On a variant of the Rudin–Keisler preorder defined via finitary maps

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The classical notion of the Rudin–Keisler preorder on ultrafilters over an infinite set $X$ is defined by letting $u \leq_{RK} v$ iff there exists a map $f : X \to X$ such that $f(u) = v$ for all $u, v \in \beta X$, where $\beta X$ is the set of ultrafilters over $X$ endowed with the standard topology, and $\tilde{f} : \beta X \to \beta X$ is the continuous extension of $f$ (see e.g. [1, 2]). By using right continuous extensions of maps of any finite arities (introduced in [3, 4]), we expand this notion as follows. Let $uR_n v$ iff there exists an $n$-ary map $f : X^n \to X$ such that $f(u, \ldots, u) = v$ for all $u, v \in \beta X$ where $\tilde{f} : (\beta X)^n \to \beta X$ is the right continuous extension of $f$. Let also $\leq$ be the union of all the $R_n$, $n \in \omega$. Clearly, $R_1$ coincides with $\leq_{RK}$, and $R_m \subseteq R_n$ whenever $m \leq n$.

We show that $R_m \circ R_n = R_{m,n}$, whence it follows that $\leq$ is a preorder. We prove that this preorder properly extend the Rudin–Keisler preorder, i.e., there are $u, v \in \beta X$ such that $u \leq v$ but not $u \leq_{RK} v$. As well-know, under the Continuum Hypothesis (CH) there exist non-principal ultrafilters over $\omega$ that are $\leq_{RK}$-minimal (see [1]). We show that an analogous fact is true for the stronger preorder $\leq$. To prove this, we state the following generalization of the Ramsey theorem, which may be of an independent interest: for any $n \in \omega$ and $n$-ary map $f : \omega^n \to \omega$ there exist $m \leq n$, $K \in [n]^m$, and an infinite $A \subseteq \omega$ such that for increasing $n$-tuples $x_0 < \ldots < x_{n-1}$ and $y_0 < \ldots < y_{n-1}$ in $A$ we have $f(x_0, \ldots, x_{n-1}) = f(y_0, \ldots, y_{n-1})$ iff $x_k = y_k$ for all $k \in K$. (The case $m = 0$ gives a constant map.)

Finally, we expand the hierarchy from the relations $R_n$ to $R_\alpha$ for arbitrary ordinals $\alpha$, and strengthen some of obtained results. In particular, we show that $R_\alpha$ are increase by inclusion as $\alpha$ grows, and included in the Comfort preorder. On $\beta\omega$, all $R_\alpha$ with $\alpha < \omega_1$ are distinct, and $R_{<\alpha} = \bigcup_{\xi<\alpha} R_\xi$ is a preorder whenever $\alpha$ is a multiplicatively indecomposable ordinal.

References


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