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CONSENSUS IN SOCIAL NETWORKS: REVISITED

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We analyze the convergence of opinions or beliefs in a general social network with non-Bayesian agents. We provide a new sufficient condition under which opinions converge to consensus and the condition is significantly more permissive than that of Lorenz (2005). This condition, which depends on properties of the network, requires agents to incorporate others' opinions into their own posterior sufficiently often.

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1 Introduction

Social networks are an important source of information for individuals and firms. The emergence of social media has led to an unprecedented level of information sharing among "friends," i.e. those who are connected and communicate. Given this, should one expect people to agree in the long run? We provide a new sufficient condition under which non-Bayesian agents in a given network converge to consensus.

In our model, agents update their opinions based on the prior opinions of their friends (and potentially themselves). Though we focus on opinions, the model can accommodate any variable on a convex set, where the convex hull of initial values is compact. For instance, instead of an opinion – a subjective probability – agents may update their belief about the value of an unknown parameter or adopt a cultural norm.

Literature on non-Bayesian learning beginning with DeGroot (1974) has agents updating their beliefs to a weighted average of their friends' beliefs. Lorenz (2005) provides a generalization of the DeGroot model by allowing the weights to depend on time and prior beliefs. The level of generality allows for many types of updating behaviour, including those that exhibit optimism or pessimism (over-weighting or under-weighting), and cognitive dissonance (giving a higher weight to those with similar beliefs). He demonstrates that aperiodic and strongly connected networks reach agreement if the weight one gives to a friend's opinion is bounded away from 0 by a positive number.[1] We provide a more permissive sufficient condition than that of Lorenz (2005). Roughly speaking, our result says that consensus is achieved unless some agents rely with an increasingly "faster" rate on their friends with the minimal opinion while some others on those with the maximal opinion.

DeMarzo et. al. (2003) considers a time-varying social network in which the agents weigh themselves differently over time. They show that opinions

[1]In particular, Lorenz (2005) requires for each agent i that if there exists y and τ such that $w_{ij}^{\tau}(y) > 0$ then there exists a $\delta > 0$ such that $w_{ij}^{t}(x) \geq \delta$ for all x and t.

converge when agents weigh other people's opinions "often enough." Our result is related to DeMarzo et. al. (2003)'s, and the two are equivalent for complete networks. In non-complete networks our condition is more restrictive. However, our condition is applicable in a wide range of environments while DeMarzo et. al. (2003)'s condition is not applicable outside of their specific model.

Mueller-Frank (2013) considers a general class of time-varying updating rules that includes rules with belief-dependent weights. The main conditions for convergence to consensus are (i) updating rules must satisfy continuity and have posteriors be strictly in between the most extreme priors in one's neighborhood and (ii) the periodwise updating functions must be of finite type. Our result does not require updating rules to be continuous or be of finite type. The assumption of continuity is especially strong in environments with endogenous network formation (Kivinen, 2017).

Several results on Bayesian updating in groups (Aumann, 1976; Geanakoplos and Polemarkis, 1982) highlight the role of common knowledge and common priors in generating consensus. When agents communicate in a network, common knowledge of the network structure is also required (Mueller-Frank, 2014). There is a subtle difference between the models on Bayesian and non-Bayesian updating. In the former, the agents have priors regarding some parameter as well as private information. Based on the agents' observed actions (which could involve revealing one's posteriors), each updates one's own prior. Here, the consensus occurs if the private information becomes "public" as time progresses. The focus is on whether agents eventually agree, and whether they learn the underlying data generating process.

The common knowledge assumption is demanding, and the agents require a powerful calculating ability to properly tease out the sources of information. In models on non-Bayesian updating the agents reveal their prior to each other, leading to an update. Private information spreads through the network but information may not be aggregated perfectly due to the lack of rationality on the agents' part. Thus, consensus may be reached but the outcome is not necessarily the same as if the private information was pooled. Molavi et. al. (2018) considers "quasi-Bayesian" learning which we consider in Section 4.

The paper is structured as follows. Next we introduce preliminary concepts and results. Section 3 contains our main results. We conclude with a discussion, which includes examples and additional results. Proofs are found in the Appendix.

2 Preliminaries

2.1 Networks and Communication

A finite set $A = \{1, \cdots, a\}$ of agents interact with each other. Each agent $i \in A$ listens to a fixed group of agents – agent i's (1-) neighborhood. A function $C : A \to 2^A$ identifies each agent's neighborhood. Specifically, $C(i)$ is i's neighborhood and it does not necessarily include i. If some agent j is in agent i's neighborhood, we say j is i's neighbor. A pair $\langle A, C\rangle$ is a network. For any integer $k \geq 2$, we iteratively define agent i's k-neighborhood $C^k(i)$ as follows: $C^k(i) = \cup_{j \in C(i)} C^{k-1}(j)$.

We say that agent j communicates to i if there exist a natural number k such that $j \in C^k(i)$. Whenever j communicates to i, the *distance* from agent j to i is $d_{ij} \equiv \min\{k \in \mathbb{Z}_+ : j \in C^k(i)\}$. Note here that d_{ij} could be different than d_{ji}.

Definition 1. *Network $\langle A, C\rangle$ is* irreducible *if every agent in A communicates with all the agents including herself.*

The diameter of a network $d(A, C)$ is the maximal distance from any agent to another, i.e., $d(A, C) = \max_{i,j \in A} d_{ij}$. A sequence of agents $i_1, i_2, \cdots, i_k$ is a *cycle* if for each $l = 1, \cdots, k$, i_{l+1} is i_l's neighbor where $i_{k+1} = i_1$. The *length* of a cycle is the number of agents in the cycle. For each agent $i \in A$, we define $V_i(A, C)$ as the lengths of cycles containing i. Formally, $V_i(A, C) \equiv \{k \in \mathbb{Z}_+ : i \in C^k(i)\}$. Notice that $V_i(A, C)$ is closed under addition: $p, q \in V_i(A, C)$ implies $p + q \in V_i(A, C)$. In other words, if i belongs to a cycle of length p and a cycle of length q then i also belongs to a cycle of length $p + q$.

Definition 2. *A network $\langle A, C \rangle$ is* aperiodic *if for each agent i, the greatest common divisor of numbers in $V_i(A, C)$ is 1.*

Let us fix a network $\langle A, C \rangle$ which is irreducible and aperiodic. We use the following notation:

$$\theta \equiv \arg\min_k \{k \in \mathbb{Z}_+ | C^\kappa(i) = A, \forall i \in A, \forall \kappa \geq k\}.$$

It is well-known that θ exists for irreducible, aperiodic networks. We start with a broad class of networks in which each agent is in her own neighborhood. In real world applications, it is hard to imagine an agent who completely disregards her own opinion.

