Syllabus
INTRODUCTION TO SDE AND NUMERICAL PROBABILITY

Approved by
the academic council of the program
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Credits	5
Contact hours	64
Self-study hours	126
Course	2

Section 1: General information about the course.

This course aims to provide a solid introduction of numerical methods based on probability theory and the general the theory and applications of Stochastic Differential Equations (SDEs). General methods to simulate random variables (discrete, real, multivariate) and techniques for solving classical and more advanced Stochastic Differential equations which arises from financial, economic or physical models will be reviewed. Numerical aspects and links between Stochastic Differential Equations with other fields in mathematics will be further discussed. The course is primarily designed for students possessing some knowledge on analysis and differential equations, and a solid background in probability theory. Although some knowledge on stochastic processes will useful, part of the course will be dedicated to review/recall the fundamentals of the theory on stochastic processes and stochastic calculus which will be used throughout the course.

Section 2. Course goals, learning objectives, expected learning outcomes.

The main objectives of the course are:

- to introduce the fundamentals of probabilistic numerical methods for the approximation of integration calculus, the simulation of given distribution and continuous time stochastic processes.
- to provide students with the knowledge of the theoretical, modeling and numerical aspects related to Stochastic Differential Equations;
- to provide students with the knowledge of fundamentals techniques to analyze the solutions of general Stochastic Differential Equations, grounding their explanations on intuitive and analytical approaches;
- To present and study some elementary models of Stochastic Differential Equations used in Finance, Physics, Economy,..
- to develop students' ability to apply the knowledge acquired during the course to study and use Stochastic Differential Equations for concrete modeling purposes, recognizing the appropriate frameworks and analytical tools related to these equations.

Prerequisites: The course requires from the students to have solid bases in probability and measure theory and some knowledge in ordinary and partial differential equations and functional analysis.

Section 3. Course Outline

№	Topic/Focus /Activity
1	**Introduction to Numerical Probability:** - Limit theorems in Probability and The Monte-Carlo methods. - Pseudo-random number generators. - Simulation of random variables: The inverse transform method; the acceptance-rejection algorithm; the Box-Muller algorithm.
2	**The Metropolis-Hasting algorithm** - Markov chains: Definitions and main properties. - State classifications: Recurrent, transient, and irreducible states; Aperiodic sets. - Invariant measure and ergodic properties of a Markov chain - General MCMC methods. - The Metropolis-Hastings algorithm: Finite and countable case.
3	**Basic Elements of Stochastic Processes.** - Time continuous Stochastic; Filtration; Stopping Time; Martingales. - - The Poisson processes and its main properties. - The Brownian motion and its main properties. - Martingales inequality.
4	**Stochastic Integration.** - Itô's integral. - Itô's formula and its applications.
5	**Strong solution to a Stochastic Differential Equation (SDE) and related properties.** - Generic form of a SDE and basic properties. - Link between Ordinary Differential Equations and Stochastic Differential Equations. - The principle of causality and the notion of strong solution to a SDE. - Construction and uniqueness and properties of a strong solution to a SDE with Lipschitz coefficients. - Explosion time and local solution. - Examples in Finance and Physics. -Discretization schemes: The Euler-Maruyama scheme. -Quantative rate of convergence.
4	**Links with partial differential equations.** - The Fokker-Planck equation - Flows of SDEs and the Feynman- Kac /Kolmogorov backward equation.
5	**Presentation of reading studies.**
6	**A short overview on Stochastic Differential Equations of McKean-Vlasov type and in optimal control.**

Description of course methodology and forms of assessment to be used

While teaching the course the following teaching methods and forms of study and control are used:

- lectures;
- home assignments;
- presentation of reading of chosen scientific papers related to the topics of the course;
- two written in-class quizzes (30mn each);
- self-study;
- teachers' consultations;
- One written test (three hours).

Assessment and grade determination:

- Average mark for quizzes [32%]
- Students Presentation [18%]

- Test [50%]

Section 4. Texts, readings and other informational resources.

