**Abstract**

**TT-95**

Theme: “Topology and chaos in the dynamics of systems, foliations, and deformations of Lie algebras”

Head: O.V. Pochinka

Department: Laboratory of topological methods in dynamics

**Purpose:** Development of methods for the qualitative theory of dynamical systems, including the construction of energy functions, topological classification, the inclusion of cascades in the flow, the search for stable arcs in the space of dynamical systems, the establishment of links between the dynamics of the system and the topology of the manifold, the algorithmization of distinguishing topological invariants. Development of new analytical and numerical methods for the study of pseudo-hyperbolic attractors, spiral quasi-attractors and mixed dynamics of multidimensional dissipative systems and their applications to specific models. Development of methods for the theory of deformations of modular Lie algebras and obtaining classification results for character algebras of characteristic 2. Development of methods for studying the qualitative behavior of foliations that are consistent with geometric structures. The study of the geometry of bundles, applications to dynamics. Development of the Chernov function calculus.

**Methods:** The development of topological methods in dynamics enriches both sciences - the qualitative theory of dynamical systems and topology. A remarkable example of such an interaction is the solution of one of the most important topological problems — the multidimensional Poincare hypothesis (in dimension greater than four). To solve this problem, S. Smale applied the theory of gradient dynamical systems induced by Morse functions. Namely, assuming that the manifold is a homotopy sphere, he proved that all critical points of a given Morse function can be successively removed and go to a function with exactly two critical points of index 0 and 1.

Using topological and geometric methods, the laboratory staff obtained deep results on the interrelation of the dynamic characteristics of flows and cascades with the topology of the manifold. The structure of manifolds admitting Morse-Smale diffeomorphisms was studied depending on the intersection structure of invariant manifolds of saddle periodic points. The interrelation of investments of invariant manifolds with the existence of Lyapunov energy functions, etc., was discovered. These results were published in a number of leading foreign and domestic journals, which indicates their compliance with the world level. The obtained reserve will allow us to further develop the relevant topics described above.

Another major achievement of the modern theory of dynamic systems, in which the laboratory staff are also recognized experts, was the discovery of mixed dynamics, which characterizes the fundamental impossibility to separate the attractors of a dynamic system from repellers. Despite the fact that such a phenomenon was discovered quite recently, we have already managed to detect a number of systems from various applications that demonstrate mixed dynamics. Thus, the development of methods for the study of mixed dynamics is an urgent task from both theoretical and practical points of view.

Thanks to the strong algebraic-geometric cell of the laboratory, it is planned to extend the concept of chaos in dynamical systems in the sense of Divani to arbitrary foliations and investigate the problem of the existence of chaos for Cartan foliations, as well as to make advancements in the classification of simple Lie algebras over fields of small characteristic.

**Results:** During the performance of the technical task, the results were obtained in several scientific areas:

1) Development of methods for the qualitative theory of dynamical systems.

2) Development of new analytical and numerical methods for the study of mixed dynamics, pseudo-hyperbolic attractors and quasi-attractors of multidimensional dissipative systems and their applications to specific models.

3) Development of methods of the theory of deformations of modular Lie algebras and obtaining classification results for characteristic algebras 2. Development of the calculus of Chernoff functions.

4) Development of methods for studying the qualitative behavior of foliations that are consistent with the geometric structures. The development of computational topology and its applications.

Direction 1) development of methods of qualitative theory of dynamical systems

1.1. New invariant (scheme of a map) of homeomorphisms of a disk with cascade of periodic orbits was introduced. It was shown that this invariant distinguishes diffeomorphisms constructed by different sequences of signatures. We construct a diffeomorphisms of 2-sphere by period doubling bifurcation two times applied to source-sink diffeomorphism with rotation in the one and another directions. The main result is the proof nonequivalence schemes of such diffeomorphisms i.e. that there is no homeomorphism sending components of one scheme to components of another scheme.