Proposition 1. *Consider any irreducible, acyclic network $\langle A, C \rangle$ such that $i \in C(i)$ for all i. Then $\theta = d(A, C)$.*

Proof. By the definition of $d(A, C)$, it must be that $\theta \geq d(A, C)$. Fix any $i, j \in A$. By definition, $j \in C^{d_{ij}}(i)$. Because $i \in C(i)$, $j \in C^k(i)$ for all $k \geq d_{ij}$. Furthermore, $d_{ij} \leq d(A, C)$ and i, j are selected arbitrarily. Thus, $\theta = d(A, C)$. □

For networks in which some agent is not in her own neighborhood, we can only identify a bound. To do so, let us define $c_i^* \equiv \min_k \{k | k, k+1 \in V_i(A, C)\}$ which is known to exist.[2]

Proposition 2. *For any irreducible, acyclic network $\langle A, C \rangle$*

$$\theta \leq \max_{i \in A}(\max_{j \in A} d_{ij} + c_i^*(c_i^* - 1)).$$

[2]See Kemeny et. al. (1966) for a proof of this result.

The proposition above provides a bound on θ for any network. This bound is not tight. To see this, consider a network in which $C(i) = A \setminus \{i\}$ and $C(j) = A$ for all $j \neq i$. One can show that $c_i^* = 2$ and $d_{ii} = 2$. Consequently, the bound is 4 but $\theta = 2$.

2.2 Beliefs and Updating

An opinion/belief of the agents is an a-dimensional vector $x \in [0,1]^a$ where x_i is agent i's opinion about some parameter. We use the following conventional notation: for each $i \in A$, $x_{-i} \equiv (x_j)_{j \neq i}$ and $x \equiv (x_i, x_{-i})$.

Time is discrete and starts at period 0. At the initial period, the agents have an exogenously given opinion, and they exchange their opinions according the network structure. Afterwards they update their opinions which become the following period's initial opinions. In the following period, the agents again exchange and update their opinions. The process repeats every period. We formalize this opinion updating process by introducing an (opinion) updating function $T : \mathbb{N} \times [0,1]^a \to [0,1]^a$ where $\mathbb{N}$ is the set of non-negative integers. Agent i's updating function is T_i and the process is a Markov chain. If the opinion is x in period t then $T(t,x)$ is the opinion in period $t+1$. We will sometimes use the notation $T^{t,1}(x)$ for $T(t,x)$ and iteratively define $T^{t,k}(x)$ as $T(t+k-1, T^{t,k-1}(x))$ for all integer $k \geq 2$. In words, $T^{t,k}(x)$ is the vector of opinions in period $t+k$ when the period t vector of opinions is x.

We are interested in how the agents' opinions evolve in the long-run. In this sense, the main focus of our study is the properties of $T^{\infty}(x) \equiv \lim_{k \to \infty} T^{0,k}(x)$ when it is well-defined. We say a network reaches consensus if $T_i^{\infty}(x) = T_j^{\infty}(x)$ for all x, i and j.

As we indicated before, the network structure must affect the updating function. Specifically, we assume that (i) one's opinion is not affected by the opinions of those who are not in the agent's neighborhood, i.e., for each x and $\bar{x}_{-C(i)}$, $T_i(t,x) = T_i(t, x_{C(i)}, \bar{x}_{-C(i)})$ for all $t \geq 0$, and (ii) if agent j is i's neighbor then j's opinion affects i's in some cases, i.e., for each $j \in C(i)$, there exists x and $\bar{x}_j$, and t such that $T_i(t,x) \neq T_i(t, \bar{x}_j, x_{-j})$. We sometimes refer to $T(t,\cdot)$ as the period-t updating function.

We assume that no agent updates her opinion outside of the extremal opinions of her neighbors.

Assumption 1. $T_i(t,x) \in [\min_{j\in C(i)} x_j, \max_{j\in C(i)} x_j]$ *for all* i *and* x.

Unless stated otherwise, Assumption 1 holds for the rest of this paper. Next we present some examples of updating functions, each of which satisfies Assumption 1.

$$T_i(t,x) = \left(\sum_{j\in C(i)} w_{ij}^t x_j^p\right)^{\frac{1}{p}} \tag{1}$$
$$\text{where } w_{ij}^t > 0, \sum_{j\in C(i)} w_{ij}^t(x) = 1$$
$$T_i(t,x) = \lambda^t \left(\sum_{j\in C(i)} w_{ij} x_j\right) + (1-\lambda^t)x_i \tag{2}$$
$$\text{where } w_{ij} > 0, \sum_{j\in C(i)} w_{ij} = 1, \lambda^t \in [0,1]$$
$$T_i(t,x) = \frac{\Pi_{j\in C(i)} x_j^{w_{ij}^t}}{\Pi_{j\in C(i)}(1-x_j)^{w_{ij}^t} + \Pi_{j\in C(i)} x_j^{w_{ij}^t}} \tag{3}$$
$$\text{where } w_{ij}^t > 0, \sum_{j\in C(i)} w_{ij}^t = 1$$
$$T_i(t,x) = \sum_{j\in C(i)} w_{ij}^t(x) x_j \tag{4}$$
$$\text{where } w_{ij}^t(x) > 0, \sum_{j\in C(i)} w_{ij}^t(x) = 1$$

The updating rule in (1) is a (weighted) L_p-norm of opinions. Notice that the weights, w_{ij}^t, vary over time. When $p = 1$ and w_{ij}^t is time invariant, this rule reduces to the one in DeGroot (1974). The updating rule in (2) is considered in DeMarzo et. al. (2003). This updating function has a very specific structure: the time-varying weight is on a constant group of friends and one's own prior. This is equivalent to varying inertia in opinions.

The updating rule in (3) is considered by Molavi et. al. (2018)[3] with time-varying weights. They study the foundations of social learning using an axiomatic approach. This updating functions is "more Bayesian" than the standard DeGroot one in the sense that it violates fewer properties of a Bayesian updating function. Lorenz (2005) studies rule (4), and notice here that the weights, $w_{ij}^t(\cdot)$, vary over time and are a function of current opinions. It is easy to see that any updating function can be written in the form of (4).

Lorenz (2005) considers the following condition: if $w_{ij}^t(x) > 0$ for some $t \geq 0$ and x then $w_{ij}^t(x) \geq \delta > 0$ for all $t \geq 0$ and x, for all $i \in A$. It is shown that if this condition is satisfied then agents' opinions converge to consensus in the long run (assuming an irreducible and aperiodic network). This sufficient condition is not satisfied for (2) when $\lambda^t \to 0$, or for (1) when $w_{ij}^t \to 0$ for some i and $j \in C(i)$. However, in these cases consensus is sometimes reached. We will introduce a general condition that subsumes Lorenz's.

