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# OPTIMAL COERCION IN PROPERTY ASSEMBLY

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Multiple units of property, each owned by a seller with a private value, are perfect complements in a redevelopment project. The paper develops a mechanism for assembly of property which (i) allows coercion of sellers, (ii) does not require any transactions unless the property assembly takes place, (iii) is ex-post budget-balanced, (iv) guarantees a minimum compensation to each seller, and (v) maximizes the joint welfare of sellers. The mechanism is shown to converge asymptotically, as the number of sellers grows, to the first-best at a high rate. The mechanism requires very little knowledge about the distributions of seller valuations, and is robust to incorrect specification of distributions of both seller or buyer valuations by the market maker.

Keywords: assembly Coercion Holdout Mechanism design

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# 1 Introduction

## 1.1 Property assembly

Property assembly is a process of purchasing multiple separately owned units of property by a single buyer, to implement a redevelopment project. Each of the units is essential for the buyer: the project cannot be implemented if even a single unit was not purchased. The primary example of property assembly is a purchase of several land tracts for construction of transport infrastructure or large buildings. Heller (2010) provides numerous additional examples of property assembly, such as patent pools, assembly of broadband spectrum, and others.

Another important, but rarely discussed, example of property assembly is demolition of aged or out-of-demand condominium buildings, with subsequent redevelopment of land; in this case, the units to be assembled are individual apartments in that building. Unlike land, no building can last forever, so the property assembly process is the most likely end of life of every condominium.

Such assembly of old condominium apartments is especially important in post-Soviet countries where most residential real estate is low-quality multi-unit buildings, "project housing" in the American jargon. Most of them were constructed by the plan economy between 1960s and 1980s and were initially state-owned. In the 1990s, most dwellers were allowed to freely privatize their units. This has created a large and liquid market for individual apartments, but also created the need for property assembly should someone decide to demolish an old building and redevelop the land.

## 1.2 The Holdout Problem

A well-known difficulty that arises during property assembly is the holdout problem. If assembly is voluntary, a seller of every property unit to be assembled is pivotal, and every such seller will exploit his veto power to extract maximum payment from the buyer. In doing so, each seller imposes a negative externality on other sellers by reducing the chance of a successful transaction. A large body of literature agrees that, as the number of units to be assembled increases, the transaction costs of property assembly increase as well. Some studies have found that, with voluntary participation of sellers, the chance of a successful transaction goes to zero as the number of sellers increases to infinity. Particularly, Cai (2003), in a model with perfect information but costly negotiations, finds that the negotiation time increases quadratically with the number of sellers. Mailath and Postlewaite (1990) solve a Bayesian-Nash game in which multiple sellers with private valuations make simultaneous offers;[1] as the number of sellers grows large, each offer converges to the upper bound of the value distribution, making the transaction asymptotically impossible. Strange (1995) considers property assembly with heterogenous landowners and concludes that the sellers of smaller land tracts will ask more per acre. Cunningham (2013) provides empirical evidence in support of this theory. Miceli and Sirmans (2007) argue that property assembly difficulties cause large-scale development projects to relocate from city centers to periphery, where property is less dispersed; such

1. Mailath and Postlewaite (1990) actually consider multiple buyers purchasing a public good, but their model can be easily modified to a property assembly game, which I do to describe their paper.

relocations cause an excessive urban sprawl. Grossman, Pincus, and Shapiro (2010) propose a "second-best" mechanism for land assembly with voluntary participation of sellers; their mechanism however is unlikely to be useful with a large number of sellers, as it belongs to the environment of Mailath and Postlewaite (1990).

In case of post-Soviet condominiums, which are typically made of dozens or even hundreds of apartments, the above conclusions imply that free-market redevelopment of these buildings, with voluntary participation of apartment owners, is virtually impossible. The prevalence of these buildings in the post-Soviet urban landscape has made impossible the emergence of urban land market, and has dramatically slowed down the evolution of the cities. Free apartment privatization of the 1990s became a blessing for the market of those apartments, but also a curse for the market of land underneath.

## 1.3 The Existing Solutions to the Holdout Problem

Given the prevalence of the need for property assembly around the world, holdout is truly a trillion-dollar problem. Not surprisingly, scholars and policymakers have long been interested in possible solutions. Many scholars agree that some form of coercion can be applied to unit owners to achieve efficiency.

Historically, many governments have been practicing property confiscation with a compensation paid, a practice known as the *eminent domain* (ED henceforth) in the United States. As of today, it remains the only practical method of coercion. In all countries, only the state can initiate such coercion. In traditional market economies such as the United States, coercion is limited to public projects such as road construction. Because of that, property assembly for private projects, such as demolition of obsolete condominium buildings, remains an unsolved problem. In post-Soviet countries, the state does practice property takings for demolition of old condominiums and for subsequent private redevelopment. While such practice was very limited until the present day, the government of Moscow is currently pushing to make it routine. In early 2017, it has announced a plan to demolish over 4000 residential multi-unit buildings, affecting up to one million residents. The procedures for demolition and compensation decision-making remain primitive and vague, which results in inefficiency and public mistrust, as evidenced by a Moscow anti-demolition rally on May 14, 2017.

A number of studies have analyzed the economics of ED. Miceli and Segerson (2007) argue that not only the ED itself, but the threat of its use, can facilitate the property assembly process by making sellers more cooperative in bargaining. Shavell (2010) concludes, not surprisingly, that ED is more useful when there are more units to be assembled. Calandrillo (2003) proposes to replace the "just compensation" by the state in case of ED taking by the "takings insurance" provided by private insurance companies. The mechanism design literature has developed some recipes on how to optimally use coercion in property assembly. A classic reference is the expected externality mechanism by d'Aspremont and Gérard-Varet (1979). This mechanism is designed for a one-time interaction between players and requires, in our context, each seller to pay the expected externality she had on other sellers.

Recently, Kominers and Weyl (2011) draw on classic mechanism design literature to propose a number of what they call *concordance mechanisms* for property assembly. In all these mechanisms, the sellers are required to pay a tax equal to the externality, ex-post or expected, that they imposed on other sellers. The authors develop tax rebate schemes to achieve budget-balancedness of their mechanisms. In some of proposed mechanisms, the budgets are not fully balanced, meaning that the government consistently makes profit on the property assembly process. Some of the proposed mechanisms are shown to achieve the second-best (i.e. accounting for private nature of seller valuations) efficiency.

A more straightforward mechanism is to make real estate taxes based on self-declared rather than market value of property, and at the same time oblige each owner to publicly offer their property at the declared value. This idea was formalized by Posner and Weyl (2017).

## 1.4 Criticism of existing approaches

The practice of government takings has well-known deficiencies. Oftentimes, the state decision to take private property is made without the seller's feedback about such decision, which may lead to welfare losses. In many countries, property takings cannot be done for private projects. In other countries, where the state mediates property takings for subsequent private redevelopment, there is little theoretical guidance on how such decisions should be made and how the sellers should be compensated. The opportunities for corruption are wide open.

The above mentioned mechanisms do condition the assembly transaction on the feedback of the parties involved, and can achieve socially optimal outcomes in some ideal conditions. These mechanisms, however, require some or all sellers to pay fees/taxes if the transaction *did not* happen. But in many cases, the value derived from owning a unit of property is sentimental rather than financial (e.g. long family history of owning the property), which means that high-value but low-income owners are likely to lose their property.

Moreover, the existing mechanisms consider property assembly as a one-time event, typically with one buyer. At the same time, in many cases (e.g. old condominium demolition) it should be modeled as a continuous event with a flow of buyers: if one buyer has failed to assemble property, another one may appear at some point in the future. The existing mechanisms would then require high-value sellers to pay taxes on a continuous basis, if they want to prevent their property from being taken.

## 1.5 The alternative

The goal of this paper is to propose a property assembly mechanism with the following features. First, property assembly is viewed as continuous event in which the set of units to be assembled is publicly offered at a certain price. The *set* of buyers, as in Cai (2003), Mailath and Postlewaite (1990), or Kominers and Weyl (2011), is replaced by a *flow* of randomly arriving buyers.

Second, to avoid continuous taxation of high-value sellers, we will require that no money changes hands unless the property assembly transaction takes place. This will ensure that a seller' low income will not force them to sell.

Third, to ensure transparency of the transaction, the mechanism will be ex-post budget-balanced: the amount paid by a buyer exactly equals the amount received by the sellers.

With these assumptions, the process of property assembly is akin to the *buy-it-now* sale, i.e. publicly offering the assembled unit of real estate at a certain price and waiting for a buyer willing to pay that price. The role of the government is to optimally aggregate signals of the sellers into the sale price of the assembled property, to determine how the sale revenues are divided between the unit sellers, and, once a buyer has been found, to force the transaction if necessary.

Because property assembly is modeled as a continuous process, a common assumption that sellers choose their strategies simultaneously and forever (as in Mailath and Postlewaite (1990)) is not realistic either. In a dynamic environment, people can naturally change their strategy, and restricting them in doing so does not serve any useful purpose. I will assume that sellers can update their strategies at any time, and therefore their strategy is based on actually observed, rather than expected, strategies of others. That allows me to replace the complex concept of Bayesian-Nash equilibrium, as in Mailath and Postlewaite (1990), by the simple concept of Nash equilibrium.

In this environment, coercion by the state is manifested in the fact that the sellers' compensation in case of property assembly may be lower than their declared value. Because the buyers' identity is not known before they participate in property assembly, no coercion can be applied to them – otherwise they would not identify themselves as buyers.

Because the coercion is applied only to the sellers, it seems unethical to target buyers' welfare as a goal of such coercion. I thus assume that the objective of the authority applying coercion is the aggregate welfare of all sellers. The most desired outcome is thus equivalent to the outcome of a bargaining problem in which all sellers act as a coalition, and the buyers have zero bargaining power.

# 2 The model and preliminary results

## 2.1 Description

### 2.1.1 The sellers

Consider an infinite-horizon continuous time environment. There are $n$ units of property, dispersed between $n$ different sellers. During each time period of duration $\mathrm{d}t$, the seller of unit $i$ receives a privately known dollar-valued utility $v_i \mathrm{d}t$ from her property, where the instantaneous value $v_i$ is randomly drawn from a distribution. The private values $v_i$ are independent from each other and are time-invariant. The sellers' time preference is given by the discount rate $r > 0$, that is, $e^{-r\mathrm{d}t}$ dollars today are equally preferred to one dollar $\mathrm{d}t$ moments of time later. Sellers are risk neutral and maximize the discounted stream of utility.

The analysis below does not require full knowledge of distributions from which $v_i$ were drawn, as obtaining empirical estimates of these distributions might be problematic. In particular, the data from sales of individual units of property is of limited value here, because property assembly typically involves sellers who do *not* want to sell their unit at market prices.

Nevertheless, some assumptions about the distributions of sellers' values have to be made. We assume that the minimum value $v_i^0$ for seller $i$ is such that the lowest possible NPV from keeping the object forever, $\frac{v_i^0}{r}$, is equal to the commonly known market value of the object. The justification for such assumption is that owners keep their units (rather than sell them) if and only if their value is above this threshold. Some elements of the model also make use of mean seller valuations, denoted $m_i \equiv Ev_i$. The variances $\sigma_i^2 = E(v_i - m_i)^2$ are also assumed to be bounded away from zero and finite, although their specific values are of little importance.

