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The geometry of twistor spaces of hypercomplex manifolds

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Overview of research area

A hypercomplex manifold is a smooth manifold M together with three integrable almost complex structures $I, J, K : TM \to TM$ which satisfy quaternionic relations

$$I^2 = J^2 = K^2 = -1, \; IJ = -JI = K.$$

A Riemannian metric g on M which is simultaneously Hermitian with respect to I, J and K is called hyperhermitian. A hyperhermitian metric on M whose Hermitian forms $\omega_I, \omega_J, \omega_K$ with respect to the structures I, J, K are closed is called hyperkähler. If in addition M is compact, simply connected and satisfies $H^{2,0}(M) = \mathbb{C}$, M is called a simple hyperkähler manifold. The simplest examples of compact hypercomplex manifolds are 2-dimensional complex tori, K3 surfaces and Hopf surfaces. According to the results of C. P. Boyer [3], these three are the only examples of compact hypercomplex manifolds of quaternionic dimension 1, and only the first two of them admit hyperkähler metrics. In higher dimensions, there are many examples of hypercomplex manifolds: according to the work of D. Joyce [8], any compact homogeneous space of compatible dimension admits left-invariant hypercomplex structures. In contrast, not many examples of compact hyperkähler structures are known: two families (Hilbert schemes of points on K3 and generalized Kummer manifolds) were found by Beauville [2], and two additional examples in quaternionic dimensions 3 and 5 were discovered by O'Grady [15, 14]. A major open problem in hyperkähler geometry is the existence of other compact examples which cannot in one way or another be reduced to these.

For both hypercomplex and hyperkähler manifolds M, there are in fact many other induced complex structures in addition to I, J and K, which together form a two-dimensional sphere:

$$S^2 = \left\{ x_1 I + x_2 J + x_3 K : x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

The topological product $M \times S^2 \cong M \times \mathbb{CP}^1$ parametrizes induced complex structures at points of M; we call it the twistor space of the hypercomplex manifold M and denote it by $\mathrm{Tw}(M)$. $\mathrm{Tw}(M)$ has a structure of a complex manifold, with respect to which the projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ to the second coordinate is holomorphic. If in additon there is a hyperkähler metric on M, it induces a natural Hermitian metric on $\mathrm{Tw}(M)$. The complex structure on $\mathrm{Tw}(M)$ is closely related to the quaternionic structure of M by means of the twistor correspondence. This correspondence takes many different forms and associates to every object on M which is in one way or another compatible with its quaternionic structure a holomorphic object on $\mathrm{Tw}(M)$, and vice versa. Thus, one of the approaches towards studying the geometry of the original manifold M is through studying its twistor space $\mathrm{Tw}(M)$. In what follows, we will denote by $I \in \mathbb{CP}^1$ an arbitrary induced complex structure on M.

One of the central subjects of the present dissertation is stability of vector bundles. It was originally defined for algebraic vector bundles on projective varieties by D. Mumford [13], and later generalized to Kähler manifolds and then to general Hermitian manifolds. Stable bundles are used in the construction of moduli spaces: when studying holomorphic vector bundles, we must restrict our attention to stable bundles in order to have a moduli space with a nice structure. Stable bundles were studied extensively in the setting of algebraic geometry (see for example [7]). While in the Kähler case the theory remains nice, non-Kähler manifolds present more difficulties, and despite certain progress (see [4, 6, 5]), not much is known in the non-Kähler case. Among the different types of non-Kähler Hermitian metrics, one in particular stands out due to its potential for simplifying the study of moduli spaces of stable vector bundles on the corresponding manifolds. We call a Hermitian metric on a manifold of complex dimension n balanced if its Hermitian form satisfies the property $d\left(\omega^{n-1}\right) = 0$. These were first introduced by M. L. Michelsohn [11], and for manifolds of complex dimension ≥ 3, they contain Kähler metrics as a strict subclass. While being more general, balanced metrics inherit many nice properties of Kähler metrics. In particular, it turns out that the theory of stability for balanced manifolds is in many ways similar to the Kähler case (see [10]). In a certain sense, having a balanced metric on a complex manifold is the next best case scenario after having a Kähler metric, at least as far as studying the moduli space of stable bundles is concerned.

Outline of results

The paper [9] of D. Kaledin and M. Verbitsky is concerned with vector bundles and connections on a hyperkähler manifold M and its twistor space $\mathrm{Tw}(M) = M \times \mathbb{CP}^1$. They show that the natural Hermitian metric on $\mathrm{Tw}(M)$ induced by the hyperkähler metric from M is balanced. Since for compact M this metric on $\mathrm{Tw}(M)$ can never be Kähler, in view of what was said in the previous section, balancedness is the best that can be hoped for, as it allows a meaningful discussion of stable bundles and various moduli spaces on $\mathrm{Tw}(M)$. They go on to study autodual connections on complex vector bundles over M. These are connections with curvature invariant under the natural action of the group $SU(2)$ on the bundle of exterior forms of M; they naturally induce holomorphic structures on the vector bundle with respect to each induced complex structure of M. In [9], they prove a version of the twistor correspondence taking an autodual connection on M to a semi-simple holomorphic vector bundle on the twistor space $\mathrm{Tw}(M)$, and describe this correspondence as an inclusion of moduli spaces. They also examine bundles E on the twistor space $\mathrm{Tw}(M)$ whose restrictions E_I to the fibres $\pi^{-1}(I)$ of the holomorphic twistor projection $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ are stable for all $I \in \mathbb{CP}^1$. They are called fibrewise stable bundles in [9], and it is shown that they are in fact also stable as bundles on $\mathrm{Tw}(M)$. The moduli space of fibrewise stable bundles is described, and a duality construction, in the spirit of S. Mukai's work on K3 surfaces [12], is given. In the present dissertation, we build onto two results from the article [9].