1.2. The structure of the splitting of four-dimensional phase space on trajectories of Morse-Smale flow admitting heteroclinic intersections was studied. Class of Morse-Smale flows on sphere  such that non-wandering set of any flow  consists of exactly four fixed points: a source, a sink, and two saddles. The wandering set of such flows contains a finite number of heteroclinic curves lying in the intersection of invariant manifolds of saddle points. The paper describes the topology of embedding invariant manifolds of saddle points of such flows, which is the first step in solving the problem of topological classification. In particular, it is proved that the closures of invariant manifolds of saddle points that do not participate in heteroclinic intersections are tamely embedded 2-sphere and arc. These manifolds are an attractor and a flow repeller. In the set of orbits belonging to the region of attraction of the attractor (repeller repulsion), a secant is constructed, which is a manifold homeomorphic to the direct product . The topology of the intersection of invariant manifolds of saddle equilibrium states with this secant is studied.

1.3. A class of Morse-Smale diffeomorphisms on a sphere of dimension four and higher is considered, for which the invariant manifolds of different saddle periodic points do not intersect. The dynamics of an arbitrary such diffeomorphism can be represented as source-drain dynamics, where the drain (source) is a connected union of one-dimensional and zero-dimensional unstable (stable) invariant manifolds of periodic points. We study the structure of the space of orbits belonging to the region of attraction of the “drain” (the repulsion region of the “source”) and the topology of embedding separatrices of saddle periodic points of codimension 1 into it.

1.4. The class of Morse-Smale diffeomorphisms without heteroclinic intersections on a sphere  of dimension  is considered. Sufficient conditions are established for the inclusion of these diffeomorphisms into a topological flow, which coincide with the necessary conditions obtained by J. Palis (which, generally speaking, is not true in the case ).

1.5. It was established that for Morse-Smale diffeomorphisms defined on a sphere , and not having heteroclinic intersections of invariant manifolds of saddle periodic points, the two-color graph is a complete topological invariant. The result obtained is also carried over to the case of Morse-Smale homeomorphisms.

1.6. The relationship between the structure of the set of equilibrium states of a gradient-like flow and the topology of the carrier manifold of dimension 4 and above has been studied. A class of manifolds that admits a generalized Hegor distribution is introduced. It is established that if a non-wandering set of a gradient-like flow consists exactly of  the nodal and  saddle equilibrium states of Morse indices 1 and , then its carrier manifold admits a generalized Hégor decomposition of genus . An algorithm for constructing gradient-like flows on closed manifolds of dimension  with a given number of nodal equilibrium states and given numbers of saddle equilibrium states of different Morse indices is given.

1.7. The results on the topological classification of Morse-Smale systems on closed manifolds are presented, including the results obtained by the authors recently.

1.8. A topological classification of one-dimensional basic sets of diffeomorphisms, satisfying S. Smale's axiom and given on orientable surfaces of negative Euler characteristic, provided with a metric of constant negative curvature, is obtained. Using the methods of Lobachevsky’s geometry, each geodesic lamination on the surface is uniquely assigned to each perfect spacious one-dimensional attractor of a A-diffeomorphism. It is established that in the absence of degree two in the attractor, there is a homotopic identity homeomorphism of the surface that maps the attractor to geodesic lamination in such a way that non-intersecting unstable manifolds from the attractor are mapped to different layers of geodetic lamination. Moreover, if non-wandering sets of homotopic A-diffeomorphisms have perfect amply spaced attractors without bundles of degree two, then the geodesic laminations corresponding to these attractors coincide. The results obtained will allow us to obtain a topological classification of constraints of A-diffeomorphisms of orientable surfaces onto one-dimensional perfect spaciously located basic sets by means of pseudo-Anosov homeomorphisms.