Because the holdout problem becomes more pronounced as the number of sellers increases, this paper pays special attention to asymptotic properties of the mechanisms analyzed. For that, a proper definition of asymptotics has to be introduced. We will consider infinite sequences of sellers, with values randomly drawn from a sequence of distributions. We assume that the distributions are of the same order of magnitude. Specifically, the sequences of above mentioned moments, $v_i^0, m_i, \sigma_i^2$ are each bounded from below and from above, and the averages of the mean and variance, $\frac{\sum_{i=1...n} m_i}{n}$ and $\frac{\sum_{i=1...n} \sigma_i^2}{n}$, each converge to a limit denoted $m$ and $\sigma^2$, respectively. The lower bound of $\sigma_i^2$ is strictly positive.

A statistic of special interest is the average sellers' valuation, $\bar{v} = \frac{\sum_{i=1...n} v_i}{n}$. Its mean is $\frac{\sum_{i=1...n} m_i}{n}$, converging to $m$, and its variance is $\frac{\sum_{i=1...n} \sigma_i^2}{n^2}$, converging to zero at the rate of $n$.

### 2.1.2 The mechanism

The government requires all sellers to publicly announce the value $\tilde{v}_i$ of their unit on a perpetual basis. This value can be updated at any time; thus, we assume that every seller $i$ chooses her $\tilde{v}_i$ knowing the exact values $\tilde{\mathbf{v}}_{-i} \equiv \{\tilde{v}_1, \ldots, \tilde{v}_{i-1}, \tilde{v}_{i+1}, \ldots, \tilde{v}_n\}$ posted by other sellers.

Once the values are revealed, the government declares the amount of compensation $x_i(\tilde{\mathbf{v}})$ payable to seller $i$ in case of property assembly, where $\tilde{\mathbf{v}}$ is the vector of all (revealed) valuations. In this paper, I only consider ex-post budget balanced mechanisms, meaning that the buyers' price $P$ is equal to the sum of sellers' compensations, for every possible vector of reported values:[2]

$$P(\tilde{\mathbf{v}}) = \sum_i x_i(\tilde{\mathbf{v}}). \tag{1}$$

A mechanism is then a set of seller compensations, as functions of $\tilde{\mathbf{v}}$: $\mathbf{x}(\cdot) = \{x_1(\cdot), \ldots, x_n(\cdot)\}$.

### 2.1.3 The buyers

There is an infinite universe of potential buyers. The arrival of buyers is a Poisson process, such that the average time interval between two buyers is unity. Each buyer demands the entire set of $n$ units, any subset has zero value to them. A buyer's value of the set is drawn from a distribution with

2. There is a weaker notion of *ex-ante* budget balance, meaning that (1) is true only in expectation. In math, assuming that the values are revealed truthfully $\tilde{\mathbf{v}} = \mathbf{v}$, $E_{\mathbf{v}} P(\mathbf{v}) = E_{\mathbf{v}} \sum_i x_i(\mathbf{v})$. In the latter type of mechanisms, the ex-post budget imbalances $P(\mathbf{v}) - \sum_i x_i(\mathbf{v})$ must be cleared by an insurance agency; the ex-ante budget balance implies its zero expected profit. In practice, such agency may be difficult to create, especially if there is lack of consensus about the distributions of seller valuations.

c.d.f. $F_n$. We assume that the c.d.f. function is smooth. The distribution is assumed to have a finite mean, for a given $n$, and to satisfy the following regularity condition:

**Assumption 1** *The quantity $z - \frac{1-F_n(z)}{F_n'(z)}$ is strictly increasing for all $z$ such that $F_n(z) < 1$.*

Such assumption is common in the mechanism design literature, e.g. in Myerson and Satterthwaite (1983). The above assumptions also imply that the distribution of values is atomless and has a connected support.

The buyers observe the price defined by (1) and purchase the set of all units if their own value exceeds it. We assume that the buyers do not attempt to bargain, which is quite plausible if the number of sellers $n$ is large. Thus, the probability of a successful transaction with a given buyer is $1 - F_n(P)$. If the transaction fails, the sellers wait until the next buyer arrives.

No one is forced to make any transfers until the purchase takes place.

The expected welfare of seller $i$, as a function of the vector of revealed valuations $\tilde{\mathbf{v}}$, can then be described as

$$w_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i) =$$

$$= \lim_{\mathrm{d}t \to 0} \mathrm{d}t\,(1 - F_n(P(\tilde{\mathbf{v}})))\,x_i(\tilde{\mathbf{v}})\,(1 - \mathrm{d}t(1 - F_n(P(\tilde{\mathbf{v}}))))\,\big(v_i \mathrm{d}t + e^{-r\mathrm{d}t} w_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i)\big)$$

$$= \lim_{\mathrm{d}t \to 0} \frac{\mathrm{d}t\,(1 - F_n(P(\tilde{\mathbf{v}})))\,x_i(\tilde{\mathbf{v}}) + (1 - \mathrm{d}t(1 - F_n(P(\tilde{\mathbf{v}}))))\,v_i \mathrm{d}t}{1 - (1 - \mathrm{d}t(1 - F_n(P(\tilde{\mathbf{v}}))))\,e^{-r\mathrm{d}t}}$$

$$= \frac{(1 - F_n(P(\tilde{\mathbf{v}})))x_i(\tilde{\mathbf{v}}) + v_i}{r + 1 - F_n(P(\tilde{\mathbf{v}}))}$$

It is also useful to define the *gains from trade* as the additional expected welfare relative to that of no-property-assembly scenario:

$$u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i) \equiv w_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i) - \frac{v_i}{r} = R_n(P(\tilde{\mathbf{v}})) \left( x_i(\tilde{\mathbf{v}}) - \frac{v_i}{r} \right), \qquad (2)$$

where $R_n(P) \equiv \frac{1-F_n(P)}{r+1-F_n(P)}$.

The *aggregate gains from trade* of all sellers can be defined as follows:

$$U(P(\cdot), \tilde{\mathbf{v}}, V) \equiv \sum_i u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i) = R_n(P(\tilde{\mathbf{v}})) \left( P(\tilde{\mathbf{v}}) - \frac{V}{r} \right), \qquad (3)$$

where $V \equiv \sum_i v_i$ is the sum of sellers' valuations.

For asymptotic analysis, we need to define how the distributions of buyer valuations compare between settings with different numbers of sellers. Denote by $\bar{v} \equiv \frac{V}{n}$ the average property value for the sellers. Likewise, denote by $p_n \equiv \frac{P}{n}$ the sale price per seller, in a setting with $n$ sellers. We assume that, in two settings with different numbers of sellers but identical sale price per seller, the proportion of buyers willing to pay that price is the same: $F_n(np) = F_1(p), \forall n, p$. In other words, doubling the number of sellers also means that the object being sold becomes, on average, twice as valuable for the buyers.[3] As a corollary, the derivatives of $F_n$ are related as follows: $F_n^{(k)}(np) = \frac{1}{n^k} F_1^{(k)}(p)$, where $F_n^{(k)}(\cdot)$ is the $k$-th derivative of $F_n(\cdot)$. We also have $R_n(np) \equiv \frac{1-F_n(np)}{r+1-F_n(np)} = R_1(p)$, and $R_n^{(k)}(np) = \frac{1}{n^k} R_1^{(k)}(p)$.

3. Mailath and Postlewaite (1990) in their Lemma 2 similarly assume that the cost of a public project is proportional to the number of participants.

## 2.2 The first best

The government's first best mechanism is the one that maximizes the sellers' gains from trade, given true valuations $\mathbf{v}$. If the government could observe $\mathbf{v}$, maximization of (3) would amount to maximization of $R_n(P)\left(P - \frac{V}{r}\right)$ over $P$, with the following first-order condition of the first-best price $P_n^a$ (assuming an interior solution):

$$R_n'(P_n^a)\left(P_n^a - \frac{V}{r}\right) + R_n(P_n^a) = 0, \tag{4}$$

which defines an implicit function $P_n^a(V)$.

**Proposition 1** *Any price $P_n^a$ that satisfies (4) delivers the global maximum of the sellers' gains from trade.*

The proof is provided in the Appendix.

We also have that $P_n^a(V)$ is uniquely defined, and that $\frac{\partial P_n^a(V)}{\partial V} > 0$.

How does $P_n^a(\cdot)$ change with $n$? Define by $p_n^a(v)$ the average first-best price as a function of the average value $\bar{v} \equiv \frac{V}{n}$, $p_n^a(\bar{v}) \equiv \frac{P_n^a(n\bar{v})}{n}$. We can rewrite (4) for a setting with $n$ sellers as

$$\begin{aligned} &R_n'(np_n^a(\bar{v}))\left(np_n^a(\bar{v}) - \frac{n\bar{v}}{r}\right) + R_n(np_n^a(\bar{v})) = \\ &= \frac{1}{n}R_1'(p_1^a(\bar{v}))\left(np_1^a(\bar{v}) - \frac{n\bar{v}}{r}\right) + R_1(p_1^a(\bar{v})) = 0, \end{aligned}$$

The above equation uniquely defines $p_n^a$ as a function of $\bar{v}$. Varying $n$ does not affect $p_n^a$, thus, the optimal price per seller depends on the average valuation $\bar{v}$, but not on the number of sellers: $p_n^a(\bar{v}) = p_1^a(\bar{v})$. In other words, the first-best aggregate price $P_n^a$ increases proportionally with $n$ as long as the average sellers' valuation $\bar{v}$ remains the same. In the analysis that follows, we drop the subscript $n$ when referring to the average first-best price $p^a(\bar{v})$.

## 2.3 Unobserved types and asymmetric information

If seller types are their private information, the vector of payments to sellers $\mathbf{x}$ must rely on the types reported by sellers $\tilde{\mathbf{v}}$. The incentive compatibility then implies that each seller $i$'s utility is maximized with respect to $\tilde{v}_i$ at $v_i$, with the following first-order condition:

$$\frac{\partial u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i)}{\partial \tilde{v}_i} = 0,$$

which implicitly defines $\tilde{\mathbf{v}}$ as a function of $\mathbf{v}$. This condition, in turn, implies that

$$\frac{\mathrm{d}u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i)}{\mathrm{d}v_i} = \frac{\partial u_i}{\partial \tilde{v}_i}\frac{\partial \tilde{v}_i}{\partial v_i} + \frac{\partial u_i}{\partial v_i} = \frac{\partial u_i}{\partial v_i} = -\frac{R_n(P(\tilde{\mathbf{v}}))}{r}, \forall \mathbf{v}. \tag{5}$$

Without loss of generality, we can focus on direct mechanisms in which the vector of reported values $\tilde{\mathbf{v}}$ always equals the vector of true values $\mathbf{v}$. Because such equality must hold for any realization of others' reported values $\tilde{\mathbf{v}}_{-i}$, any direct mechanism is also a dominant-strategy mechanism. In such mechanism, by integrating (5) over $v_i$, we can present the gains from trade as follows (cf.(2)):

$$R_n(P(\mathbf{v}))\left(x_i(\mathbf{v}) - \frac{v_i}{r}\right) \equiv u_i \equiv u_i^0(\mathbf{v}_{-i}) - \frac{H_i(\mathbf{v})}{r}, \tag{6}$$

where

$$H_i(\mathbf{v}) \equiv \int_{v_i^0}^{v_i} R_n(P(\{z, \mathbf{v}_{-i}\}))\mathrm{d}z \tag{7}$$

and where $u_i^0(\mathbf{v}_{-i})$ is the gains from trade of seller $i$ when his own value is equal to minimum possible, while the values of others are given by $\mathbf{v}_{-i}$.