The first result concerns metric properties of the twistor space. As mentioned above, it is proved in [9] that the twistor space of a hyperkähler manifold is balanced. In the present work, we generalize this result as follows.

Theorem 3.2.3. *Let (M, I, J, K, g_M) be a compact hyperhermitian manifold of complex dimension n. Its twistor space $\mathrm{Tw}(M)$ is balnced.*

Since an arbitrary hypercomplex manifold always adimts hyperhermitian metrics, it follows directly from this theorem that the twistor space $\mathrm{Tw}(M)$ of a general compact hypercomplex manifold is balanced. Thus, the balanced metric on $\mathrm{Tw}(M)$ (which is constructed indirectly) exists regardless of any assumptions on the metric structure of M. The central point in the proof of the theorem is the fact that on a Hermitian manifold of complex dimension n, given a strictly positive $(n-1,n-1)$-form η, one can always extract its "$(n-1)^{\mathrm{st}}$ root". In other words, there is always a strictly positive $(1,1)$-form ω such that $\omega^{n-1}=\eta$, and moreover, such ω is unique. The problem then reduces to finding a closed strictly positive $(n-1,n-1)$-form on the twistor space $\mathrm{Tw}(M)$ of a compact hypercomplex manifold, which is assembled as a linear combination of forms constructed from a hyperhermitian metric from M. This result was published in the paper [17].

The second result pertains to stability of vector bundles on the twistor space $\mathrm{Tw}(M)$ of a hyperkähler manifold M. As was mentioned in an earlier paragraph, in the article [9], Kaledin an Verbitsky show that a fibrewise stable bundle on $\mathrm{Tw}(M)$ is also stable in the usual sense. In fact, their proof gives a stronger result. We call a holomorphic vector bunlde E on $\mathrm{Tw}(M)$ generically fibrewise stable if all but a finite number of its restrictions E_I to the fibres $\pi^{-1}(I)$ of the projection $\pi:\mathrm{Tw}(M)\to\mathbb{CP}^1$ are stable. As follows from the results of Kaledin and Verbitsky, a generically fibrewise stable bundle E on $\mathrm{Tw}(M)$ doesn't have any nonzero subsheaves of lower rank; such E is called irreducible. We prove the following partial converse to this result.

Theorem 4.2.1. *Let M be a compact simple hyperkähler manifold and E a holomorphic vector bundle on its twistor space $\mathrm{Tw}(M)$. If E is generically fibrewise stable, then it is irreducible. The converse is true in case E has rank 2 or 3, and also for bundles E of arbitrary rank that are generically fibrewise simple.*

Here, E is called generically fibrewise simple if all but a finite number of its restrictions E_I to the fibres $\pi^{-1}(I)$ of the projection $\pi:\mathrm{Tw}(M)\to\mathbb{CP}^1$ satisfy $\mathrm{Hom}(E_I,E_I)=\mathbb{C}$. The first step towards the proof of this theorem is the following result.

Theorem 4.1.2. *Let M be a compact simple hyperkähler manifold with hyperkähler metric g and twistor space $\mathrm{Tw}(M)$, and E a holomorphic vector bundle on $\mathrm{Tw}(M)$ of rank r. Then the sets*

$$\left(\mathbb{CP}^1\right)^{\mathrm{st}}=\left\{I\in\mathbb{CP}^1 : E_I \text{ is stable}\right\},\ \left(\mathbb{CP}^1\right)^{\mathrm{sst}}=\left\{I\in\mathbb{CP}^1 : E_I \text{ is semi-stable}\right\}$$

are Zariski open in $\mathbb{CP}^1$.

This fact is a generalization of a result of A. Teleman from [16], where it is shown that for a holomorphic vector bundle E on the product of complex manifolds $X\times Y$, the set of fibres of the projection $X\times Y\to Y$ to which E restricts stably is Zariski open in Y, subject to some conditions on X and Y. Note that the twistor projection $\pi:\mathrm{Tw}(M)\to\mathbb{CP}^1$ is not a special case of this, since the complex structure on $\mathrm{Tw}(M)$ is not the product of the complex structures from M and $\mathbb{CP}^1$; however, the argument from [16] can be adapted to prove Theorem 4.1.2. As for Theorem 4.2.1, it is proved by contradiction. Starting with an irreducible bundle E on $\mathrm{Tw}(M)$ and assuming that it is not generically fibrewise stable, we can conclude from Theorem 4.1.2 that for each $I \in \mathbb{CP}^1$ there exist destabilizing subsheaves $\mathcal{F}_I \subseteq E_I$, and moreover, they can be chosen to all have the same rank s. The ultimate strategy is to try to assemble them into a global subsheaf $\mathcal{F} \subseteq E$ on $\mathrm{Tw}(M)$, which would contradict the irreducibility of E. Via a version of the Plücker embedding for vector bundles, subsheaves $\mathcal{F}_I \subseteq E_I$ give rise to line subsheaves $L_I \subseteq \Lambda^s(E_I)$ whose image lies in the cone of exterior monomials $C^s(E_I) \subseteq \Lambda^s(E_I)$, and moreover we can choose the L_I to all be restrictions of a single line bundle L on $\mathrm{Tw}(M)$. The problem now reduces to finding a section of