1. I. Karatzas and S. E. Shreve (1998): Brownian motion and Stochastic Calculus.
2. B. Oksendal (1998): Stochastic Differential Equations: An Introduction with Applications.
3. P. Protter (1990): Stochastic Integration and Differential Equations.
4. G. Pages (2016): Introduction to Numerical Probability for Finance (Online course: https://www.lpsm.paris//dw/lib/exe/fetch.php?media=users:pages:probnum_gilp_pf16.pdf)
5. P.E. Kloeden and E. Platen (1999): Numerical Solutions of Stochastic Differentials Equations.
6. P. Glasserman (2003): Monte-Carlo Methods in Financial Engineering.
7. R. Durrett (2012): Essentials of Stochastic Processes.

This bibliography will be completed with more specific publications related to the subject of the course.

Section 5. Samples of some basic problems:

- Propose an algorithm to simulate a Cauchy distribution with parameters 1,1.

- Analyze the rate of convergence of the Monte-Carlo approximation of the integral over [0,1] of the function exp(|x|^2).

- Given (B(t); 0≤t≤ T) is a standard Brownian motion, show that the process M(t) = exp($\sqrt{2\mu}$B(t)-μt) is a square integrable martingale.

- Given (B(t); 0≤t≤ T) is a standard Brownian motion, give the explicit expression of the solution : dU(t) = -βU(t)dt+ rdB(t), U(0)~N(0,1) and of its time marginal densities.

- Using an appropriate change of probability measure, demonstrate the existence of a weak solution to the SDE: dX(t) = sign(X(t)) $\sqrt{|X|(t)}$dt + dW(t); X(0) = 0.

- Demonstrate that the solution to the SDE dX(t) = (4-X(t))dt + $\sqrt{X(t)}$dW(t); X(0) = 1; never hits 0.

- Formulate in terms of a martingale problem, the existence problem of the SDE dX(t) = b(t,X(t))dt + σ(t,X(t))dW(t); X(0) = x, where (W(t); 0≤t≤ T) is a d-dimensional Brownian motion, b:(0,∞)x$R^m \longrightarrow R^m$ and σ:(0,∞)x$R^m \longrightarrow R^{mxd}$.
- Give sufficient conditions on b to ensure the existence of a weak solution to the SDE: dX(t) = b(t,X(t))dt + σdW(t) for σ>0.

Grading system and how both the course and final test will be graded

Written tests are graded out of 100 points. Then the results for the written tests and quizzes are used to calculate the final mark using the weights specified in section 3 and the resulting mark is converted into 10-points scale.

Make-up policies and form of the make-up

If a student didn't attend a quiz and the medical document is provided to the study office in time then the weight of this test is transferred to the Test 2 so that the formula for the Fall term in this case is

- Average mark for quizzes and presentation [40%]
- Test [60%]

If a student didn't attend Test in the Fall term and the medical document is provided to the study office in time then he can write Test (a different version of the test) on another day (the same applies for all students missing Test). If a student didn't attend this Test for the second time he/she gets zero score for Test (even if the medical document is provided). If a student doesn't attend the midterm test in the Spring term or doesn't attend a quiz or doesn't submit a home assignment in time then he/she gets zero score for the corresponding activity.

If the Fall term mark is below 4 out of 10 then the student can sit one written re-take exam (Commission) in the end of January/beginning of February set in accordance with the HSE's Internal Regulations. This exam covers all the material studied in the Fall term.

Section 6. Academic Integrity

The Higher School of Economics strictly adheres to the principle of academic integrity and honesty. Accordingly, in this course there will be a zero-tolerance policy toward academic dishonesty. This includes, but is not limited to, cheating, plagiarism (including failure to properly cite sources), fabricating citations or information, tampering with other students' work, and presenting a part of or the entirety of another person's work as your own. HSE uses an automated plagiarism-detection system to ensure the originality of students' work. Students who violate university rules on academic honesty will face disciplinary consequences, which, depending on the severity of the offense, may include having points deducted on a specific assignment, receiving a failing grade for the course, being expelled from the university, or other measures specified in HSE's Internal Regulations.