## 2.4 Optimal non-coercive mechanism

We now investigate what can be achieved by a mechanism that respects the participation constraints, i.e. allows sellers to opt out of the property assembly process. A seller would stay out iff her gains from trade are negative; to keep the sellers in, such mechanism would then have to satisfy $u_i(\mathbf{x}(\cdot), \mathbf{v}, v_i) = u_i^0(\mathbf{v}_{-i}) - \frac{H_i(\mathbf{v})}{r} \geq 0, \forall i, \mathbf{v}$. Because $H_i(\mathbf{v})$ increases with $v_i$ from zero to a positive value, the latter constraint is equivalent to

$$u_i^0(\mathbf{v}_{-i}) - \frac{H_i(\{\infty, \mathbf{v}_{-i}\})}{r} \geq 0, \forall i, \mathbf{v}_{-i}. \tag{8}$$

Solving for $u_i^0(\mathbf{v}_{-i})$ in (6), substituting it into (8), and aggregating across all sellers, we obtain

$$R_n(P(\mathbf{v}))\left(P(\mathbf{v}) - \frac{V}{r}\right) + \sum_i \frac{H_i(\mathbf{v}) - H_i(\{\infty, \mathbf{v}_{-i}\})}{r} \geq 0. \tag{9}$$

We now turn to the second-best non-coercive mechanism. It is characterized by maximizing the aggregate gains from trade (3), averaged over the (generally unknown) distribution of seller values, subject to the constraint (9). Note that both (3) and (9) depend on the sum of values $V$, but not on individual values $\mathbf{v}$, thus we can assume that the second-best aggregate price is a function of aggregate value $V$ only. We denote such price, in a setting with $n$ sellers, by $P_n^b(V)$.

Because of that, the function $H_i(\mathbf{v})$ in such mechanism can be redenoted from (7), for a setting with $n$ sellers, as $H_i(\mathbf{v}) = H_n^b(V) - H_n^b(\sum_{j\neq i} v_j + v_i^0)$, where

$$H_n^b(V) \equiv \int_0^V R_n(P_n^b(V))\mathrm{d}V. \tag{10}$$

With new notations, (9) divided by $n$ becomes

$$R_n(np_n^b(\bar{v}))\left(p_n^b(\bar{v}) - \frac{\bar{v}}{r}\right) + \frac{H_n^b(n\bar{v}) - H_n^b(\infty)}{r} \geq 0, \tag{11}$$

where $p_n^b(\bar{v}) \equiv \frac{P_n^b(n\bar{v})}{n}$.

We now repeat the impossibility result of Mailath and Postlewaite (1990) in the setting of this paper.

**Proposition 2** *For all but a countable number of realizations of $\bar{v}$, the probability of transaction converges to zero with $n$.*

**Proof.** Suppose the contrary, that there exists a subset $\mathcal{V} \subseteq R_+$ satisfying $|\mathcal{V}| \equiv \int_{v\in\mathcal{V}} \mathrm{d}v > 0$ and such that, for every $\bar{v} \in \mathcal{V}$ there exists an infinite increasing sequence $n_1, \ldots, n_k, \ldots$ with bounded away from zero probability of transaction. Then, there exists $\epsilon > 0$ such that $R_{n_k}(n_k p_{n_k}^b(\bar{v})) > \epsilon, \forall k, \forall \bar{v} \in \mathcal{V}$. By integrating the latter inequality over all $\bar{v} \in \mathcal{V}$ and recalling (10), we obtain $H_{n_k}^b(\infty) - H_{n_k}^b(n_k v_0) \geq \int_{v\in\mathcal{V}} R_{n_k}(n_k p_{n_k}^b(v))\mathrm{d}n_k v \geq n_k \epsilon |\mathcal{V}|$, where $v_0 = \inf \mathcal{V}$. Thus, the component $\frac{H_{n_k}^b(n_k v_0) - H_{n_k}^b(\infty)}{r}$ in (11) is smaller than $-n_k \frac{\epsilon}{r}|\mathcal{V}|$ which converges to negative infinity with $k$.

At the same time, the first component of (11) is

$$R_{n_k}(n_k p^b_{n_k}(\bar{v}))\left(p^b_{n_k}(\bar{v})-\frac{\bar{v}}{r}\right)=R_1(p^b_{n_k}(\bar{v}))\left(p^b_{n_k}(\bar{v})-\frac{\bar{v}}{r}\right)<$$
$$<\frac{1-F_1(p^b_{n_k}(\bar{v}))}{r}p^b_{n_k}(\bar{v}). \tag{12}$$

Because the buyer's value is assumed to have a mean, $1-F_1(x)$ must approach zero at a rate faster than $n$, meaning that $(1-F_1(x))x$ decreases for large enough $x$, and therefore has a finite upper bound. Thus, the first (positive) component of (11) is bounded from above, while the second goes to negative infinity, violating the inequality for all sufficiently large $n_k$. ■

# 3 Coercive mechanisms: preliminaries

## 3.1 Description

The above discussion implies that coercion is necessary for optimality. Coercion is the removal of participation constraint (8), resulting in negative gains from trade for some sellers. While many experts agree that *some* coercion is necessary in property assembly, there is also a consensus that *too much* coercion is not acceptable. For example, the Fifth Amendment to the U.S. constitution, the most famous piece of legislation regarding property takings, states "[No] private property [shall] be taken for public use, without just compensation." In legal practice, the "just compensation" is usually the market value of the confiscated unit. Compensations paid in other countries are usually calculated in the same way.

In the current model, the assumed market value of unit $i$ is $\frac{v_i^0}{r}$. To make a coercive mechanism consistent with existing legislation, it is then necessary that

$$x_i(\mathbf{v}) \geq \frac{v_i^0}{r}, \forall i, \forall \mathbf{v}. \tag{13}$$

Throughout the paper, we refer to (13) as the *constitutional constraint*, which is a weaker substitute of the participation constraint (8).

I also include some notion of equity between sellers. To make sure that a large proportion of gains from trade is not regularly channeled to one or few sellers, I assume there exists an upper bound $\bar{x}$ on the expected payment to each seller $i$, given realizations of the values of others:

$$E_{v_i} x_i(\mathbf{v}) \leq \bar{x}. \tag{14}$$

This assumption is justified by the fact that the expected value $m_i$ of seller $i$ is bounded from above. Also note that, even if this restriction was not imposed, only a vanishing fraction of sellers could afford an unbounded compensation per seller, because the expected aggregate gains from trade are always of the same order of magnitude as the expected aggregate value.

## 3.2 Fixed compensation mechanism

It is easy to find a coercive mechanism that meets the above specified constraints and delivers relatively high levels of welfare. For example, consider a *fixed compensation mechanism* (FCM) which completely ignores sellers' inputs and offers them compensations $\{x_1, \dots, x_n\}$ that respect (13) and (14).

The sellers' joint gains from trade are then $R(P)\left(P - \frac{V}{r}\right)$, where $P = \sum_i x_i$. Maximizing the expected joint gains from trade yields the optimal per-seller price $p^c$ such that (cf.(4))

$$R_n'(np_n^c)\left(np_n^c - \frac{n\bar{m}_n}{r}\right) + R_n(np_n^c) = 0, \tag{15}$$

where $\bar{m}_n \equiv \frac{\sum_{i=1}^n m_i}{n}$ is the expectation of $\bar{v}$ over sellers' valuations. By assumptions of the paper, it converges to $m$.

How good is this mechanism? Define the *loss* function as the difference between the first-best gains from trade per seller and the one delivered by the mechanism:

$$L_n^c(\bar{v}) \equiv R_n(np^a(\bar{v}))\left(p^a(\bar{v}) - \frac{\bar{v}}{r}\right) - R_n(np_n^c)\left(p_n^c - \frac{\bar{v}}{r}\right).$$

The loss function satisfies $L_n^c(\bar{m}_n) = 0$, $L_n^c(\bar{v}) \geq 0$, $L_n^c{}'(\bar{v}) = -\frac{R_n(np^a(\bar{v})) - R_n(np_n^c)}{r}$ increasing in $v$, and $L_n^c{}'(\bar{m}_n) = 0$.

Although the asymptotic loss for a given $\bar{v}$ is generally not zero, the *ex-ante* loss (i.e. its expectation over $\bar{v}$) is, because the difference $\bar{v} - \bar{m}_n$ converges in probability to zero by the law of large numbers. In other words, even a mechanism as simple as fixed compensation is asymptotically efficient, provided that the aggregate compensation was chosen correctly.

We now analyze how fast the expected loss converges to zero. We approximate $L_n^c(\bar{v})$ by the second-order Taylor expansion around $\bar{m}_n$: $L_n^c(\bar{v}) =$

$\frac{1}{2}L^{c\prime\prime}(\bar{m}_n)(\bar{v}-\bar{m}_n)^2+o((\bar{v}-\bar{m}_n)^2)$. Then,

$$E_{\bar{v}}L^c(\bar{v}) \approx \frac{1}{2}L^{c\prime\prime}(\bar{m}_n)E_{\bar{v}}(\bar{v}-\bar{m}_n)^2 = \frac{1}{2}L^{c\prime\prime}(\bar{m}_n)\frac{\sum_{i=1}^n \sigma_i^2}{n^2},$$

which converges to zero at the rate of $n$. In simple words, as the number of sellers doubles, the magnitude of the ex-ante aggregate loss remains unchanged, so that the ex-ante loss per seller is cut in half.

## 3.3 Own-compensation mechanism

We now consider a mechanism in which the compensation $x_i$ to seller $i$ may depend on her own reported type $\tilde{v}_i$, as well as on compensations to other sellers $x_j, j \neq i$, which $i$ takes as given, but not directly on the types of other sellers. Then, a seller can manipulate his own compensation, but cannot directly manipulate that of others.

Because a seller $i$ can influence the probability of transaction only through own compensation, the incentive compatibility constraint is as follows:

$$\frac{\partial u_i(\mathbf{x}(\cdot),\tilde{\mathbf{v}},v_i)}{\partial \tilde{v}_i} = \left(R_n'(P)\left(x_i - \frac{v_i}{r}\right) + R_n(P)\right)\frac{\partial x_i}{\partial \tilde{v}_i} = 0.$$

Focusing on direct mechanisms where $\tilde{\mathbf{v}} = \mathbf{v}$, the latter equality can be true only if at least one of the following is true:

$$R_n'(P)\left(x_i - \frac{v_i}{r}\right) + R_n(P) = 0, \tag{16}$$

meaning that seller's $i$ utility (conditional on choices of others) is maximized, or $\frac{\partial x_i}{\partial v_i} = 0$, meaning that the compensation is (locally) constant.

Denote by $x_i^*(v_i, \mathbf{x}_{-i})$ the compensation determined by (16). It is strictly increasing in $v_i$. Seller $i$ wants his compensation $x_i$ to be as close as possible to $x_i^*(\cdot)$. Other sellers want $x_i$ to be as low as possible if their own gains from trade are positive (to increase the transaction probability through lower aggregate price), and as high as possible if their own gains from trade are negative.

