ALGEBRAIC GEOMETRY. FINAL EXAM PROBLEMS

For the full credit, please, write complete solutions of any two problems from the list below and send me a scan of your paper by 9 pm, Tuesday, March 26.

Problem 1. Let $X \subset \mathbb{P}^n$, $n > 1$, be a smooth hypersurface of degree d.
(a) Compute $\dim H^0(X, \omega_X)$. Here ω_X denotes the sheaf of top degree differential forms on X.
(b) Prove that if $d > n$ then X is not rational.
(c) Show that if $d = n + 1$ then $\omega_X = \mathcal{O}_X$.

Problem 2. Let X be a projective scheme over a field. For a coherent sheaf E, its Euler characteristic $\chi(E)$ is defined to be

$$\chi(E) := \sum_{i=0}^{\infty} \dim H^i(X, E).$$

(a) Show that, for any short exact sequence $0 \to E_0 \to E_1 \to E_2 \to 0$ of coherent sheaves, one has that

$$\chi(E_0) - \chi(E_1) + \chi(E_2) = 0.$$

(b) Let X be a smooth projective connected curve over an algebraically closed field k. Recall that the genus of X is set to be $g_X := \dim H^0(X, \omega_X)$. Let E be a line bundle over X. Show that $\chi(E) = 1 - g_X + \deg E$. This is the Riemann-Roch formula. (Hint: use Serre's duality to check it for $E = \mathcal{O}_X$. Then use part (a).)
(c) In notation of part (b) show that $\deg \omega_X = 2g_X - 2$.

Problem 3. Let X, Y be smooth projective connected curves over an algebraically closed field k. Assume that $g_X < g_Y$. Show that every morphism $f : X \to Y$ is constant. (Hint: if $\operatorname{char} k = 0$ the proof is easier.)

Problem 4. Let X and Y be smooth connected projective varieties over a field k. Assume X and Y are birationally equivalent and that X has k-point. Show that Y has a k-point. (Hint: there are dense open subschemes $U \subset X$, $W \subset Y$, and an isomorphism $U \xrightarrow{\sim} W$. Let $x \in X(k)$. Show that there is a curve on X passing through x which is smooth at x, and which intesects U.)

Problem 5. (a) Let V be a finite-dimensional vector space over a field k, $S(V)$ the symmetric algebra (*i.e.*, the algebra of polynomials on V^*). Show that $Ext^{\cdot}_{S(V)}(k, k)$ as an algebra (with respect to the Yoneda product) is isomorphic to $\bigwedge^{\cdot} V^*$. Here k is considered as a module over $S(V)$ such that $V\,k = 0$.
(b) Let X be a smooth scheme over a field k, $i : X \to X \times X$ the diagonal embedding. Show that

$$\mathcal{E}xt^q_{\mathcal{O}_{X\times X}}(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \xrightarrow{\sim} i_*(\bigwedge^q T_X),$$

where T_X is the tangent sheaf on X.

Problem 6. Suppose we have a coherent sheaf E on a projective scheme X over a field and that we have an endomorphism $f : E \to E$.
(a) Show that if f is injective then it is an isomorphism.
(b) Give a counterexample to the above statement if X is not projective.