1.9. Work was done on the topological classification of one-dimensional basic sets of diffeomorphisms satisfying the S. Smale axiom A and defined on orientable surfaces of negative Euler characteristic, equipped with a metric of constant negative curvature. Using the methods of Lobachevsky’s geometry, each geodesic lamination on the surface is uniquely assigned to each perfect spacious one-dimensional attractor of a A-diffeomorphism. It is established that in the absence of degree two in the attractor, there is a homotopic identity homeomorphism of the surface that maps the attractor to geodesic lamination in such a way that non-intersecting unstable manifolds from the attractor are mapped to different layers of geodetic lamination. Moreover, if non-wandering sets of homotopic A-diffeomorphisms have perfect amply spaced attractors without bundles of degree two, then the geodesic laminations corresponding to these attractors coincide. The results obtained will allow us to obtain a topological classification of constraints of A-diffeomorphisms of orientable surfaces onto one-dimensional perfect spaciously located basic sets by means of pseudo-Anosov homeomorphisms.

1.10. It has been established that the application of S. Smale’s surgery to Anosov's algebraic endomorphism, which is a finite-leafing covering of degree at least two, does not lead to the construction of an DE-endomorphism, the non-wandering set of which contains a one-dimensional stretching attractor.

1.11. It is proved that gradient-like flows on closed surfaces are topologically conjugate if and only if they are topologically equivalent. The class of  smooth gradient-like flows (Morse flows) on a closed surface is a subclass of the set of Morse-Smale flows: they are all coarse. Their nonwandering set consists of a finite number of hyperbolic fixed points and a finite number of hyperbolic limit cycles, such flows also do not have trajectories connecting saddle points. It is well known that the class of topological equivalence of the Morse-Smale flow can be described combinatorially, for example, by the oriented Peixoto graph or the Oshemkov-Sharko molecule. However, the description of the topological conjugacy class of such a system already requires at least the introduction of continuous invariants (modules) corresponding to the periods of limit cycles. Therefore, one equivalence class contains a sonotinium of topological conjugacy classes. Gradient-like flows are Morse-Smale flows without limit cycles.

1.12. Received a complete topological classification of -stable flows on surfaces. Structurally stable (coarse) flows on surfaces have only a finite number of singular points and a finite number of closed trajectories; all of them are hyperbolic, and there are also no trajectories connecting saddle points. Violation of the last point leads to -stable flows on surfaces that are not structurally stable. However, in the present paper we prove that the topological classification of such flows can also be reduced to a combinatorial problem. The complete topological invariant is a multigraph, and we present a polynomial algorithm for distinguishing such graphs up to isomorphism. We also obtained a graph criterion for the orientability of a manifold and a formula related to a graph for calculating its Eulerian characteristic. In addition, we give a polynomial algorithm for verifying orientability and calculating the Euler characteristic.

1.13. In 1978, J. Palis discovered the presence of a continuum of topologically non-conjugate flows (cascades) in the vicinity of a system with heteroclinic tangency - the presence of modules. V. Di Melu and S. Van Strin in 1987 characterized the class of diffeomorphisms of surfaces with a finite number of modules. It turned out that the condition of the finiteness of the modules imposes a restriction on the length of the chain of saddles involved in heteroclinic tangency, such saddles in the chain cannot be more than three. Surprisingly, this effect is not detected for continuous dynamic systems. In this paper, we consider gradient flows of the height function of a vertical oriented surface of the genus . Such flows have a chain consisting of  saddle points. In the present paper it is established that the number of modules of such flows is equal to . This result is a direct consequence of the sufficient conditions for the topological conjugacy of flows in the vicinity of such systems established in this article. A complete topological invariant of topological equivalence for such systems is a four-color graph that carries information about the relative position of the cells. Equipping the edges of the graph with analytical parameters — modules associated with saddle ligaments — provides sufficient conditions for the topological conjugacy of flows of the class under consideration.