**Proposition 3** *For each seller $i$ and all realizations of value $\mathbf{v}$, the optimal compensation $x_i(v_i, \mathbf{x}_{-i})$ never exceeds $x_i^*(v_i, \mathbf{x}_{-i})$.*

In other words, coercion can be manifested in lowering the compensation paid to a seller, but never in increasing it. The proof is in the Appendix.

Therefore, if a compensation to seller $i$ is not equal to $x_i^*(v_i, \mathbf{x}_{-i})$, it must be lower than that and not dependent on $v_i$. Denote such compensation $\hat{x}_i(\mathbf{x}_{-i})$; this is the maximum that seller $i$ can get given $\mathbf{x}_{-i}$.

Such compensation scheme can presented in a very simple and intuitive way: every seller $i$ is offered a maximum compensation $\hat{x}_i(\mathbf{x}_{-i})$ in case of property assembly transaction, but is allowed to ask less than that in order to increase the chance of transaction. Sellers with sufficiently low value $v_i$, such that $x_i^*(v_i, \mathbf{x}_{-i}) < \hat{x}_i(\mathbf{x}_{-i})$ will then take advantage of the opportunity.

**Proposition 4** *For any sequence of mechanisms with increasing number of sellers $n$, with per-seller compensations $p_n$ converging to $p$ such that $R_1(p) > 0$, we have that the compensation to each seller $i$ equals $\hat{x}_i(\mathbf{x}_{-i})$ when $n$ is large enough.*

In other words, as the number of sellers increases, to achieve a non-vanishing probability of transaction, all sellers earn a fixed compensation that does not depend on their (reported) value. Thus, an own-compensation mechanism with non-vanishing gains from trade is identical to a fixed-compensation mechanism when the number of sellers is large enough.

**Proof.** Suppose the contrary, that for each element in an infinite sequence $n_1, \ldots, n_k, \ldots$ there exists a seller $i$ whose compensation is not equal to $\hat{x}_i(\mathbf{x}_{-i})$, meaning that it must be equal to $x_i^*(v_i, \mathbf{x}_{-i})$ as defined by (16). By assumption, $R_{n_k}(n_k p_{n_k})$ is not zero for large $k$, and so is its derivative, so we can rewrite (16) for a seller $i$ with the lowest possible value $v_i^0$ as follows:

$$x_i^*(v_i^0, \mathbf{x}_{-i}) = \frac{R_{n_k}(n_k p_{n_k})}{-R'_{n_k}(n_k p_{n_k})} + \frac{v_i^0}{r} = n_k \frac{R_1(p_{n_k})}{-R'_1(p_{n_k})} + \frac{v_i^0}{r} \to_{k\to\infty} \infty.$$

Therefore, (14) is violated for all sufficiently large $k$. Because the compensation to seller $i$ is non-decreasing in $v_i$, (14) is also violated for all $v_i$, for all sufficiently large $k$. ■

## 3.4 Second-best coercive mechanism

What are the maximum gains from trade that can be achieved, accounting for the private nature of seller types and for the constraints of Section 3.1? In principle, for any given realization $\mathbf{v}$ of seller types, it is possible to specify a mechanism that delivers the first-best. But then such mechanism would have to deviate from the first-best at other realizations of $\mathbf{v}$, in order to meet the incentive compatibility constraint (6) and the constitutional constraint (13). To achieve the "second-best", a mechanism would then have to weigh welfare losses at different realizations of $\mathbf{v}$, with weights derived from the probability distribution of seller values. However, using such distributions is against the commitment of this paper to prior-free mechanisms due to unobservable nature of seller type distributions.

Moreover, even if the seller type distributions were known, there exists a computational difficulty of assessing the optimal mechanism. Note that the individual rationality constraint (6,7) is essentially a partial differential equation (PDE). Therefore, the problem of finding a second-best mechanism is a problem of optimization with PDE constraints. The number of dimensions in that problem is equal to the number of sellers, i.e. is infinite if one attempts to characterize the asymptotic properties of the mechanism. Optimization with PDE constraints, even with a finite number of dimensions, is notorious for its difficulty, with few analytical results that can be derived.

For these two reasons, we avoid taking the path of calculating the "second-best" mechanism. Instead, the next section characterizes what I call a *optimal feasible mechanism* that is shown to be relatively easy to compute, and which is much closer in welfare terms to the first-best than the fixed-compensation mechanism.

# 4 Optimal feasible mechanism

The philosophy of the optimal feasible mechanism (OFM) is simply to seek a computationally feasible approximate solution to the system (4,6,13). The traditional approach in the mechanism design literature is to meet the constraint (6) exactly, making the value reports truthful, at the expense of the deviation from the first-best (4). But in the setting of this paper, as the previous section emphasizes, that would amount to solving an $n$-dimensional system of partial differential equations, which we seek to avoid. Instead, we take an alternative route: for each reported vector of values $\tilde{v}$, we will seek to meet (4) exactly (except in cases where (13) is binding) while the incentive compatibility constraint (6) will be approximated. The fact that (6) does not hold exactly means that the value reports are not exactly truthful. Because of that, (4) will maximize welfare for the wrong vector of seller values, which leads to welfare losses. The magnitude of these welfare losses is assessed in Section 4.1.3.

## 4.1 Approximating the IC constraint

Suppose the mechanism is such that the buyer's price is first-best, $P_n^a(V)$. Then, the function $H_i(\mathbf{v})$ in (6) can be redefined as follows:

$$H_i(\mathbf{v}) = H_n^a(V) - H_n^a(v_i^0 + \sum_{j \neq i} v_j),$$

where $H_n^a(V) \equiv \int_0^V R_n(P_n^a(V))$. With new notations, (6) becomes

$$R_n(P_n^a(V)) \left( x_i(\mathbf{v}) - \frac{v_i}{r} \right) \equiv u_i^0(\mathbf{v}_{-i}) - \frac{H_n^a(V) - H_n^a(V + v_i^0 - v_i)}{r}. \quad (17)$$

Aggregate (17) across all sellers and rearrange to obtain

$$\sum_i u_i^0(\mathbf{v}_{-i}) \equiv R_n(P_n^a(V)) \left( P_n^a(V) - \frac{V}{r} \right) + n \frac{H_n^a(V)}{r} - \sum_i \frac{H_n^a(V + v_i^0 - v_i)}{r}. \quad (18)$$

Take a cross-derivative of both sides of (18) with respect to $v_1, \ldots, v_n$. The left-hand side is then necessarily zero, because each element of the left-hand side depends on all $v_1, \ldots, v_n$ except one. At the same time, the right-hand side is generally non-zero, making the first-best allocation for all realizations of $\mathbf{v}$ generally impossible.

Denote the right-hand side of (18) by $G_n(\mathbf{v})$; our goal is to approximate it by a sum of $n$ functions, such that function $i$ depends on all elements of $\mathbf{v}$ except $v_i$. Different orders $k = 1 \ldots n$ of approximation are available. For each setting with $n$ sellers, define the *weight* of seller $i$ as the mean value of that seller, relative to the sum of means: $s_{i,n} \equiv \frac{m_i}{\sum_j m_j}$. As the number of sellers increases, the weight of each seller decreases to zero at the rate of $n$.

Throughout this section, we will assume that the constitutional constraint (13) is not binding for all sellers, i.e. each of the seller values is below a certain threshold. The solution is generalized in section 4.2.

#### 4.1.1 The approximation: definition

The first-order approximation of $u_i^0(\mathbf{v}_{-i})$ can be introduced as follows (dropping the subscript $n$):

$$\tilde{u}_i^{[1]}(\mathbf{v}) = u_i^{[1]}(\mathbf{v}_{-i}) + \eta_i^{[1]}(\mathbf{v}),$$

where $u_i^{[1]}(\mathbf{v}_{-i}) \equiv s_i G(\{m_i, \mathbf{v}_{-i}\})$ and $\eta_i^{[1]}(\mathbf{v}) \equiv s_i \epsilon_i^{[1]}(\mathbf{v})$ such that $\epsilon_i^{[1]}(\mathbf{v}) \equiv G(\mathbf{v}) - G(\{m_i, \mathbf{v}_{-i}\})$. It is trivial to verify that $\sum_i \tilde{u}_i^{[1]}(\mathbf{v}) = G(\mathbf{v})$.

This approximation is not perfect because $u_i^0$ is meant to be independent of $v_i$, while $\tilde{u}_i^{[1]}$ indeed depends on it via $\epsilon_i^{[1]}$. A better second-order approximation can be achieved by approximating the latter for each seller $i$ by a sum of $n-1$ functions,

$$\epsilon_i^{[1]}(\mathbf{v}) = \sum_{i_2 \neq i} \frac{s_{i_2}}{1-s_i} \left[\epsilon_i^{[1]}(\{m_{i_2}, \mathbf{v}_{-i_2}\}) + \epsilon_{i,i_2}^{[2]}(\mathbf{v})\right], \tag{19}$$

and by attributing each of these functions to a seller other than $i$. The second-order approximation of $u_i^0(\mathbf{v}_{-i})$ is then $\tilde{u}_i^{[2]}(\mathbf{v}) = u_i^{[2]}(\mathbf{v}_{-i}) + \eta_i^{[2]}(\mathbf{v})$ such that

$$u_i^{[2]}(\mathbf{v}_{-i}) \equiv u_i^{[1]}(\mathbf{v}_{-i}) + s_i \sum_{i_2 \neq i} \frac{s_{i_2}}{1-s_{i_2}} \epsilon_{i_2}^{[1]}(\{m_i, \mathbf{v}_{-i}\})$$

and $\eta_i^{[2]}(\mathbf{v}) \equiv s_i \sum_{i_2 \neq i} \frac{s_{i_2}}{1-s_i} \epsilon_{i,i_2}^{[2]}(\mathbf{v})$. The latter is defined by (cf.(19)) $\epsilon_{i_1,i_2}^{[2]}(\mathbf{v}) \equiv \epsilon_{i_1}^{[1]}(\mathbf{v}) - \epsilon_{i_1}^{[1]}(\{m_{i_2}, \mathbf{v}_{-i_2}\})$. It is, again, straightforward to verify that $\sum_i \tilde{u}_i^{[2]}(\mathbf{v}) = G(\mathbf{v})$.

By induction, the $k$-th order approximation of $u_i^0(\mathbf{v}_{-i})$ is

$$\tilde{u}_i^{[k]}(\mathbf{v}) = u_i^{[k]}(\mathbf{v}_{-i}) + \eta_i^{[k]}(\mathbf{v}) \tag{20}$$

such that

$$\begin{aligned} u_i^{[k]}(\mathbf{v}_{-i}) \equiv\ & u_i^{[k-1]}(\mathbf{v}_{-i}) + \\ & + s_i \sum_{i_2 \neq i} \frac{s_{i_2}}{1-s_{i_2}} \cdots \sum_{i_k \neq i, i_2, \ldots, i_{k-1}} \frac{s_{i_k}}{1-\sum_{l=2\ldots k} s_{i_l}} \epsilon_{i_2,\ldots,i_k}^{[k-1]}(\{m_i, \mathbf{v}_{-i}\}) \end{aligned}$$

and

$$\eta_i^{[k]}(\mathbf{v}) = s_i \sum_{i_2 \neq i} \frac{s_{i_2}}{1 - s_i} \cdots \sum_{i_k \neq i, i_2, \ldots, i_{k-1}} \frac{s_{i_k}}{1 - s_i - \sum_{l=2\ldots k-1} s_{i_l}} \epsilon_{i, i_2, \ldots, i_k}^{[k]}(\mathbf{v}). \tag{21}$$

The $k$-th order error is given by

$$\epsilon_{i_1, \ldots, i_k}^{[k]}(\mathbf{v}) = \epsilon_{i_1, \ldots, i_{k-1}}^{[k-1]}(\mathbf{v}) - \epsilon_{i_1, \ldots, i_{k-1}}^{[k-1]}(\{m_{i_k}, \mathbf{v}_{-i_k}\}).$$

Because in the above approximation sequence each $i_1, \ldots, i_k$ must be unique, the highest-possible order of approximation is $n$.