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & \mathbb{P}\left(\pi_*\left[\mathrm{Hom}(L, \Lambda^s E)\right]\right) \arrow[dl] \\ \mathbb{CP}^1 & \end{tikzcd}$$

where Y is the closed analytic subset of $\mathbb{P}\left(\pi_*\left[\mathrm{Hom}(L, \Lambda^s E)\right]\right)$ consisting of morphisms $L_I \to \Lambda^s(E_I)$ with image in $C^s(E_I)$. For bundles E of rank 2 or 3, this is not a problem, because in this case $C^s(E_I) = \Lambda^s(E_I)$, but if $\mathrm{rk}\, E > 3$, it's not clear that such a section should exist. However, in that case, one can always find a multisection, that is, a section of Y over a branched covering $f : X \to \mathbb{CP}^1$, and then passing to the fibred product

$$\begin{tikzcd} Z \arrow[r, "\varphi"] \arrow[d, "\rho"'] & \mathrm{Tw}(M) \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & \mathbb{CP}^1 \end{tikzcd}$$

one can construct a proper subsheaf of the pullback bundle $\varphi^*(E)$ on Z, and after some work, use the generic fibrewise simplicity assumption on E to get a contradiction to its irreducibility, proving Theorem 4.2.1. In addition to this, the dissertation contains an explicit example of a stable vector bundle on $\mathrm{Tw}(M)$ for M a K3 surface all of whose restrictions to the fibres of $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ are non-stable; this result is the subject of the article [18] of the author of the dissertation.

Directions of further research and applications

There are different ways in which the results of this dissertation might be extended and used in further research. Firstly, the result about balanced metrics on twistor spaces of compact hypercomplex manifolds is useful since it gives a lot of new examples of balanced manifolds. Indeed, as was mentioned in a previous section, there are a lot of examples of compact hypercomplex manifolds, hence Theorem 3.2.3 gives a lot of new examples of balanced manifolds, namely, their twistor spaces. Balanced metrics are a nice generalization of Kähler metrics, and new examples of such are always of interest in their own right. Secondly, since the presence of a balanced metric makes the study of stable bundles easier, one can ask whether any of the results from [9] about stability and moduli spaces for hyperkähler manifolds generalize to the hypercomplex setting. While a full generalization is probably too much to hope for, one can start to investigate certain special types of hypercomplex manifolds, such as HKT-manifolds. A hyperhehermitian metric on a hypercomplex manifold (M, I, J, K) is called an HKT metric if it satisfies the property $\partial_I \Omega_I = 0$, where $\Omega_I = \omega_J + \sqrt{-1}\omega_K$. These are generalizations of hyperkähler metrics that enjoy many interesting properties (see [1] for a survey of HKT-manifolds). Perhaps, in the presence of an HKT-structure on M, the proof of Theorem 3.2.3 can be simplified to yield an explicit balanced metric on $\mathrm{Tw}(M)$, which can then be used to study moduli spaces of stable bundles on $\mathrm{Tw}(M)$ and M.

As for Theorem 4.2.1, it is the opinion of the author that its full converse is in fact true, without any assumptions on the rank of E or otherwise. In other words, an irreducible bundle E on the twistor space $\mathrm{Tw}(M)$ of a simple hyperkähler manifold M should be generically fibrewise stable. In view of Theorem 4.2.1, which establishes this for the case that E is generically fibrewise simple, a possible way of approaching the full statement would be to show that any irreducible E is generically fibrewise simple. One can again argue by contradiction, and assume that there is a morphism $F : E \to E(D)$ (where D is a divisor on $\mathrm{Tw}(M)$) which is not a multiplication by a meromorphic function from $\mathbb{CP}^1$. It has no eigenvalues over $\mathbb{CP}^1$, but one can again take a branched covering $f : X \to \mathbb{CP}^1$ and pass to the fibred product of $f : X \to \mathbb{CP}^1$ and $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$, over which the pullback of F does have eigenvalues, and so the pullback of E has subsheaves, from which one can try to get a contradiction to the irreducibility of E. If the full converse of Theorem 4.2.1 is proven, it would give a very neat characterization of irreducible bundles on $\mathrm{Tw}(M)$ as generically fibrewise bundles. Irreducible bundles occur only on non-algebraic manifolds and are difficult to study, so such a characterization would nicely complement the duality results from [9].

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