3 Main Results

3.1 Sufficiency

To introduce our condition, we need to define the following two variables:

$$\alpha_i^t(x) = \begin{cases} 1 & \text{if } |C(i)| = 1 \text{ or if } \max_{j \in C(i)} x_j = \min_{j \in C(i)} x_j \\ \frac{T_i(t,x) - \min_{j \in C(i)} x_j}{\max_{j \in C(i)} x_j - \min_{j \in C(i)} x_j} & \text{in all other cases} \end{cases}$$

[3]In Malavi et. al. (2018) the weights w_{ij}^t are time-independent, and $\sum_{j \in C(i)} w_{ij}^t$ need not equal 1.

and

$$\beta_i^t(x) = \begin{cases} 1 & \text{if } |C(i)| = 1 \text{ or if } \max_{j \in C(i)} x_j = \min_{j \in C(i)} x_j \\ \frac{\max_{j \in C(i)} x_j - T_i(t,x)}{\max_{j \in C(i)} x_j - \min_{j \in C(i)} x_j} & \text{in all other cases} \end{cases}.$$

Observe here that

$$\begin{aligned} T_i(t,x) &= (1 - \alpha_i^t(x)) \min_{j \in C(i)} x_j + \alpha_i^t(x) \max_{j \in C(i)} x_j \\ &= \beta_i^t(x) \min_{j \in C(i)} x_j + (1 - \beta_i^t(x)) \max_{j \in C(i)} x_j. \end{aligned}$$

If we think of $T_i(t,x)$ as the convex combination of the extremal opinions in i's neighborhood, then $\alpha_i^t(x)$ and $\beta_i^t(x)$ are the weights i places on the maximal and minimal opinions, respectively.

Let $\underline{\alpha}^t$ be the lowest weight given by any agent to the maximal opinion in her neighborhood, i.e., $\underline{\alpha}^t \equiv \inf_{i \in A \& x \in [0,1]^a} \alpha_i^t(x)$. In addition, for any integer $k \geq 1$, let $\underline{\alpha}^{t,k} \equiv \prod_{\tau=t}^{t+k-1} \underline{\alpha}^\tau$. Similarly, we define $\underline{\beta}^t$ and $\underline{\beta}^{t,k}$. Observe here that $\underline{\alpha}^{t,k} + \underline{\beta}^{t,k} \leq 1$ for all integers $t \geq 0$ and $k \geq 1$ in irreducible networks because $\alpha_i^t(x) = 1 - \beta_i^t(x)$ for all i and x with $\max_{j \in C(i)} x_j \neq \min_{j \in C(i)} x_j$.

In the lemma below, we consider how the extremal opinions behave.

Lemma 1. *Let $\langle A, C \rangle$ be an irreducible, aperiodic network. Then for all x and $t \geq 0$,*

$$\max_{j \in A} T_j^{t,\theta}(x) - \min_{j \in A} T_j^{t,\theta}(x) \leq (1 - \underline{\alpha}^{t,\theta} - \underline{\beta}^{t,\theta}) \left(\max_{j \in A} x_j - \min_{j \in A} x_j \right).$$

If the network is complete, i.e., if $C(i) = A$ for all $i \in A$, then the definitions of $\underline{\alpha}^t$ and $\underline{\beta}^t$ give the lemma above with $\theta = 1$. In non-complete networks, the intuition behind the lemma is as follows: because the network is irreducible and aperiodic, all the agents communicate with one another after θ periods. This means that both maximal and minimal (initial) opinions affect each agent's opinion in θ periods. The lowest weight one assigns to the maximal opinion in her neighborhood in period τ is $\underline{\alpha}^\tau$. Thus, each agent must assign at least the weight of $\underline{\alpha}^{t,\theta}$ to the period-t maximal opinion in the whole network after θ periods. Thus, $T_i^{t,\theta}(x) \geq (1 - \underline{\alpha}^{t,\theta}) \min_{i \in A} x_j + \underline{\alpha}^{t,\theta} \max_{i \in A} x_j$ for all i. A similar logic yields that $T_i^{t,\theta}(x) \leq \underline{\beta}^{t,\theta} \min_{j \in A} x_j + (1 - \underline{\beta}^{t,\theta}) \max_{j \in A} x_j$. By rearranging terms, we obtain that between periods t and $t+\theta$, the distance between extremal opinions shrinks at least by $\underline{\alpha}^{t,\theta} + \underline{\beta}^{t,\theta}$ fraction.

Theorem 1. *Let $\langle A, C \rangle$ be an irreducible, aperiodic network. Then consensus is reached if there exists a sequence $\{t_k\}$ such that (i) $t_{k+1} - t_k \geq \theta$ for all k and (ii)*

$$\lim_{\tau \to \infty} \sum_{k=1}^{\tau} (\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}) = +\infty$$

To prove this theorem, note that the extremal opinions in the network cannot move further apart over time because (by Assumption 1) no agent's updated opinion falls outside of the interval formed by the extremal opinions in the agent's neighborhood. The lemma preceding the theorem means that after τ blocks of θ periods (where block k starts at period t_k), the extremal opinions will be at most $\prod_{k=1}^{\tau}(1 - \underline{\alpha}^{t_k,\theta} - \underline{\beta}^{t_k,\theta})$ fraction of the distance between extremal opinions in the initial period. We complete the proof by showing that this maximal fraction goes to 0 as the number of blocks increases as long as the sum of $(\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta})$ over k converges to infinity.

Checking our condition could be somewhat impractical because one has to look for a sequence $\{t_k\}$ with a certain characteristics. However, because we provide only a sufficient condition, one may want to impose the restriction that the blocks of θ periods must be consecutive. Specifically, if $\lim_{\tau \to \infty} \sum_{k=0}^{\tau} (\underline{\alpha}^{k\theta,\theta} + \underline{\beta}^{k\theta,\theta}) = \infty$ then consensus is reached. Although this new sufficient condition is more practical than the original, it is narrower in scope.

Our sufficient condition means that unless some agents rely on the minimal opinion while others on the maximal opinion at an increasingly "faster rate," consensus is reached in irreducible, aperiodic networks. It is easy to see that our condition is significantly more general than that of Lorenz (2005). He considers updating functions in the form of (4) and requires conditions on each weight $w_{ij}^t(x)$ to guarantee consensus. Our condition is only in terms of the weights assigned to the extremal opinions. In fact, as long as one of these is bounded below or is converging to 0 slowly then our condition is satisfied. Consequently, our condition subsumes the condition from Lorenz (2005).

It is also easy to see that consensus occurs in the long term if $\underline{\alpha}^{t,\theta} + \underline{\beta}^{t,\theta} = 1$ for some t. In non-complete networks, this condition requires that either everyone updates her opinion to the maximal one in each period between t and $t + \theta$ or everyone to the minimal one. In complete networks, the condition could mean one more scenario in which everyone weighs the maximal and minimal opinions in the same way. Finally, we note here that our sufficient condition is satisfied when at least one of the following conditions are satisfied: $\sum_{k=1}^{\infty} \underline{\alpha}^{t_k,\theta} = +\infty$ or $\sum_{k=1}^{\infty} \underline{\beta}^{t_k,\theta} = +\infty$.

A simple corollary follows from Theorem 1.

Corollary 1. *Suppose the network is complete, i.e., $C(i) = A$ for all $i \in A$. Then convergence to consensus occurs if* $\lim_{t\to\infty} \sum_{k=1}^{t} (\underline{\alpha}^k + \underline{\beta}^k) = +\infty$.

The corollary follows from the fact that complete networks have $\theta = 1$. This weaker condition is essentially the condition of DeMarzo et. al. (2003), though applicable to a larger set of updating functions. We will return to this issue in Section 4.2.