#### 4.1.2 The approximation: asymptotic properties

We now assess the asymptotic properties, as the number of sellers $n$ goes to infinity, of the approximation error $\eta_i^{[k]}$ of arbitrary order $k$. First of all, we study the function $G_n(\mathbf{v})$ and its components. Observe that

$$H_n^a(n\bar{v}) \equiv n H_1^a(\bar{v}), \forall \bar{v}. \tag{22}$$

For proof, first observe that $H_n^a(0) = 0, \forall n$. Next, differentiate both sides of (22) w.r.t. $\bar{v}$ to observe identity of the derivatives:

$$\frac{\partial H_n^a(n\bar{v})}{\partial \bar{v}} = n \frac{\partial H_n^a(n\bar{v})}{\partial n\bar{v}} = n R_n(n p^a(\bar{v})) = n R_1(p^a(\bar{v})) = n \frac{\partial H_1^a(\bar{v})}{\partial \bar{v}}, \forall \bar{v}.$$

By Taylor-expanding (22), we have

$$H_n^a(V + v_i^0 - v_i) = nH_1^a(\bar{v} - \frac{1}{n}(v_i - v_i^0))$$
$$= nH_1^a(\bar{v}) - nR_1(p^a(\bar{v}))\frac{1}{n}(v_i - v_i^0) + n\sum_{l=2\ldots\infty}\frac{1}{l!}R_1^{(l-1)}(p^a(\bar{v}))\left(-\frac{v_i - v_i^0}{n}\right)^l. \tag{23}$$

Using this information, we can rewrite $G_n(\mathbf{v})$ as follows:

$$\begin{aligned} G_n(\mathbf{v}) &= nR_1(p^a(\bar{v}))\left(p^a(\bar{v}) - \frac{\sum_i v_i^0}{nr}\right) \\ &- \frac{n}{r}\sum_{l=2\ldots\infty}\frac{1}{l!}R_1^{(l-1)}(p^a(\bar{v}))\sum_{i=1}^{n}\left(-\frac{v_i - v_i^0}{n}\right)^l. \end{aligned} \tag{24}$$

The first component on the right-hand side of (24) is asymptotically proportional to $n$, while other components are of smaller and ever-decreasing order of magnitude. This allows us to approximate $G_n(\mathbf{v})$ by the first component of (24),

$$G_n(\mathbf{v}) \approx ng_n(\bar{v}), \tag{25}$$

where $g_n(z) \equiv R_1(p^a(z))\left(p^a(z) - \frac{\sum_i v_i^0}{nr}\right)$.

Next, we calculate the asymptotic order of magnitude of $\epsilon_i^{[1]}$. From (25),

$$G_n(\{m_i, \mathbf{v}_{-i}\}) \approx ng_n(\bar{v} - \frac{1}{n}(v_i - m_i)) = ng_n(\bar{v}) - g_n'(\bar{v})(v_i - m_i) + o(v_i - m_i),$$

thus $\epsilon_i^{[1]}$ can be approximated by

$$\epsilon_i^{[1]}(\mathbf{v}) \approx g_n'(\bar{v})(v_i - m_i),$$

with asymptotic rate of $n^0$, i.e. converging to a constant value.

By induction, we conjecture that $\epsilon_{i_1,\ldots,i_k}^{[k]}(\mathbf{v})$ can be approximated by

$$\epsilon_{i_1,\ldots,i_k}^{[k]}(\mathbf{v}) \approx n^{-k+1} g_n^{(k)}(\bar{v}) \prod_{l=1\ldots k} (v_{i_l} - m_{i_l}), \tag{26}$$

where $g_n^{(k)}(\cdot)$ is the $k$-th derivative of $g_n(\cdot)$. Suppose this is true for $\epsilon_{i_1,\ldots,i_{k-1}}^{[k-1]}(\mathbf{v})$. Then,

$$\epsilon_{i_1,\ldots,i_{k-1}}^{[k-1]}(\{m_{i_k}, \mathbf{v}_{-i_k}\}) = n^{-k+2} \prod_{l=1\ldots k-1} (v_{i_l} - m_{i_l}) \left[ g_n^{(k-1)}(\bar{v} - \frac{1}{n}(v_{i_k} - m_{i_k})) \right]$$
$$= n^{-k+2} \prod_{l=1\ldots k-1} (v_{i_l} - m_{i_l}) \left[ g_n^{(k-1)}(\bar{v}) - \frac{1}{n} g^{(k)}(v_{i_k} - m_{i_k}) + o\left(\frac{v_{i_k} - m_{i_k}}{n}\right) \right].$$

By referring to the definition of $\epsilon_{i_1,\ldots,i_k}^{[k]}(\mathbf{v})$, we can approximate it by (26).

Next, observe from (21) that every $\eta_i^{[k]}(\mathbf{v})$ is a weighted average of $\epsilon_{i,i_2,\ldots,i_k}^{[k]}(\mathbf{v})$, for various sequences of $i_2 \ldots i_k$, multiplied by the weight $s_{i,n}$. Because each element $\epsilon_{i,i_2,\ldots,i_k}^{[k]}(\mathbf{v})$ of the weighted sum has order of magnitude $n^{-(k-1)}$ while the weight $s_{i,n}$ has order of magnitude $n^{-1}$, we conclude that $\eta_i^{[k]}(\mathbf{v}) = O(n^{-k})$.

### 4.1.3 Inefficiency due to approximation error

Because each function $u_i^0(\mathbf{v}_{-i})$, independent from $v_i$, is approximated by $\tilde{u}_i^{[k]}(\mathbf{v})$, which does depend on $v_i$ via $\eta_i^{[k]}(\mathbf{v})$, sellers will deviate from reporting their true value $v_i$. This section characterizes the asymptotic properties of the reported value deviation and of the associated deviation of welfare from the first-best.

The compensation to seller $i$ as a function of the vector of reported types $\tilde{\mathbf{v}}$ is determined from (17), assuming that the types are reported truthfully, the aggregate price is first-best, transaction probability is positive, and $u_i^0(\mathbf{v}_{-i})$ is approximated by $\tilde{u}_i^{[k]}(\mathbf{v})$:

$$x_i(\tilde{\mathbf{v}}) = \frac{\tilde{u}_i^{[k]}(\tilde{\mathbf{v}}) - \frac{1}{r}(H_n^a(\tilde{V}) - H_n^a(\tilde{V} - \tilde{v}_i + v_i^0))}{R_n(P_n^a(\tilde{V}))} + \frac{\tilde{v}_i}{r}. \tag{27}$$

With this compensation, the utility (2) of seller $i$ becomes (recalling the definition (20) of $\tilde{u}_i^{[k]}$)

$$u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i) = R_n(P_n^a(\tilde{V}))\left(x_i(\tilde{\mathbf{v}}) - \frac{v_i}{r}\right)$$
$$= R_n(P_n^a(\tilde{V}))\left(\frac{\tilde{v}_i - v_i}{r}\right) + u_i^{[k]}(\tilde{\mathbf{v}}_{-i}) + \eta_i^{[k]}(\tilde{\mathbf{v}}) - \frac{H_n^a(\tilde{V}) - H_n^a(\tilde{V} + v_i^0 - \tilde{v}_i)}{r}.$$

The first order condition of optimal report $\tilde{v}_i$ is then

$$\frac{\partial u_i(\tilde{\mathbf{v}}, v_i)}{\partial \tilde{v}_i} = R_n'(P^a(\tilde{V}))\frac{\partial P^a(\tilde{V})}{\partial V}\left(\frac{\tilde{v}_i - v_i}{r}\right) + \frac{\partial \eta_i^{[k]}(\tilde{\mathbf{v}})}{\partial \tilde{v}_i} = 0,$$

which results in the following optimal report:

$$\tilde{v}_i = v_i - \frac{r \frac{\partial \eta_i^{[k]}(\tilde{\mathbf{v}})}{\partial \tilde{v}_i}}{R_n'(P^a(\tilde{V}))\frac{\partial P^a(\tilde{V})}{\partial V}}.$$

We can also assess the deviation of the aggregate reported value from the true one,

$$\tilde{V} - V = -\frac{r \sum_i \frac{\partial \eta_i^{[k]}(\tilde{\mathbf{v}})}{\partial \tilde{v}_i}}{R_n'(P^a(\tilde{V}))\frac{\partial P^a(\tilde{V})}{\partial V}}. \tag{28}$$

Observe that $\sum_i \frac{\partial \eta_i^{[k]}(\tilde{\mathbf{v}})}{\partial \tilde{v}_i}$ is a weighted average of $\frac{\partial \epsilon_{i_1,\ldots,i_k}^{[k]}(\tilde{\mathbf{v}})}{\partial \tilde{v}_{i_1}}$ for various sequences $i_1, \ldots, i_k$. The latter can be approximated by (cf.(26))

$$n^{-k+1}\left[\frac{1}{n} g_n^{(k+1)}(\bar{v}) \prod_{l=1\ldots k}(v_{i_l} - m_{i_l}) + g_n^{(k)}(\bar{v})\right].$$

With constant $k$ and rising $n$, the first term in square brackets vanishes with $n$ but the second does not, thus the numerator in (28) is $O(n^{-k+1})$. The denominator of (28) is asymptotically constant, thus we have $\tilde{V} - V = O(n^{-k+1})$. The expectation of $\tilde{V} - V$ over realizations of $V$ has the same order of magnitude.

To assess the welfare loss associated with deviation from truthful reporting of types, we Taylor-expand the aggregate gains from trade $U_n^a(\tilde{V}, V) = R_n(P_n^a(\tilde{V}))\left(P_n^a(\tilde{V}) - \frac{V}{r}\right)$ around $U_n^a(V, V)$, noting that $\frac{\partial U_n^a(V,V)}{\partial \tilde{V}} = 0$, to obtain $U_n^a(\tilde{V}, V) - U_n^a(V, V) = \frac{1}{2}\frac{\partial^2 U_n^a(V,V)}{\partial \tilde{V}^2}(\tilde{V} - V)^2 + o(\tilde{V} - V)^2$. The aggregate gains from trade and all its derivatives are of order $n$, thus the aggregate welfare loss is $O(n^{-2k+3})$, so that per-seller welfare loss is $O(n^{-2k+2})$.