1.14. A review of existing graph invariants for gradient-like flows on surfaces up to topological equivalence was made and we develop effective algorithms for distinguishing them (recall that a flow on a surface is called a gradient-like flow if its nonwandering set consists of a finite set of hyperbolic fixed points, and there are no trajectories connecting the saddle points). In addition, we developed a parameterized algorithm for the Fluitas invariant, which is linear when the number of sources is fixed. As a result, we show that the classes of topological equivalence and topological conjugacy coincide for gradient-like flows; therefore, all the proposed invariants and distinction algorithms are also applicable to topological classification, taking into account the time of movement along trajectories. Thus, as the main result of this article, we have obtained many ways to recognize the equivalence class and conjugacy class of an arbitrary gradient-like flow on an arbitrary surface for polynomial time.

1.15. Earlier, the Smale-Vietoris class of diffeomorphisms was introduced, which contains the Smale DE mappings. Here we have considered the class of Smale-Vietoris A-diffeomorphisms, which is determined using basic A-endomorphisms of varieties whose dimension is less than the dimension of the carrying varieties of A-diffeomorphisms. It is shown that there is a one-to-one correspondence between the base sets of the base A-endomorphism and the Smale-Vietoris A-diffeomorphism. For a back-invariant basis set of a basic A-endomorphism, an exact description of the corresponding nontrivial basis set of a Smale-Vietoris A-diffeomorphism is given. On the basis of the description obtained, a bifurcation is constructed between different types of solenoidal basic sets.

1.16. The problem of the existence of a simple arc connecting two structurally stable systems on a closed manifold is included in the list of the fifty most important problems of dynamical systems. For Morse-Smale flows on an arbitrary closed manifold, this problem was solved by S. Newhouse and M. Peixoto in 1980. As follows from the works of S. Matsumoto, P. Blanchard, V. Grines, E. Nozdrinova, O. Pochinki, for the Morse-Smale cascades, obstacles to the existence of such an arc exist on closed manifolds of any dimension. In these papers, necessary and sufficient conditions for belonging to the same simple isotopic class for gradient-like diffeomorphisms on a surface or a three-dimensional sphere were found. The present result is the next step in this direction. Namely, the author has established that all coarse orientation-changing diffeomorphisms of a circle lie in one component of a simple connection, whereas the simple isotopy class of a coarse orientation-preserving transformation of a circle is completely determined by the rotation number of Poincaré.

1.17. The modern qualitative theory of dynamical systems is closely intertwined with a fairly young topology science. Many strange topology constructs are sooner or later found in the dynamics of discrete or continuous dynamical systems. Here we have shown that Artin-Fox wild arc naturally occurs in invariant sets of dynamical systems.

Direction 2) the development of new analytical and numerical methods for the study of mixed dynamics, pseudo-hyperbolic attractors and quasi-attractors of multidimensional dissipative systems and their applications to specific models

2.1. Studied the dynamic properties of a Celtic stone moving along a plane. Two-parameter families of corresponding nonholonomic models are considered in which bifurcations are studied, leading to a change in the types of stable modes of motion of a stone, as well as to the emergence of chaotic dynamics. It is shown that in such models multistability phenomena are observed, when stable modes of various types (regular and chaotic) can coexist in the phase space of the system. It is also shown that the chaotic dynamics of the nonholonomic model of the Celtic stone can be very diverse. Here, in the respective regions of the parameters, they are observed as spiral strange attractors of various types, including the so-called. Shilnikov's discrete attractors, and mixed dynamics, when the attractor and repeller intersect and almost coincide. A new scenario of the emergence of mixed dynamics as a result of the reversible bifurcation of fusion of stable and unstable limit cycles is found.

2.2. The features of spiral attractors in the three-dimensional model of Rosenzweig-MacArthur, which describes the dynamics in the food chain “prey-predator-predator”, were studied. It is well known that with the values ​​of the parameters, when this system is fast-slow, spiral attractors having the shape of a teacup arise in it. We show that such attractors arise as a result of the Shilnikov bifurcation scenario, the first stage of which is associated with the emergence of a soft Andronov-Hopf bifurcation, and the last stage results in a homoclinic attractor containing a Shilnikov loop to a saddle-focal equilibrium state with a two-dimensional unstable manifold. It is shown that homoclinic spiral attractors, together with the fast-slow dynamics of the system, lead to the emergence of a new type of packet activity in the system. Intervals of fast oscillations for this type of activity alternate with slow motions of two types: low amplitude oscillations around the saddle-focal equilibrium and motion near the stable slow variety of the fast subsystem. We demonstrate that the packet activity of the indicated type can be both chaotic and regular.

