Syllabus

1. Course Description

a. Title of a Course : Research Seminar "Elements of Functional Analysis"(A.Yu. Pirkovskii)

b. Pre-requisites : basic courses of analysis and linear algebra.

c. Course Type : optional

d. Abstract : Markov chains form the simplest class of random processes for which the future does not depend on the past but depends only on the present state of the process. Being rather simple, at the same time Markov chains have very deep and beautiful mathematics. They are known as probably the most important class of random processes, in particular, because of the numerous applications in mathematics, physics, biology, economics, etc. Indeed, once a stochastic process is given, it is natural to simplify it by assuming that the future does not depend on the past, and very often this approximation works well. The present course is the introduction to the theory of Markov chains. It will concern with their most important properties and the most known applications. The course is aimed at the 3rd and 4th year students, but is also possible for 1st and 2nd year students.

2. Learning Objectives

The seminar is intended to introduce the subject area to the students, and to offer them an opportunity to prepare and give a talk.

3. Learning Outcomes

Successful participants imporve their presentation skills and prepare for participation in research projects in the subject area.

4. Course Plan

1) Examples of models leading to Markov chains.
2) Markov chains with finite number of states.
3) Ergodic properties of Markov Chains.

4) Applications of Markov Chains: random walks, birth-death processes, etc.
5) Entropy of Markov Chains.
6) Markov Chains with infinite number of states.

5. Reading List

a. Required

Meyn, Tweedie; Markov Chains and Stochastic Stability; Cambridge University Press, 2009 http://probability.ca/MT/

b. Optional

S. Kuksin, A. Shirikyan. Mathematics of two-dimensional turbulence. CUP, 2012.

https://webusers.imj-prg.fr/~sergei.kuksin/MyBooks/bookArmen.pdf

6. Grading System

final grade = 0.35×(midterm colloquium)+0.35×(problem sets)+0.3×(final exam)+(bonus points)

The colloquium and exam are oral.

To get the maximal grade for the problem sets it suffices to solve 3/4 of all problems.

Extra problems, the problems marked by the letter B, work in the class --- will be counted as bonus (up to 3 points). .

A half-integer grade is rounded to the bigger nearest integer, another fractional grade is rounded to the nearest integer.

7. Guidelines for Knowledge Assessment

**3.8.** Show that a normed space $X$ is separable iff there exists a dense vector subspace $X_0 \subset X$ of an at most countable dimension.

**3.9.** Prove that the dimension of an infinite-dimensional Banach space is uncountable.

**3.10.** Show that $c_0$, $C[a,b]$, $\ell^p$, $L^p[a,b]$, $L^p(\mathbb{R})$ ($p < \infty$) are separable, while $\ell^\infty$, $C_b(\mathbb{R})$, $L^\infty[a,b]$, $L^\infty(\mathbb{R})$ are not separable.

**3.11.** **(a)** Prove that every normed space $X$ can be isometrically embedded into $\ell^\infty(S)$ for some $S$.
**(b)** Prove that every separable normed space $X$ can be isometrically embedded into $\ell^\infty$.

**3.12.** Let $X$ be a normed space.
**(a)** Prove that if $X^*$ is separable, then so is $X$.
**(b)** Is the converse true?
**(c)** Prove that there is no topological isomorphism between $(\ell^\infty)^*$ and $\ell^1$.

**3.13.** Prove that
**(a)** a Hilbert space is reflexive;
**(b)** $c_0$ is not reflexive;
**(c)** $\ell^1$ is not reflexive;
**(d)** $L^1(X,\mu)$ is not reflexive (unless it is finite-dimensional);
**(e)** $C[a,b]$ is not reflexive.

**3.14** (*the dual map*). Let $X$ and $Y$ be normed spaces, and let $T\colon X \to Y$ be a bounded linear map. Define $T^*\colon Y^* \to X^*$ by $T(f) = f \circ T$. Show that $T^*$ is bounded, and that $\|T^*\| = \|T\|$.

8. Methods of Instruction

Students are individually assigned papers and textbook excerpts to give a seminar talk.

9. Special Equipment and Software Support : no requirements