### 4.2 The constitutional constraint

This section investigates how the approximately incentive compatible mechanism characterized in Section 4.1 is affected by the constitutional constraint (13). Section 2.4 has determined that coercion is necessary for asymptotically positive gains from trade, and therefore at least some sellers will have negative gains from trade (17) for some realizations of $\mathbf{v}$. For any such seller $i$, because the right-hand side of (17) is non-increasing in own value $v_i$, it is bounded away from zero (downwards) as $v_i$ goes to its upper bound. At the same time, the left-hand side of (17) satisfies

$$\begin{aligned} R_n(P_n^a(V))\left(x_i(\mathbf{v}) - \frac{v_i}{r}\right) &\underbrace{\geq}_{(13)} -R_n(P_n^a(V))\frac{v_i - v_i^0}{r} \\ = -\frac{1 - F_n(P_n^a(V))}{r + 1 - F_n(P_n^a(V))}\frac{v_i - v_i^0}{r} &\geq -\frac{(1 - F_n(P_n^a(V)))(v_i - v_i^0)}{r^2} \\ &\geq_{P_n^a(V) \geq V} -\frac{(1 - F_n(V))(v_i - v_i^0)}{r^2} \to_{v_i \to \infty} 0. \quad (29) \end{aligned}$$

The latter limit is true because $1 - F_n(V) = o(v_i^{-1})$; otherwise, the distribution of buyers' values would not have a mean, contradicting the assumption of Section 2.1.3.

Therefore, for large enough $v_i$, the left and right-hand sides of (17) disagree under the first-best mechanism due to the constitutional constraint (13). We now characterize a correction to the mechanism of Section 4.1 that allows to respect the constitutional constraint (13), and assess how the correction alters the asymptotic properties of the mechanism.

Section 4.1 has shown that achieving $P_n^a(V)$ exactly is generally impossible even when (13) is non-binding for all sellers, because any mechanism attempting that would result in sellers' deviation from truthful revelation of values. At the same time, such deviation can be made arbitrarily small as the number of sellers increases. The current section assumes, for analytical tractability, that the sellers always reveal their values truthfully.

Combine (17) with the first inequality in (29) to obtain

$$u_i^0(\mathbf{v}_{-i}) - \frac{H_n^a(V) - H_n^a(V + v_i^0 - v_i)}{r} + R_n(P_n^a(V))\frac{v_i - v_i^0}{r} \geq 0. \qquad (30)$$

The derivative of the left-hand side of (30) with respect to $v_i$ is equal to

$$R_n'(P_n^a(V))\frac{\partial P_n^a(V)}{\partial V}\frac{v_i - v_i^0}{r} \leq 0,$$

with strict inequality if the probability of transaction is positive and $v_i > v_i^0$. This means that if (30) is violated for some $v_i$, it is also violated for all values above $v_i$. Suppose that, whenever the compensation prescribed by Section 4.1 violates the constitutional constraint for seller $i$, the lowest possible compensation $\frac{v_i^0}{r}$ is given. Then, such compensation will be offered whenever the (reported) value $v_i$ is equal or above the threshold $\hat{v}_i$ that turns (30) into equality. Because the minimum compensation does not depend on $v_i$, we then have $\frac{\partial x_i(\mathbf{v})}{\partial v_i} = 0$ for all $v_i \geq \hat{v}_i$. At the same time, full differentiation of (17) with respect to $v_i$ yields $R_n'(P(\mathbf{v}))\left(x_i(\mathbf{v}) - \frac{v_i}{r}\right)\frac{\partial P(\mathbf{v})}{\partial v_i} + R_n(P(\mathbf{v}))\frac{\partial x_i(\mathbf{v})}{\partial v_i} = 0$, which further means that $\frac{\partial P(\mathbf{v})}{\partial v_i} = 0$ for all $v_i \geq \hat{v}_i$. We then also have $\sum_{j \neq i}\frac{\partial x_j(\mathbf{v})}{\partial v_i} = \frac{\partial P(\mathbf{v})}{\partial v_i} - \frac{\partial x_i(\mathbf{v})}{\partial v_i} = 0$. While it is theoretically possible that a change in $v_i$ leads to redistribution of compensations $x_j, j \neq i$, without changing the sum, it is not clear what such redistribution could achieve. We assume that $\frac{\partial x_j(\mathbf{v})}{\partial v_i} = 0, \forall j, \forall v_i \geq \hat{v}_i$. In other words, a seller $i$ who reports a value above $\hat{v}_i$ is treated by the mechanism as if he has reported the value $\hat{v}_i$.

### 4.2.1 The mechanism and its properties

With these considerations, we can specify the optimal feasible mechanism as follows: deliver the first-best aggregate price $P_n^a(\cdot)$, baseline utility $u_i^0(\cdot)$ approximated by (cf.(20)) $\tilde{u}_i^{[k]}(\cdot)$, and compensations $x_i(\cdot)$ that satisfy (17), as if the seller $i$'s valuation was

$$v_i^*(\mathbf{v}) = \min\{v_i, \hat{v}_i(\mathbf{v}_{-i})\},$$

rather than the true value $v_i$. Here $\hat{v}_i(\mathbf{v}_{-i})$ is defined by (cf.(30))

$$u_i^0(\mathbf{v}_{-i}^*) - \frac{H_n^a(\sum_{j \neq i} v_j^* + \hat{v}_i) - H_n^a(\sum_{j \neq i} v_j^* + v_i^0)}{r} + R_n(P_n^a(\sum_{j \neq i} v_j^* + \hat{v}_i))\frac{\hat{v}_i - v_i^0}{r} \equiv 0. \tag{31}$$

Such mechanism does not create incentives to deviate from truthful-telling, other than those stemming from imperfect approximation of $u_i^0(\mathbf{v}_{-i})$ and outlined in Section 4.1. We now provide some additional properties of the mechanism.

**Proposition 5** *For any given realization of* $\mathbf{v}$*, coercion cannot be applied to all sellers at the same time.*

**Proof.** Divide all sellers into two groups: $i \in N_1$ have $v_i^* = v_i < \hat{v}_i$ and $i \in N_2$ have $v_i^* = \hat{v}_i \leq v_i$. In $N_2$, each seller $i$ receives the minimum compensation $\frac{v_i^0}{r}$, thus the aggregate compensation to those in $N_1$ is $P_n^a(\sum_{\forall i} v_i^*) - \frac{\sum_{i \in N_2} v_i^0}{r}$. From the properties of $P_n^a(\cdot)$ we have $P_n^a(\sum_{\forall i} v_i^*) > \frac{\sum_{\forall i} v_i^*}{r} = \frac{\sum_{i \in N_1} v_i}{r} + \frac{\sum_{i \in N_2} \hat{v}_i}{r}$, so the aggregate surplus (compensation minus own value) of group $N_1$ is

$$P_n^a(\sum_{\forall i} v_i^*) - \frac{\sum_{i \in N_2} v_i^0}{r} - \frac{\sum_{i \in N_1} v_i}{r} \geq P_n^a(\sum_{\forall i} v_i^*) - \frac{\sum_{i \in N_2} \hat{v}_i}{r} - \frac{\sum_{i \in N_1} v_i}{r} > 0.$$

Therefore, $N_1$ is non-empty (otherwise it would have zero surplus), and at least some of its members have positive surplus. ■

**Proposition 6** *A seller whose value is lowest possible, $v_i^0$, enjoys nonnegative gains from trade.*

**Proof.** Consider (30) with $v_i = v_i^0$ to conclude $u_i^0(\mathbf{v}_{-i}) \geq 0$. ■

How does the compensation $x_i$ depend on the (reported) type $v_i$ when $v_i \leq \hat{v}_i$, given $\mathbf{v}_{-i}$? Fully differentiating (17) with respect to $v_i$ and rearranging, we obtain[4] $\frac{\partial x_i}{\partial v_i} = \frac{-R_n'(P_n^a)}{R_n(P_n^a)} \left(x_i - \frac{v_i}{r}\right) \frac{\partial P_n^a(V)}{\partial V}$. Because $\frac{\partial P_n^a(V)}{\partial V} > 0$ and $\frac{-R_n'(P_n^a)}{R_n(P_n^a)} > 0$, the sign of $\frac{\partial x_i}{\partial v_i}$ is equal to the sign of $x_i - \frac{v_i}{r}$, which is known to be decreasing from a positive to a negative value. In other words, compensation $x_i$ is increasing in type $v_i$ when $i$'s gains from trade are positive, is flat when zero, and is decreasing when the gains from trade are negative.

4. The analysis is provided for the case when $v_j \leq \hat{v}_j$ for all $j$ (i.e. the constitutional constraint is non-binding for all sellers), thus $v_j^* = v_j$ is independent from $v_i$. The other scenario, when $v_j^* = \hat{v}_j$ for some $j$, is omitted.

What is the intuition behind this result? In non-coercive mechanisms (Myerson and Satterthwaite (1983) is a classic reference), compensation always increases with seller's value. This is because a higher requested compensation has two effects on welfare: more cash in case of a transaction (positive effect), but a lower probability that the transaction indeed happens (negative effect). For sellers with higher value, the negative welfare effect is weaker (they have less to lose if the transaction fails) while the positive one is the same, thus they ask more.

When coercion is applied to a seller, the above logic is turned upside down. Coercion means that a lower probability of transaction has positive, not negative, effect on welfare. To preserve truthful revelation of types, the other effect should be reversed too: a higher reported type of a seller (which optimally entails a lower probability of transaction) should result in lower, not higher, compensation. Otherwise the sellers being coerced would report infinitely high types.

#### 4.2.2 Asymptotic properties

This section analyzes the asymptotic properties of the threshold $\hat{v}_i(\mathbf{v}_{-i})$, as the number of sellers grows large, and welfare losses caused by the constitutional constraints. From (23), the difference $H_n^a(\sum_{j\neq i} v_j^* + \hat{v}_i) - H_n^a(\sum_{j\neq i} v_j^* + v_i^0)$ in (31) can be approximated by

$$R_1\left(p^a\left(\frac{\sum_{j\neq i} v_j^* + \hat{v}_i}{n}\right)\right)(v_i - v_i^0)$$
$$- \frac{1}{2n}R_1'\left(p^a\left(\frac{\sum_{j\neq i} v_j^* + \hat{v}_i}{n}\right)\right)(v_i - v_i^0)^2 + o(n^{-1}),$$

the first component of which is equal to $R_n(P_n^a(\sum_{j\neq i} v_j^* + \hat{v}_i))(v_i - v_i^0)$ in (31) and thus cancels out. Therefore, the threshold $\hat{v}_i(\mathbf{v}_{-i})$ satisfies

$$u_i^0(\mathbf{v}_{-i}^*) + \frac{1}{2nr} R_1' \left( p^a \left( \frac{\sum_{j\neq i} v_j^* + \hat{v}_i}{n} \right) \right) (\hat{v}_i - v_i^0)^2 + o(n^{-1}) = 0,$$

which allows us to approximate the threshold $\hat{v}_i$ as follows:

$$\hat{v}_i \approx \left( \frac{2nru_i^0(\mathbf{v}_{-i}^*)}{-R_1'\left(p^a\left(\frac{\sum_{j\neq i} v_j^* + \hat{v}_i}{n}\right)\right)} \right)^{\frac{1}{2}} + v_i^0. \tag{32}$$

Here $u_i^0(\mathbf{v}_{-i}^*)$ is approximated by $\tilde{u}_i^{[k]}(\{\hat{v}_i, \mathbf{v}_{-i}^*\})$, which depends on $\hat{v}_i$ via $\eta_i^{[k]}(\cdot)$, but Section 4.1.2 shows the latter is $O(n^{-k})$ and therefore has a vanishing impact on (32). $\tilde{u}_i^{[k]}(\cdot)$ also changes with $n$, but can be shown to converge to a constant. $p^a(\cdot)$ in (32) is asymptotically constant and independent of $\hat{v}_i$, too. Therefore, we have that $\hat{v}_i = O(n^{\frac{1}{2}})$.