4 Discussion

4.1 Sufficient Condition in Specific Models

We now consider how our condition translates to specific updating functions we considered in the previous section. The next example is a generalization of a weighted average.

Example 1 (L_p-updating function)**.** *If every agent has the same updating function in (1), then the weights do not depend on the current opinion. Thus, let* $\underline{w}_i^t \equiv \min_{j\in C(i)} w_{ij}^t$*,* $\underline{w}^t \equiv \min_{i\in A} \underline{w}_i^t$*, and* $\underline{w}^{t,\theta} \equiv \prod_{\tau=t}^{t+\theta-1} \underline{w}^\tau$ *for all* $t \geq 0$*. In this case, our sufficient condition is satisfied if there exists* $\{t_k\}$ *with* $t_{k+1} - t_k \geq \theta$ *and* $\sum_k \underline{w}^{t_k,\theta} = \infty$*. To see this, observe that when* $p = 1$ *we have that* $\underline{\alpha}^t = \underline{\beta}^t = \underline{w}^t$*. Thus,* $\sum_k \underline{w}^{t_k,\theta} = \infty$ *is equivalent to* $\sum_k(\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}) = \infty$*. Let* $p \in (0,1)$*. Then we know that by Jensen's inequality,*

$$\sum_{j\in C(i)} w_{ij}^t x_j^p \leq \left(\sum_{j\in C(i)} w_{ij}^t x_j\right)^p.$$

Subsequently,

$$T_i(t,x) = \left(\sum_{j\in C(i)} w_{ij}^t x_j^p\right)^{1/p} \leq \sum_{j\in C(i)} w_{ij}^t x_j \leq \underline{w}^t \min_{j\in C(i)} x_j + (1-\underline{w}^t) \max_{j\in C(i)} x_j.$$

Thus, $\sum_k \underline{w}^{t_k,\theta} = \infty$ *implies that* $\sum_k \underline{\beta}^{t_k,\theta} = \infty$*. A similar proof works for the* $p > 1$ *or* $p < 0$ *cases.*

Theorem 1 is not necessarily useful for Bayesian models, as Bayesian updating functions are often difficult to characterize and there exists a large literature characterizing Bayesian consensus under common knowledge and common priors. However, Molavi et. al. (2018) provides a model of quasi-Bayesian updating in which Theorem 1 can be useful. The following example illustrates that if $x^0 \in [0,1)^a$ or $x^0 \in (0,1]^a$ (but not the union)[4] and Assumption 1 holds then Theorem 1 applies.

Example 2 (Quasi-Bayesian Updating)**.** *If every agent's updating function is the form of (3), then the weights do not depend on the current opinion. In this case, we will show that, unless the initial opinions satisfy both*

[4]Cromwell's rule states that subjective beliefs should always be in $(0,1)$. This is a standard assumption in applying Bayes's rule.

$\min_{i\in A} x_i^0 = 0$ *and* $\max_{i\in A} x_i^0 = 1$*, consensus is reached. Clearly, if either* $\min_{i\in A} x_i^0 = 0$ *or* $\max_{i\in A} x_i^0 = 1$ *(but not both) then opinions converge to 0 or 1, respectively (for irreducible, aperiodic networks). Thus, let us concentrate on opinions where* $0 < \min_{i\in A} x_i^0 < \max_{i\in A} x_i^0 < 1$.

As in the previous example let us define $\underline{w}^t$ *and* $\underline{w}^{t,\theta}$ *for all* $t \geq 0$*. In this case, our sufficient condition is satisfied if there exists* $\{t_k\}$ *with* $t_{k+1} - t_k \geq \theta$ *and* $\sum_k \underline{w}^{t_k,\theta} = \infty$.

To prove this, let $Z_i^t = \frac{x_i^t}{(1-x_i^t)}$ *and* $z_i^t = lnZ_i^t$*. The updating function can be rewritten as* $Z_i^{t+1} = \Pi_{j\in C(i)}(Z_j^t)^{w_{ij}^t}$ *and therefore:*

$$z_i^{t+1} = \sum_{j\in C(i)} w_{ij}^t z_i^t$$

Notice that this has the same structure as time-varying DeGroot (1974), which is a special case of (1). The only difference is that $z_i^t \in (-\infty, +\infty)$*, which is not a compact set. However, our proof for Theorem 1 is valid when* $[\min_{i\in A} z_i^0, \max_{i\in A} z_i^0]$ *is a compact set, which occurs when* $\min_{i\in A} x_i^0$, $\max_{i\in A} x_i^0 \in (0,1)$*. Thus, consensus is reached as long as there exists* $\{t_k\}$ *with* $t_{k+1} - t_k \geq \theta$ *and* $\sum_k \underline{w}^{t_k,\theta} = \infty$ *as we have shown in Example 1.*

4.2 Necessity

A natural question is whether the sufficient condition is also necessary. Unfortunately, the answer to this question is negative. Below we present an example of a complete network which reaches consensus despite violating the sufficient condition we have identified.

Example 3 (Complete Network Consensus)**.** *There are only two agents 1 and 2 who behave as in the time-dependent DeGroot model. The weight one places on the other's opinion is a function of time and how different the* *opinions of the two agents are. Specifically, for* $i = 1, 2$, $j \neq i$ *and* x,

$$w_{ij}^t(x) = \begin{cases} \frac{3}{4} & \text{if } |x_i - x_j| \leq 0.5 \\ 1 - \frac{1}{4}\frac{1}{2^t} & \text{if } |x_i - x_j| > 0.5 \end{cases}$$

and

$$w_{ii}^t(x) = 1 - w_{ij}^t(x).$$

Observe here that $\underline{\alpha}^t = \underline{\beta}^t = \frac{1}{4}\frac{1}{2^t}$. *Hence,* $\sum_t(\underline{\alpha}^t + \underline{\beta}^t) < \infty$. *Observe here that thanks to DeGroot (1974), consensus will emerge if the agents' opinions become closer than 0.5. This happens by period 1 no matter how different the opinions were in period 0. Thus, consensus always emerges.*

The example demonstrates that our sufficient condition is not necessary for reaching consensus. One may argue that the sufficient condition can be adjusted to cover the example above which has a very specific structure. Indeed, after one period, all the initial opinions enter the interval on which our sufficient condition holds. One approach to identify the necessary conditions may involve (i) identifying intervals of opinions where our sufficient condition is satisfied and (ii) determining if the opinions outside the interval enter it after some periods. However, both seem to be highly arbitrary. Thus, such an approach does not appear to be fruitful for identifying the necessary conditions for reaching consensus in general models such as ours.

