Syllabus

1. Course Description

a. Title of a Course : Symmetric Functions (E. Yu. Smirnov).

b. Pre-requisites : the standard courses in algebra and discrete mathematics or combinatorics. Some knowledge of representation theory of symmetric and general linear groups is not required, but helpful.

c. Course Type : optional

d. Abstract : The theory of symmetric functions is one of the central branches of algebraic combinatorics. Being a rich and beautiful theory by itself, it also has numerous connections with the representation theory and algebraic geometry (especially geometry of homogeneous spaces, such as flag varieties, toric and spherical varieties). In this course we will mostly focus on the combinatorial aspects of the theory of symmetric functions and study the properties of Schur polynomials. In representation theory they appear as characters of representations of GL(n); they are also closely related with the geometry of Grassmannians. The second half of the course will be devoted to Schubert polynomials, a natural generalization of Schur polynomials, defined as «partially symmetric» functions. Like the Schur functions, they also have a rich structure and admit several nice combinatorial descriptions; geometrically they appear as representatives of Schubert classes in the cohomology ring of a full flag variety. Time permitting, we will also discuss K-theoretic (non-homogeneous) analogues of Schur and Schubert polynomials.

2. Learning Objectives

The seminar is intended to introduce the subject area to the students, and to offer them an opportunity to prepare and give a talk.

3. Learning Outcomes

Successful participants imporve their presentation skills and prepare for participation in research projects in the subject area.

4. Course Plan

1) Symmetric polynomials. The ring of symmetric functions. Bases of the ring of symmetric functions: el-ementary, complete, monomial

symmetric functions, power sums. Transition formulas between these bases.

2) Schur functions. Algebraic definition. Jacobi-Trudi formula. Combinatorial definition, equivalence with the algebraic definition. Young tableaux.
3) Applications to combinatorics: enumerating plane partitions. MacMahon formula.
4) Richardson - Schensted - Knuth (RSK) correspondence. Jeu de taquin.
5) Multiplication of Schur functions. Pieri rule. Littlewood - Richardson rule. Knutson - Tao puzzles*.
6) Symmetric group, its Coxeter presentation. The Bruhat order. The Lehmer code and the essential set of a permutation.
7) «Partially symmetric» polynomials. Divided difference operators. Schubert polynomials.
8) Properties of Schubert polynomials. Monk's formula, Lascoux transition formula.
9) Combinatorial presentation of Schubert polynomials. Pipe dreams. Positivity. Fomin - Kirillov theorem.
10) Flagged Schur functions, determinantal formulae.
11) Generalizations*: double Schubert polynomials, Stanley symmetric functions, Grothendieck polynomials.

5. Reading List

a. Required: A. Knutson; Schubert polynomials and symmetric functions; Lisbon, 2012 http://pi.math.cornell.edu/~allenk/schubnotes.pdf

b. Optional

Edward A. Bender and S. Gill Williamson, Foundations of Combinatorics with Applications, Dover Publications, 2006, http://www.math.ucsd.edu/~ebender/CombText/index.html

6. Grading System

The formula for marking is 0.3 cumulative + 0.7 final exam, where cumulative is proportional to number of tasks solved.

7. Guidelines for Knowledge Assessment

Exercise 1. Use the Lindström–Gessel–Viennot trick to prove the second Jacobi–Trudi formula: $s_\lambda = \det(e_{\lambda_i - i + j})$.

Exercise 2. Let a Young diagram λ be presented as a union of *hooks* embedded one into another with "arms" of lengths $a_1, \ldots, a_k$ and "legs" of lengths $\ell_1, \ldots, \ell_k$. Let $s_{(a|\ell)}$ be the Schur polynomial for the hook with arm and leg a and ℓ, respectively. Prove the *Giambelli formula*:

$$s_\lambda = \det(s_{(a_i|\ell_j)})_{i,j=1}^k.$$

Exercise 3. Show that the Pieri formulas are implied by the Littlewood–Richardson rule.

Exercise 4. Show that the Kostka number $K_{\lambda\mu}$ is positive iff $\lambda \geq \mu$, i.e. μ is obtained from λ by "dropping some boxes down".

Exercise 5. Prove the identities

$$K_\lambda = \sum_{\lambda \in \mu \otimes 1} K_\mu, \qquad (1 + |\lambda|)K_\lambda = \sum_{\mu \in \lambda \otimes 1} K_\mu, \qquad \sum_{|\lambda| = n} K_\lambda^2 = n!.$$

Exercise 6. Let $p_k = x_1^k + \cdots + x_n^k$ be the *Newton power sums*. Show that

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} + \cdots + (-1)^{k-1} e_{k-1} p_1 + (-1)^k k e_k = 0.$$

(we formally set $e_k = 0$ for $k > n$).

8. Methods of Instruction

Students are individually assigned papers and textbook excerpts to give a seminar talk.

9. Special Equipment and Software Support : no requirements