The welfare loss associated with departure from the first-best mechanism is due to the difference between the true average value and the "truncated" one,

$$\frac{\sum_{i=1}^n v_i^*(\mathbf{v}) - v_i}{n} = -\frac{\sum_{i=1}^n I(v_i \geq \hat{v}_i(\mathbf{v}_{-i}))(v_i - \hat{v}_i(\mathbf{v}_{-i}))}{n}. \tag{33}$$

Asymptotically, such difference depends on the (generally unknown) distributions of seller valuations: the thicker are the tails of the distributions, the more likely that $v_i^* \neq v_i$, the further away from zero is (33). Specifically, for distributions without tails (i.e. bounded from above), $\hat{v}_i$ eventually becomes greater than the distribution upper bound for all $i$, and the constitutional constraint does not reduce welfare for large enough $n$.

For distributions that have tails, if all elements of the sum in (33) converge to zero at the same rate, then so does (33) which is their average. Suppose the sellers' distribution is Pareto, known for its heavy tails, such that $\Pr(v_i \geq x) = \left(\frac{v_i^0}{x}\right)^\alpha$. Then the expectation of $v_i^*(\mathbf{v}) - v_i$ in (33), conditional on $\mathbf{v}_{-i}$, is given by $\frac{\alpha}{\alpha-1}(v_i^0)^\alpha \hat{v}_i(\mathbf{v}_{-i})^{-(\alpha-1)}$, which is $O(n^{-\frac{\alpha-1}{2}})$. The discrepancy between the buyer's price per seller $p^a\left(\frac{\sum_1^n v_i^*}{n}\right)$ and the optimal one $p^a(\bar{v})$ is also $O(n^{-\frac{\alpha-1}{2}})$. The expected welfare effect of the constitutional constraint is $O(n^{-(\alpha-1)})$.

# 5 Numerical examples

We now illustrate the optimal feasible mechanism with two numerical examples. In both examples, seller values were randomly drawn from Lomax distribution (also known as type-II Pareto distribution) such that $\Pr(v_i \geq x) = \left(\frac{2m_i}{2m_i+x}\right)^3, \forall x \geq 0$, with means equal to $m_i$. In Example 1, there are four "type-A" sellers with $m_i = 1$ and two "type-B" sellers with $m_i = 2$. In Example 2, there are 40 type-A sellers and 20 type-B sellers. For comparison purposes, the first four type-A sellers and two type-B sellers in Example 2 have the same values as those in Example 1.

We assume that the prospective buyers show up, on average, once per year. The discount rate is 5%. Thus, $r = 0.05$. We approximate $u_i^0$ by $\tilde{u}_i^{[k]}$ with $k = 3$.

The buyers' distribution of values is uniform between zero and some $P_m$ such that the mean buyer's value $\frac{P_m}{2}$ is equal to the aggregate mean of all sellers NPVs, $\frac{\sum_i m_i}{r}$, in both examples. Specifically, $P_m = 320$ in Example

| Seller | $v_i$ | Report bias, $\tilde{v}_i - v_i$ | | Compensation, $x_i$ | | | Gains from trade, $u_i$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Ex1 | Ex2 | Ex1 | Ex2 | FCM | Ex1 | Ex2 | FCM |
| $A_1$ | 0.7439 | 0.1511 | 0.0001 | 40.050 | 35.972 | 35.367 | 14.67 | 14.22 | 14.31 |
| $A_2$ | 1.8319 | 0.1223 | 0.0001 | 39.371 | 35.960 | 35.367 | 1.59 | -0.46 | -0.89 |
| $A_3$ | 4.8276 | 0.0917 | 0.0001 | 30.618 | 35.537 | 35.367 | -38.43 | -41.13 | -42.74 |
| $A_4$ | 0.6683 | 0.1539 | 0.0001 | 40.026 | 35.970 | 35.367 | 15.54 | 15.24 | 15.37 |
| $A_{\min}$ | 0.0094 | | 0.0001 | | 35.936 | 35.367 | | 24.10 | 24.57 |
| $A_{\max}$ | 5.1759 | | 0.0001 | | 35.452 | 35.367 | | -45.89 | -47.60 |
| $B_1$ | 3.0314 | 0.2241 | 0.0002 | 77.881 | 71.872 | 70.734 | 10.06 | 7.58 | 7.06 |
| $B_2$ | 0.4510 | -0.096 | 0.0001 | 69.693 | 71.779 | 70.734 | 35.37 | 42.31 | 43.11 |
| $B_{\min}$ | 0.0146 | | 0.0001 | | 71.718 | 70.734 | | 48.15 | 49.20 |
| $B_{\max}$ | 11.213 | | 0.0001 | | 69.353 | 70.734 | | -104.43 | -107.24 |

Table 1: Mechanism outcomes for individual sellers. Ex($i$) is optimal feasible mechanism for Example $i$, FCM is fixed compensation mechanism. All values are in dollars.

1 and $P_m = 3200$ in Example 2.

For comparison purposes, we also illustrate the fixed compensation mechanism.

Results for individual sellers are reported in Table 1. In Example 2, we show only sellers whose values coincide with those from Example 1, as well as those with minimal and maximal values within each type. In all examples, the constitutional constraint was not binding for all sellers. Average outcomes are given in Table 2. Figure 1 illustrates how the compensation $x_{A_1}$ and the aggregate price $P$ respond to variation of the reported type $\tilde{v}_{A_1}$ by seller $A_1$.

All above tables and figures confirm the findings of the paper. The compensation $x_i$ first rises and then declines with the reported type $\tilde{v}_i$, but $x_i$ varies less with $\tilde{v}_i$ as the number of sellers grows. At the same time, the aggregate price $P$ rises with $\tilde{v}_i$, with the dependence becoming increasingly linear. In both optimal and fixed-compensation mechanisms, the deviation

| | Example 1 | | Example 2 | |
|---|---|---|---|---|
| | OFM | FCM | OFM | FCM |
| Average value, $\bar{v}$ | 1.9257 | | 1.5443 | |
| 1st best average price, $p^a(\bar{v})$ | 49.1714 | | 47.8165 | |
| 1st best average gains from trade | 6.4958 | | 11.4133 | |
| Average report bias, $\frac{\tilde{V}-V}{n}$ | 0.1079 | N/A | 0.0001 | N/A |
| Average price, relative to 1st best | 0.4352 | -2.0157 | 0.0003 | -0.6608 |
| Avg gains from trade, relative to 1st best | -0.0296 | -0.4594 | $-1.4 \times 10^{-8}$ | -0.0494 |

Table 2: Average mechanism outcomes. OFM is optimal feasible mechanism, FCM is fixed compensation mechanism. All values are in dollars.

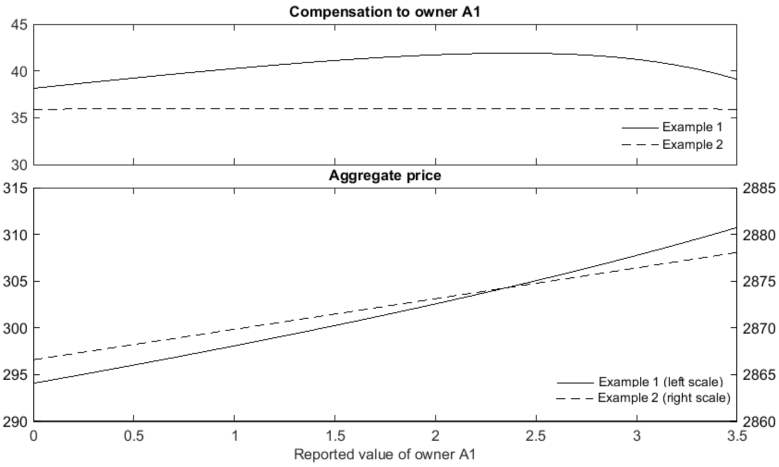


Figure 1: Effects of varying reported value $\tilde{v}_{A_1}$ by seller $A_1$.

of all relevant parameters from the first-best decreases with the number of sellers, but in the OFM this deviation is always smaller and converges to zero at a much faster rate.

The last line of Table 2 also allows to access by how much the OFM outperforms the FCM. In Example 1, the gap between OFM and the first-best is $0.4594/0.0296 = 15.5$ times smaller than the gap between the FCM and the first-best. In Example 2, the ratio is $0.0494/(1.4 \times 10^{-8}) = 3.5 \times 10^6$ times.

# 6 Discussion

## 6.1 Misspecification

How does the optimal feasible mechanism perform if the government has incorrectly estimated the sellers' mean values $m_i$ or the buyers' distribution of values $F_n$? For comparison purposes, first observe that, in each of these misspecification cases, the fixed-compensation mechanism results in a biased aggregate price (15). The bias generally does not go away with rising number of sellers.

In the OFM, seller means $m_i$ affect only the approximation of $u_i^0(\mathbf{v}_{-i})$. The effect is twofold: through the shares of each seller $s_{i,n}$, and through the approximation error $\epsilon^{[k]}_{i_1,\ldots,i_k}(\mathbf{v})$. If all seller means are misspecified by the same factor, such that the estimated mean $\hat{m}_i$ equals $qm_i$ for some $q > 0$, then the shares $s_{i,n}$ are unchanged, and the only effect is through $\epsilon^{[k]}_{i_1,\ldots,i_k}(\mathbf{v})$. The latter is approximated by (26); replacing the true $m_i$ in the latter by its estimate $\hat{m}_i$ yields $\epsilon^{[k]}_{i_1,\ldots,i_k}(\mathbf{v}) \approx n^{-k+1} g^{(k)}_n(\bar{v}) \prod_{l=1\ldots k}(v_{i_l} - \hat{m}_{i_l})$, which is biased from zero by $n^{-k+1} g^{(k)}_n(E\bar{v}) \prod_{l=1\ldots k}(\hat{m}_{i_l} - m_{i_l})$. The bias converges to zero at the same rate as the error itself does, and therefore, although the reported value deviation (28) now has generally non-zero expectation, it converges to zero at the same rate as before. We can conclude that misspecified sellers' average values do not compromise asymptotic properties of the OFM.