4.3 Counter-Example and Non-Convergence

Based on the previous subsection, one needs to explore if our sufficient condition can be weakened. We look for possible directions based on DeMarzo et. al. (2003) which identify both necessary sufficient conditions for reaching consensus in a specific setting. They consider the updating functions in the form of (2) and show that consensus is reached if $\sum_{t=1}^{+\infty} \lambda^t = +\infty$. Our condition would require the existence of a sequence $\{t_k\}$ with $t_{k+1} - t_k \geq \theta$ and $\sum_{k=1}^{+\infty} \lambda^{t_k,\theta} = \infty$ where $\lambda^{t,\theta} \equiv \prod_{\tau=t}^{t+\theta-1} \lambda^\tau$. Thus, our condition is more restrictive than that of DeMarzo et. al. (2003). The two conditions however are equivalent in complete networks. This observation raises the following question: can our condition be replaced in Theorem 1 by $\sum_{t=1}^{+\infty}(\underline{\alpha}^t + \underline{\beta}^t) = \infty$? Our answer is negative: the following counter-example demonstrates this.

Example 4. *There are four agents and agent 1 listens to agents 1 and 2, agent 2 to agents 1, 2 and 3, agent 3 to agents 2, 3 and 4, and agent 4 to agents 3 and 4. The updating functions are as follows (for $\epsilon < \frac{1}{4}$):*

$$T_i(t,x) = \begin{cases} (1-\delta_i^t)\min_{j\in C(i)} x_j + \delta_i^t \max_{j\in C(i)} x_j & \text{if } i = 1,2 \\ \delta_i^t \min_{j\in C(i)} x_j + (1-\delta_i^t)\max_{j\in C(i)} x_j & \text{if } i = 3,4 \end{cases}$$

where

$$\delta_1^t = \delta_4^t = \begin{cases} \frac{\epsilon}{2(2^{t-2}-(2^t-1)\epsilon)} & \text{if } t \text{ is even} \\ 0.5 & \text{if } t \text{ is odd} \end{cases}$$

and

$$\delta_2^t = \delta_3^t = \begin{cases} \frac{\epsilon}{2^{t-2}-(2^t-1)\epsilon} & \text{if } t \text{ is even.} \\ \frac{2^{t-2}-(2^t-1)\epsilon}{2^{t-1}-(2^{t+1}-1)\epsilon} & \text{if } t \text{ is odd} \end{cases}$$

Let us consider the sequence $\{T^{0,t}(0,0.5,0.5,1)\}$. One can calculate that

$$T^{0,t}(0,0.5,0.5,1) = \begin{cases} \left(\frac{2^t-1}{2^{t-1}}\epsilon, 2\epsilon, 1-2\epsilon, 1-\frac{2^t-1}{2^{t-1}}\epsilon\right) & \text{if } t \text{ is odd} \\ \left(\frac{2^t-1}{2^{t-1}}\epsilon, \frac{1}{2}, \frac{1}{2}, 1-\frac{2^t-1}{2^{t-1}}\epsilon\right) & \text{if } t \text{ is even.} \end{cases}$$

One can easily see that the first and last agent's opinion converges to 2ϵ and $1-2\epsilon$, respectively. However, the opinions of agents 2 and 3 do not converge.

Observe here that $\underline{\alpha}^t = \underline{\beta}^t = \min_{i=1,\cdots,4}\{\delta_i^t\}$. Furthermore, $\sum_t \underline{\alpha}^t = \sum_t \underline{\beta}^t = \infty$ because the even numbered $\underline{\alpha}^t$s and $\underline{\beta}^t$s converge to 0 while the odd numbered ones to 0.5. However, as we already mentioned above, the agents do not converge to a consensus. Our sufficient condition is not satisfied here. To see this, observe that $\theta = 3$ in this example. Thus, any three consecutive periods will have at least one odd period and $\underline{\alpha}^t$ and $\underline{\beta}^t$ decrease by 4th between *any two consecutive odd periods. Subsequently, whatever 3 period blocks we choose, both* $\underline{\alpha}^{t,\theta}$ *and* $\underline{\beta}^{t,\theta}$ *decrease at least by half between two blocks, which is a too fast of a decrease.*

To understand the mathematical structure of the example, consider a sequence of numbers $\{a_k\}_{k=1,2,3,...}$ comprising of two subsequences: $\{a_{2k-1}\}_{k=1,2,3,...}$ and $\{a_{2k}\}_{k=1,2,3,...}$. If it possible to choose these subsequences such that $\sum_{k=1}^{+\infty} a_{2k-1} = +\infty$ and $\sum_{k=1}^{+\infty} a_{2k} < +\infty$, and still have $\sum_{k=1}^{+\infty} a_{2k-1}a_{2k} < +\infty$, then one can violate the sufficient condition of Theorem 1 while keeping the weaker condition similar to DeMarzo et. al. (2003). In the example above $\{\underline{\alpha}^k\}_{k=1,2,3,...}$ and $\{\underline{\beta}^k\}_{k=1,2,3,...}$ satisfy an analogous condition.

To understand how the example works, notice that agents behave very differently depending on whether it is an odd period or and even period. Let us focus on agents 1 and 2 because their behavior is symmetric to other two's. Intuitively, agent 2 is bouncing between 0.5 and a point close to agent 1's opinion. When agent 2 is close to agent 1, then agent 1 puts a lot of weight on agent 2's prior. However, when agent 2 is at 0.5 then agent 1 almost ignores agent 2. In fact, agent 1 becomes more and more isolated over time *in odd period only.* Similarly, agent 2 puts a lot of weight on agent 1 when they are far apart but pays more attention to agent 3 when they are close. In this way, agent 2 and 3 act as counter-weight to one another.

It should be noted that Example 4 produces a type of convergence but not a consensus. Agents 2 and 3 each converge to a set of opinions that are cycled through, and these sets intersect with the limiting sets of other agents. Because agents 2 and 3 are the more centrist agents, this prevents the agents with extreme opinions from converging. This observation raises a question: can our sufficient condition be weakened to guarantee consensus if we assume convergence? The following theorem says that if opinions converge and $\sum_{t=1}^{+\infty} \underline{\alpha}^t + \underline{\beta}^t = +\infty$ then there exist two individuals who converge to agreement on an extreme opinion. Define x_i^* as agent i's limiting opinion, $\underline{x}^*$ as the minimum limiting opinion, and $\bar{x}^*$ as the maximum limiting opinion.

Theorem 2. *Let $\langle A, C\rangle$ be an irreducible network. If the opinion of each agent $i \in A$ converges to a singleton x_i^*, and $\sum_{t=1}^{\infty}(\underline{\alpha}^t + \underline{\beta}^t) = +\infty$ then there exists $j, k \in A$ such that $x_j^* = x_k^*$ and $x_j^*, x_k^* \in \{\underline{x}^*, \bar{x}^*\}$.*

The proof works because there is a gap between limiting points for every period after some $\bar{t}$ and extreme individuals are their own extreme prior. While the condition does not rule out disagreement altogether, it does establish that there cannot be complete disagreement (ie. $x_i^* \neq x_j^*$ for all $i, j \in A$). In conclusion, Theorem 2 allows us to rule out certain types of disagreement by weakening our original condition and assuming convergence.

4.4 Discontinuity

An important feature of Theorem 1 is that continuity of T is not required to guarantee consensus. Given that there are many applications that involve discontinuous updating functions, we explore the conditions under which discontinuity derails consensus. Mueller-Frank (2013) establishes conditions that, together with continuity, guarantee consensus. In particular, the updating function can be time variant but must be of only finite types, and must satisfy the following assumption.

