

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