What if the mechanism designer's estimate of buyers distribution of values $\hat{F}(\cdot)$ (dropping the subscript $n$) is incorrect? Denote by $\hat{R}(P) \equiv \frac{1-\hat{F}(P)}{r+1-\hat{F}(P)}$ the estimate of $R(P)$. Then, if all sellers' reported values $\tilde{v}_i$ were true, the estimate $\hat{P}^a(\tilde{V})$ of the first-best price $P^a(V)$ that satisfies

$$\hat{R}'(\hat{P}^a)\left(\hat{P}^a - \frac{\tilde{V}}{r}\right) + \hat{R}(\hat{P}^a) = 0 \tag{34}$$

would generally deviate from true first-best (4). However, if the sellers know the true distribution $F(\cdot)$, a bias in the mechanism will generally cause them to deviate from reporting true types. Denote by $\hat{H}^a(\tilde{V}) = \int_0^{\tilde{V}} \hat{R}(\hat{P}^a(z))\mathrm{d}z$ the mechanism designer's estimate of $H^a(\tilde{V})$. Ignoring the errors of approximation of $u^0_i(\mathbf{v}_{-i})$, and assuming the constitutional constraint (13) does not bind any seller, the compensation to seller $i$ is calculated as (cf.(27))

$$x_i(\tilde{\mathbf{v}}) = \frac{u^0_i(\tilde{\mathbf{v}}_{-i}) - \frac{1}{r}(\hat{H}^a(\tilde{V}) - \hat{H}^a(\tilde{V} - \tilde{v}_i + v^0_i))}{\hat{R}(\hat{P}^a(\tilde{V}))} + \frac{\tilde{v}_i}{r}.$$

Such seller maximizes $R(\hat{P}^a(\tilde{V}))\left(x_i(\tilde{\mathbf{v}}) - \frac{v_i}{r}\right)$, which results in the following first-order condition of optimal $\tilde{v}_i$:

$$\left(\frac{R'(\hat{P}^a(\tilde{V}))}{R(\hat{P}^a(\tilde{V}))} - \frac{\hat{R}'(\hat{P}^a(\tilde{V}))}{\hat{R}(\hat{P}^a(\tilde{V}))}\right)\frac{R(\hat{P}^a(\tilde{V}))}{\hat{R}(\hat{P}^a(\tilde{V}))}\left[u_i^0(\mathbf{v}_{-i}) - \frac{1}{r}(\hat{H}^a(\tilde{V}) - \hat{H}^a(\tilde{V} - \tilde{v}_i + v_i^0))\right] + \frac{R'(\hat{P}^a(\tilde{V}))}{r}(\tilde{v}_i - v_i) = 0.$$

Aggregating the above across all sellers, and recalling that the term in square brackets aggregated across all $i$ equals $\hat{R}(\hat{P}^a(\tilde{V}))\left(\hat{P}^a(\tilde{V}) - \frac{\tilde{V}}{r}\right)$, we obtain

$$\left(\frac{R'(\hat{P}^a(\tilde{V}))}{R(\hat{P}^a(\tilde{V}))} - \frac{\hat{R}'(\hat{P}^a(\tilde{V}))}{\hat{R}(\hat{P}^a(\tilde{V}))}\right) R(\hat{P}^a(\tilde{V}))\left(\hat{P}^a(\tilde{V}) - \frac{\tilde{V}}{r}\right) + \frac{R'(\hat{P}^a(\tilde{V}))}{r}(\tilde{V} - V) = 0.$$

The latter expression can be modified to

$$R'(\hat{P}^a(\tilde{V}))\left(\hat{P}^a(\tilde{V}) - \frac{V}{r}\right) - \frac{\hat{R}'(\hat{P}^a(\tilde{V}))}{\hat{R}(\hat{P}^a(\tilde{V}))} R(\hat{P}^a(\tilde{V}))\left(\hat{P}^a(\tilde{V}) - \frac{\tilde{V}}{r}\right) = 0.$$

Because $\hat{P}^a(\tilde{V})$ is chosen to achieve (34), we can also conclude $R'(\hat{P}^a(\tilde{V}))\left(\hat{P}^a(\tilde{V}) - \frac{V}{r}\right) + R(\hat{P}^a(\tilde{V})) = 0$, which means (cf.(4)) $\hat{P}^a(\tilde{V}) = P^a(V)$. In other words, when the government makes a mistake in estimating the distribution of buyers' types, the sellers correct their reported values so the aggregate sale price remains unchanged.

We can conclude that, unlike the FCM, the OFM is robust to mistakes by the government in estimation of the mechanism's inputs.

## 6.2 Collusion

Does the OFM encourage collusion? Because the ultimate purpose of the mechanism is to maximize the sellers' joint gains from trade, collusion between sellers is unlikely. Does a buyer have an incentive to collude with a subset of sellers, for example by secretly purchasing their units and manipulating their prices? Because manipulation with reported type $\tilde{v}_i$ generally affects payments to sellers other than $i$, a buyer who got control of unit $i$ might have an incentive to engage in such manipulation. However, if a buyer attempts to manipulate the reported values of a sizeable fraction of all units, then the remaining independent sellers would observe an unusual distribution of reported types. From that, they would infer that somebody is certain to assemble their property, which would lead them to reevaluate the probability of transaction and adjust their reported types so that the buyer's price is increased. This would offset the potential gains from manipulation. Further research is required to assess the potential magnitude of manipulation and its welfare consequences.

## 6.3 Auctions

When the proposed mechanism is initiated for the first time for a given set of units to be assembled, multiple buyers may show up simultaneously. In this case, an auction can be administered, with the reservation price being equal to the one determined by the OFM. Because the final transaction price is then determined by competition between buyers, rather than by inputs of sellers, the difference between the final and the reservation prices can be divided between sellers in an arbitrary predetermined way, for example proportionately to their mean values $m_i$.

### 6.4 Fixed buyer's price

A scenario opposite to the previous section is when there is only one buyer with non-random commonly known purchase price. In particular, this scenario is relevant when the purchase is initiated by a government with a transparent budget. The mechanism of this paper depends heavily on the random nature of buyers' valuations, and cannot be used in this scenario. Further research is required to address the issue.

## A Proofs

**Proof of Proposition 1** Because the proof is provided for a given number of sellers, we drop the subscript $n$ throughout the proof for clarity of exposition. First, observe that Assumption 1 implies

$$F^{(2)}(P)(1 - F(P)) + 2(F'(P))^2 > 0, \forall P. \tag{35}$$

We have the following relationships between $R(P)$ and its derivatives:

$$R'(P) = -\frac{rF'(P)}{(r+1-F(P))^2},$$
$$R^{(2)}(P) = -\frac{r}{(r+1-F(P))^2}\left(F^{(2)}(P) + \frac{2(F'(P))^2}{r+1-F(P)}\right) \underbrace{<}_{(35)} \frac{2(R'(P))^2}{R(P)}. \tag{36}$$

Calculate the second derivative of (3) with respect to $P$ at every $P^a$ that satisfies (4) as follows:

$$R^{(2)}(P^a)\left(P^a - \frac{V}{r}\right) + 2R'(P^a) \underbrace{<}_{(36)} 2R'(P^a)\left(\frac{R'(P^a)}{R(P^a)}\left(P^a - \frac{V}{r}\right) + 1\right) \underbrace{=}_{(4)} 0$$

Therefore, at any point $P^a$ where the first derivative of $U(P, \mathbf{v}, V)$ with respect to $P$ is zero, the second derivative is negative. Because $U(P, \mathbf{v}, V)$ is a continuous function of its arguments, such point $P^a$ is unique for a given $\mathbf{v}$ and corresponds to the global maximum.■

**Proof of Proposition 3** Suppose the contrary, that for some $v_i, \mathbf{x}_{-i}$, we have $x_i(v_i, \mathbf{x}_{-i}) > x_i^*(v_i, \mathbf{x}_{-i})$. Consider changing $x_i(v_i, \mathbf{x}_{-i})$ by a marginal amount $-\mathrm{d}x < 0$. Because the compensation becomes closer to $x_i^*(v_i, \mathbf{x}_{-i})$, seller $i$ is better off.

If all other sellers enjoy positive gains from trade, they are better off, too, because lowered $x_i$ increases the probability of transaction. Thus, we have achieved a Pareto-improvement, compromising the initial assumption of optimality.

Suppose there exists a seller $j$ with value $v_j$ who enjoys negative gains from trade: $x_j(v_j, \mathbf{x}_{-j}) < \frac{v_j}{r}$. Because $x_j^*(v_j, \mathbf{x}_{-j}) > \frac{v_j}{r}$, we also have that $x_j(v_j, \mathbf{x}_{-j}) < x_j^*(v_j, \mathbf{x}_{-j})$. Consider changing $x_j(v_j, \mathbf{x}_{-j})$ by a marginal amount $\mathrm{d}x > 0$. Seller $j$ is better off, as the compensation becomes closer to his ideal $x_j^*(v_j, \mathbf{x}_{-j})$. Because $x_i$ and $x_j$ have changed by the same amount but in opposite directions, the total transaction price is unaffected, so the welfare of sellers other than $i, j$ is unaffected, too. We have achieved a Pareto-improvement, again, thus the initial allocation could not be optimal.■

# References


Cai, Hongbin. 2003. "Inefficient Markov perfect equilibria in multilateral bargaining." *Economic Theory* 22 (3): 583–606.

Calandrillo, Steve. 2003. "Eminent Domain Economics: Should 'Just Compensation' Be Abolished, and Would 'Takings Insurance' Work Instead?" *Ohio State Law Journal* 64 (2): 451–530.

Cunningham, Chris. 2013. "Estimating the holdout problem in land assembly."

d'Aspremont, Claude, and Louis-André Gérard-Varet. 1979. "Incentives and incomplete information." *Journal of Public economics* 11 (1): 25–45.

Grossman, Zachary, Jonathan Pincus, and Perry Shapiro. 2010. "A second-best mechanism for land assembly." *Department of Economics, UCSB.*

Heller, Michael. 2010. *The Gridlock Economy: How Too Much Ownership Wrecks Markets Stops Innovation, and Costs Lives.*

Kominers, Scott Duke, and E Glen Weyl. 2011. "Concordance among holdouts." In *Proceedings of the 12th ACM conference on Electronic commerce,* 219–220. ACM.

Mailath, George J, and Andrew Postlewaite. 1990. "Asymmetric information bargaining problems with many agents." *The Review of Economic Studies* 57 (3): 351–367.

Miceli, Thomas J, and Kathleen Segerson. 2007. "A bargaining model of holdouts and takings." *American Law and Economics Review* 9 (1): 160–174.

Miceli, Thomas J, and CF Sirmans. 2007. "The holdout problem, urban sprawl, and eminent domain." *Journal of Housing Economics* 16 (3): 309–319.

Myerson, Roger B, and Mark A Satterthwaite. 1983. "Efficient mechanisms for bilateral trading." *Journal of economic theory* 29 (2): 265–281.

Posner, Eric A., and E. Glen Weyl. 2017. "Property Is Only Another Name for Monopoly." *Journal of Legal Analysis* 9 (1): 51–123.

Shavell, Steven. 2010. "Eminent Domain versus Government Purchase of Land Given Imperfect Information about Owners' Valuations." *Journal of Law and Economics* 53 (1): 1–27.

Strange, William C. 1995. "Information, holdouts, and land assembly." *Journal of Urban Economics* 38 (3): 317–332.

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Консолидация нескольких объектов недвижимости, каждый из которых имеет отдельного продавца с ненаблюдаемой личной ценностью объекта, необходима для реализации проекта реконструкции. Данная работа предлагает механизм для консолидации собственности, который (а) допускает принуждение отдельных продавцов, (б) не требует никаких платежей до момента консолидации, (в) имеет сбалансированный бюджет, (г) гарантирует компенсацию каждому продавцу не меньше установленного минимума, (д) максимизирует суммарное благосостояние продавцов. Показана асимптотическая (при увеличении числа продавцов) сходимость механизма к теоретическому максимуму благосостояния с большой скоростью сходимости. Механизм требует минимального объема информации о распределении личной ценности объекта для продавцов и устойчив к ошибкам создателя механизма в оценке распределений личной ценности как продавцов, так и покупателей.

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