Assumption 2. *T satisfies Assumption 1 and*

$$T_i(t, x) \in (\min_{j \in C(i)} x_j, \max_{j \in C(i)} x_j)$$

for all i and x such that $\underline{x}_i \neq \bar{x}_i$.

Assumption 2 is a stronger version of Assumption 1, which requires beliefs be *strictly* between the extreme beliefs among one's neighbours. The following example assumes that the updating function is time invariant and satisfies Assumption 2. However, it is discontinuous which leads to disagreement.

Example 5. *Let there be two agents, $A = \{1,2\}$, each of which has the following updating functions:*

$$T_1(x) = \begin{cases} \delta_1 \frac{1}{4} + (1-\delta_1)x_1 & \text{if } x_1 < \frac{1}{4} \text{ and } x_2 > \frac{3}{4} \\ 0.5x_1 + 0.5x_2 & \text{otherwise} \end{cases}$$

$$T_2(x) = \begin{cases} \delta_2 \frac{3}{4} + (1-\delta_2)x_2 & \text{if } x_1 < \frac{1}{4} \text{ and } x_2 > \frac{3}{4} \\ 0.5x_1 + 0.5x_2 & \text{otherwise} \end{cases}$$

Notice that, with the exception of continuity, the assumptions of Mueller-Frank (2013) hold. However, if $(\delta_1, \delta_2) \in (0,1)^2$ and the initial opinions $x^0 = (x_1^0, x_2^0)$ satisfy $x_1^0 < \frac{1}{4}$ and $x_2^0 > \frac{3}{4}$ then opinions converge to a non-consensus point $\left(\frac{1}{4}, \frac{3}{4}\right)$.

Example 5 demonstrates that a "small" amount of discontinuity can lead to non-consensus for a large class of initial opinions. Furthermore, it is clear that even though the updating function has a specific structure there are many updating functions (i.e., values of (δ_1, δ_2)) that can lead to non-consensus. Can a "smaller" amount of discontinuity prevent consensus? The minimal amount of discontinuity required to get non-consensus is for two agents' updating functions to be discontinuous. This follows from our condition $\sum_{k=1}^{+\infty} \underline{\alpha}^{k,\theta} + \underline{\beta}^{k,\theta} = +\infty$: one agent's discontinuity can lead $\sum_{k=1}^{+\infty} \underline{\alpha}^{k,\theta} < +\infty$ or $\sum_{k=1}^{+\infty} \underline{\beta}^{k,\theta} < +\infty$ but not both.

4.5 Conclusion

A general sufficient condition was established to guarantee convergence to consensus in a social network, and this condition was related to the properties of the network. It was demonstrated that the condition applies to many models currently used in the literature, and it collapses to a weaker condition if the network topology is restricted to a complete network. Furthermore, certain types of disagreement can be ruled out when the set of updating functions is restricted to those that converge.

Discontinuous updating is analyzed and it was shown that, when our condition is violated, consensus is not robust to small amounts of discontinuity. In particular, if there exists one discontinuous point on the updating functions of at least two agents then long-run disagreement can arise for some set of initial beliefs of positive measure.

Given that social networks exhibit the “small world” property (ie. small diameters), our results suggest that widespread disagreement implies that there are some people do not respond to others’ beliefs. This raises a question: how can one produce (stable) disagreement in a highly connected network with agents who listen to one another? We leave the answer to this question for future research.

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Appendix

Proof of Proposition 2. Fix any $i, j \in A$. By definition, $j \in C^{d}_{ij}(i)$ and $j \notin C^k(i)$ for any $k < d_{ij}$. Kemeny et. al (1966) show that any $k \geq c_i^*(c_i^* - 1)$ is in $V_i(A, C)$ or equivalently, $i \in C^k(i)$. Then $j \in C^{d_{ij}+k}(i)$ for all $k \geq c_i^*(c_i^* - 1)$ because $C^{d_{ij}+k}(i) = \cup_{\ell \in C^k(\ell)} C^{d_{ij}}(\ell)$. Because this is true for all i and j, we obtain

$$\theta \leq \max_{i \in A}(\max_{j \in A} d_{ij} + c_i^*(c_i^* - 1)).$$

□

To prove Lemma 1 we first introduce some notation and definitions. Let $\underline{T} : [0,1]^n \to [0,1]^n$ $\underline{T} : [0,1]^n \to [0,1]^n$ be a function such that

$$\underline{T}_i(t, x) = (1 - \underline{\alpha}^t) \min_{j \in C(i)} x_j + \underline{\alpha}^t \max_{j \in C(i)} x_j$$

for all i and x. We define $\underline{T}^{t+\tau,t}(x)$ in the same way as we defined $T^{t+\tau,t}(x)$. The following lemma plays a key role in the proof of Lemma 1.

Lemma 2. *(a) For all natural number $\tau \geq 1$ and $t \geq 0$, $\underline{T}^{t+\tau,t}(x)$ is monotonic.*

(b) For all natural number $k \geq 1$ and x, $T^{t,k}(x) \geq \underline{T}^{t,k}(x)$.

(c) Let $\langle A, C \rangle$ be irreducible and aperiodic. Then for any x and $j \in A$,

$$\min_{j \in A} T_j^{t,\theta}(x) \geq (1 - \underline{\alpha}^{t,\theta}) \min_{j \in A} x_j + \underline{\alpha}^{t,\theta} \max_{j \in A} x_j.$$

Proof. (a) Because

$$\underline{T}_i(t, x) = \underline{\alpha}^t \max_{j \in C(i)} x_j + (1 - \underline{\alpha}^t) \min_{j \in C(i)} x_j,$$

we have $\underline{T}(x) \geq \underline{T}(x^*)$ whenever $x \geq x^*$. Furthermore, the monotonicity of

$\underline{T}(\tau, x)$ for all τ and the definition of $\underline{T}^{t+\tau,t}(\cdot)$ imply that $\underline{T}^{t+\tau,t}(\cdot)$ is monotonic.

(b) By the definition of $\underline{T}(t, x)$, we have that $T(\tau, y) \geq \underline{T}(\tau, y)$ for all non-negative natural number τ and y. Subsequently, $T(t, x) \geq \underline{T}(t, x)$ and $T(t+1, T(t, x)) \geq \underline{T}(t+1, T(t, x))$ for all t. By combining these with the monotonicity of $\underline{T}(t, \cdot)$, we obtain that

$$T^{t,2}(x) = T(t+1, T(t, x)) \geq \underline{T}(t+1, T(t, x)) \geq \underline{T}(t+1, \underline{T}(t, x)) = \underline{T}^{t,2}(x).$$

One can extend the argument above and obtain that

$$T^{t,k}(x) \geq \underline{T}^{t,k}(x)$$

for each natural number $k \geq 1$.

(c) Recall that θ satisfies the following condition: $j \in C^{\theta}(i)$ for all $i, j \in A$. We now show that for any x,

$$\min_{j \in A} \underline{T}_j^{t,\theta}(x) \geq (1 - \underline{\alpha}^{t,\theta}) \min_{j \in A} x_j + \underline{\alpha}^{t,\theta} \max_j x_j.$$

Let $\bar{i}$ be an agent for whom $x_{\bar{i}} = \max_{j \in A} x_j$. We know that each $i \in A$ listens to $\bar{i}$ in θ steps, i.e., $\bar{i} \in C^{\theta}(i)$. Let y be an opinion such that $y_i = \min_{j \in A} x_j$ for all $i \neq \bar{i}$ and $y_{\bar{i}} = x_{\bar{i}} = \max_{j \in A} x_j$. Clearly, $x \geq y$. Thus, by the monotonicity of $\underline{T}^{t,\tau}(\cdot)$,

$$\underline{T}^{t,\tau}(x) \geq \underline{T}^{t,\tau}(y)$$

for all τ. We now concentrate on $\underline{T}(t, y)$. If i does not listen to $\bar{i}$ (i.e., if $\bar{i} \notin C(i)$), then $\underline{T}_i(t, y) = \min_{j \in A} x_j$. On the other hand, if i listens to only $\bar{i}$ (i.e., $\{\bar{i}\} = C(i)$), then $\underline{T}_i(t, y) = y_{\bar{i}} = \max_{j \in A} x_j$. If i listens to some other agents in addition to $\bar{i}$ (i.e., $\{\bar{i}\} \subset C(i)$), then

$$\begin{aligned}\underline{T}_i(t,y) &= (1-\underline{\alpha}^t) \min_{j \in C(i)} y_j + \underline{\alpha}^t \max_{j \in C(i)} y_j \\ &= (1-\underline{\alpha}^t) \min_{j \in A} x_j + \underline{\alpha}^t \max_{j \in A} x_j.\end{aligned}$$

Let y^1 be an opinion such that $y_i^1 = \min_{j \in A} x_j$ if $\bar{i} \notin C(i)$ and $y_i^1 = (1-\underline{\alpha}^t) \min_{j \in A} x_j + \underline{\alpha}^t \max_{j \in A} x_j$ if $\bar{i} \in C(i)$. Observe that $\underline{T}(t,y) \geq y^1$ for all i. Thus, by the monotonicity of $\underline{T}^{t,\tau}(\cdot)$ for all τ, $\underline{T}^{t,2}(x) \geq \underline{T}^{t,2}(y) \geq \underline{T}(t+1,y^1)$. We now turn our attention to $T(t+1,y^1)$. If i does not listen to $\bar{i}$ in two steps (i.e., if $\bar{i} \notin C^2(i)$), then $\underline{T}_i(t+1,y^1) = \min_{j \in A} x_j$. On the other hand, if i listens to $\bar{i}$ in two steps (i.e., $\bar{i} \in C^2(i)$), then

$$\begin{aligned}\underline{T}_i(t+1,y^1) &= (1-\underline{\alpha}^{t+1}) \min_{j \in C(i)} y_j^1 + \underline{\alpha}^{t+1} \max_{j \in C(i)} y_j^1 = \\ &\geq (1-\underline{\alpha}^{t+1}) \min_{j \in A} x_j + \underline{\alpha}^{t+1}((1-\underline{\alpha}^t) \min_{j \in A} x_j + \underline{\alpha}^t \max_{j \in A} x_j) \\ &= (1-\underline{\alpha}^{t,2}) \min_{j \in A} x_j + \underline{\alpha}^{t,2} \max_{j \in A} x_j\end{aligned}$$

Let y^2 be an opinion such that $y_i^2 = \min_{j \in A} x_j$ if $\bar{i} \notin C^2(i)$ and $y_i^2 = (1-\underline{\alpha}^{t,2}) \min_{j \in A} x_j + \underline{\alpha}^{t,2} \max_{j \in A} x_j$ if $\bar{i} \in C^2(i)$. Observe that $\underline{T}(t+1,y^1) \geq y^2$. Thus, by the monotonicity of $\underline{T}^{t,\tau}(\cdot)$ for all τ, $\underline{T}^{t,3}(x) \geq \underline{T}^{t,3}(y) \geq \underline{T}^{t+1,2}(y^1) \geq \underline{T}(t+2,y^2)$. We now turn our attention to $\underline{T}(t+2,y^2)$. If i does not listen to $\bar{i}$ in three steps (i.e., if $\bar{i} \notin C^3(i)$), then $\underline{T}_i(t+2,y^2) = \min_{j \in A} x_j$. On the other hand, if i listens to $\bar{i}$ in three steps (i.e., $\bar{i} \in C^3(i)$), then

$$\begin{aligned}\underline{T}_i(t+2,y^2) &= \min_{j \in C(i)} y_j^2 + \underline{\alpha}^{t+2} \left(\max_{j \in C(i)} y_j^2 - \min_{j \in C(i)} y_j^2 \right) \\ &\geq (1-\underline{\alpha}^{t,3}) \min_{j \in A} x_j + \underline{\alpha}^{t,3} \max_{j \in A} x_j.\end{aligned}$$

By following the same procedure iteratively, let us define $y_i^{\theta-1}$. Observe that $\underline{T}(t+\theta-2, y^{\theta-2}) \geq y^{\theta-1}$. Thus, by the monotonicity of $\underline{T}^{t,\tau}$ for all τ, $\underline{T}^{t,\theta}(x) \geq \underline{T}^{t,\theta}(y) \geq \underline{T}^{t+1,\theta-1}(y^1) \geq \cdots \geq \underline{T}^{t+\theta-1,1}(y^{\theta-1}) = \underline{T}(\theta-1, y^{\theta-1})$. We now turn our attention to $T(\theta-1, y^{\theta-1})$. We know that each i listens to $\bar{i}$ in θ periods. Thus,

$$
\begin{aligned}
\underline{T}_i(\theta-1, y^{\theta-1}) &= \min_{j\in C(i)} y_j^{\theta-1} + \underline{\alpha}^{\theta-1}\left(\max_{j\in C(i)} y_j^{\theta-1} - \min_{j\in C(i)} y_j^{\theta-1}\right) \\
&\geq (1-\underline{\alpha}^{t,\theta}) \min_{j\in A} x_j + \underline{\alpha}^{t,\theta} \max_{j\in A} x_j
\end{aligned}
$$

This means that $\min_{i\in A} \underline{T}_i^{t,\theta}(x) \geq (1-\underline{\alpha}^{t,\theta}) \min_{j\in A} x_j + \underline{\alpha}^{t,\theta} \max_{j\in A} x_j$. By combining this with (b) of this lemma, we obtain (c). □

Proof of Lemma 1. Parts b and c of Lemma 2 yield that

$$\min_{i\in A} T_i^{t,\theta}(x) \geq (1-\underline{\alpha}^{t,\theta}) \min_{j\in A} x_j + \underline{\alpha}^{t,\theta} \max_{j\in A} x_j.$$

Similarly, one can show that

$$\max_{i\in A} T_i^{t,\theta}(x) \leq \underline{\beta}^{t,\theta} \min_{j\in A} x_j + (1-\underline{\beta}^{t,\theta}) \max_{j\in A} x_j.$$

Consequently,

$$\max_{i\in A} T_i^{t,\theta}(x) - \min_{i\in A} T_i^{t,\theta}(x) \leq (1-\underline{\alpha}^{t,\theta} - \underline{\beta}^{t,\theta})(\max_{j\in A} x_j - \min_{j\in A} x_j).$$

□

Proof of Theorem 1. Fix any x. Set $x^0 = x$ and $x^t = T^{0,t}(x)$ for all $t \geq 1$. Now consider the sequence $\{x^t\}$. Let $\underline{x}^t = \min_{i\in A} x_i^t$ and $\bar{x}^t = \max_{i\in A} x_i^t$. To prove the theorem it suffices to show $\lim_{t\to\infty}\{\bar{x}^t - \underline{x}^t\} \to 0$. Because $T_i(\tau, x) \in [\min_{j\in A} x_j, \max_{j\in A} x_j]$ for all i and τ, $\{\bar{x}^t - \underline{x}^t\}$ is a non-increasing sequence. Thus, we only need to show that the distance between extremal opinions converges to 0 for some subsequence. Let $\{t_k\}$ be a subsequence with $t_{k+1} - t_k \geq \theta$ for all k and $\lim_{\tau\to\infty} \sum_{k=1}^{\tau} (\underline{\alpha}^{t_k} + \underline{\beta}^{t_k}) = \infty$. Because $\{\bar{x}^t - \underline{x}^t\}$ is non-increasing and $\{t_k\}$ satisfies $t_{k+1} - t_k \geq \theta$ for all k, Lemma 1 gives that for all $\tau \geq 2$,

$$\begin{aligned}
\bar{x}^{t_\tau} - \underline{x}^{t_\tau} &\leq \max_{i\in A} T_i^{t_{\tau-1},\theta}(x^{t_\tau-1}) - \min_{i\in A} T_i^{t_{\tau-1},\theta}(x^{t_{\tau-1}}). \\
&\leq (1 - \underline{\alpha}^{t_{\tau-1},\theta} - \underline{\beta}^{t_{\tau-1},\theta}) \left(\bar{x}^{\tau_{\tau-1}} - \underline{x}^{\tau_{\tau-1}}.\right) \\
&\leq \prod_{k=1}^{\tau-1} (1 - \underline{\alpha}^{t_k,\theta} - \underline{\beta}^{t_k,\theta}) \left(\bar{x}^0 - \underline{x}^0\right).
\end{aligned}$$

Consequently, we complete the proof by showing that $\lim_{\tau\to\infty} \prod_{k=1}^{\tau} (1 - \underline{\alpha}^{t_k,\theta} - \underline{\beta}^{t_k,\theta}) \to 0$ when $\lim_{\tau\to\infty} \sum_{k=1}^{\tau} (\underline{\alpha}^{t_k} + \underline{\beta}^{t_k}) = \infty$. It is easy to see that for any $\tau \geq 1$,

$$\prod_{k=1}^{\tau} (1 - \underline{\alpha}^{t_k,\theta} - \underline{\beta}^{t_k,\theta}) \leq \left(1 - \frac{\sum_{k=1}^{\tau} \left(\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}\right)}{\tau}\right)^{\tau}.$$

In addition,

$$\lim_{\tau\to\infty} \left(1 - \frac{\sum_{k=1}^{\tau} \left(\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}\right)}{\tau}\right)^{\tau} \leq \exp\left(-\sum_{k=1}^{l} \left(\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}\right)\right) \quad \forall l \in \mathbb{N}.$$

□

Furthermore, because $\lim_{l\to\infty} \sum_{k=1}^{l} \left(\underline{\alpha}^{t_k,\theta} + \underline{\beta}^{t_k,\theta}\right) = \infty$, the previous three inequalities give that

$$\lim_{\tau\to\infty} \prod_{k=1}^{\tau} (1 - \underline{\alpha}^{t_k,\theta} - \underline{\beta}^{t_k,\theta}) = 0.$$

Proof of Theorem 2. Assume that agents converge to different limit points, and only one agent converges to $\underline{x}^*$ and one to $\bar{x}^*$. We will derive a

contradiction.
Because convergence occurs by assumption, after a certain time $\underline{t} > 0$ then each agent is in the neighbourhood of their own limit point. For $a > 1$ this means $\underline{x}^* \neq \bar{x}^*$.
Denote i as the agent with $x_i^* = \underline{x}^*$. By the assumption of irreducibility, there exists a $j \in C(i)$ such that $x_i^* < x_j^*$. We examine the behaviour of i's opinion after time $\underline{t}$.
Case 1: $\underline{x}_i^t > x_i^t$. Due to the constriction assumption, and the assumption that $\underline{x}_i^t$ is outside the neighbourhood of x_i^*, a contradiction is established as i's posterior must lie outside the neighbourhood of their limit point x_i^*.
Case 2: $\underline{x}_i^t = x_i^t$ and, without loss of generality, $\bar{x}_i^t$ is in the neighbourhood of x_j^*. First notice that, for some $\kappa > 0$:

$$x_j^{t+1} - x_i^{t+1} > \kappa$$

This is because the neighborhoods of the limiting points are non-intersecting for large enough $\underline{t}$, and are thus separated by a positive distance. Therefore, we can find a z such that $x_i^t < z < x_j^t$ for all $t > \underline{t}$ and $z - x_i^t > \kappa - \epsilon > 0$ for small enough $\epsilon > 0$. Therefore:

$$\begin{aligned}
\kappa - \epsilon < z - x_i^{t+1} &= z - (1 - \alpha_i^t(x^t))x_i^t - \alpha_i^t(x^t)\bar{x}_i^t \\
&\leq z - (1 - \alpha_i^t(x^t))x_i^t - \alpha_i^t(x^t)x_j^t \\
&< z - (1 - \alpha_i^t(x^t))x_i^t - \alpha_i^t(x^t)z \\
&\leq (1 - \underline{\alpha}^t)(z - x_i^t) \\
&= \Pi_{k=\underline{t}}^t(1 - \underline{\alpha}^k)(z - x_i^k)
\end{aligned}$$

By the same logic of Theorem 2, that the distance between z and x_i^t shrinks to zero in the limit as $t \to +\infty$ when $\sum_{t=\underline{t}}^{+\infty} \underline{\alpha}^t = +\infty$. This produces a contradiction because it is assumed that there is a gap $\kappa - \epsilon > 0$ between i and j's neighbourhoods.

A similar proof can be done for the maximum individual, giving our result $\sum_{t=\underline{t}}^{+\infty} \underline{\alpha}^t + \underline{\beta}^t = +\infty$.

□

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Мы анализируем сходимость мнений (убеждений) в социальной сети с небайесовскими агентами. Мы формулируем новое достаточное условие, значительно ослабляющее условие в работе Lorenz (2005), при котором мнения участников сети сходятся к консенсусу. Наше условие требует от агентов учитывать мнения других участников при формировании своих апостериорных убеждений с достаточной частотой, в зависимости от свойств социальной сети.

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