2.3. A scenario of the emergence of mixed dynamics in reversible two-dimensional diffeomorphisms is proposed. The key point of the scenario is a spasmodic increase in the size of the strange attractor and the strange repeller, resulting from the heteroclinic bifurcations of invariant saddle point manifolds belonging to the attractor and the repeller. As a result of such bifurcations, the strange attractor collides with the boundary of its region of attraction, and the strange repeller with the boundary of its region of “repellation”, after which an intersection of the attractor and the repeller occurs. After the implementation of the scenario, the dissipative chaotic dynamics associated with the existence of a separable strange attractor and strange repeller instantly becomes mixed when the attractor and repeller are fundamentally inseparable. The possibility of implementing the proposed scenario is demonstrated on one of the well-known problems of the dynamics of a rigid body, namely, on the nonholonomic model of the Suslova top.

2.4. Presents an example of a strange attractor of a new type. It is shown that this attractor belongs to the class of wild pseudohyperbolic attractors. Such an attractor was found in a four-dimensional system, which can be represented as an extension of the Lorentz system.

2.5. A new method for constructing three-dimensional flow systems with various chaotic attractors was proposed. Using the proposed method, an example of a three-dimensional flow system with an asymmetric Lorenz attractor is constructed. In contrast to the classical Lorenz attractor, the discovered attractor does not have symmetry. However, an asymmetric attractor, like the classical one, belongs to the class of “real” chaotic, more precisely, pseudo-hyperbolic attractors, whose theory was developed by D. Turayev and L. P. Shilnikov. Any trajectory of a pseudo-hyperbolic attractor has a positive Lyapunov exponent and this property is preserved for attractors of close systems. At the same time, unlike hyperbolic, pseudohyperbolic attractors admit homoclinic tangency. However, bifurcations of such tangencies do not lead to the birth of stable periodic orbits. In numerical experiments, when constructing, for example, diagrams of the Lyapunov senior exponent, in the vicinity of a pseudohyperbolic attractor, there are no stability windows that correspond to the appearance of regular attractors.

2.6. To search for the asymmetric Lorenz attractor, we used the << saddle card >> method. Using the construction of diagrams for the Lyapunov senior exponent, we show that in the neighborhood of the detected attractor there are indeed no stability windows. In addition, we establish the pseudo-hyperbolicity of the indicated attractor using the LMP method recently presented in the work of Gonchenko, Kazakov and Turaev.

Direction 3) Development of methods for the theory of deformations of modular Lie algebras and obtaining classification results for characteristic algebras 2. Development of the Chernov function calculus

3.1. Global deformations of a Lie algebra of type  over an algebraically closed field of characteristic 2 have been studied. Global deformations of a given Lie algebra give a new simple 34-dimensional Lie algebra of characteristic 2. The classification of simple Lie algebras over algebraically closed fields of characteristic  has now been completed: all simple Lie algebras are either classical or isomorphic to Lie algebras of Cartan type corresponding to various differential forms, or to Melikyan algebras in the case . The problem of classifying simple Lie algebras over fields of characteristic 2 seems to be particularly difficult. At present, a large number of finite-dimensional simple Lie algebras have been constructed, with respect to which it is not known whether they are actually new, unparalleled with large characteristics of the ground field.

3.2. It is shown that in series  and  the space of local deformations are nontrivial only for type  Lie algebras and  in characteristic 2. Global deformations  are already described. This paper closes the topic of deformations of Lie algebras of the  and  series.

3.3. We studied (for an arbitrary fixed ) Cauchy problem for the Schrödinger equation in space  over a field 

 (1)

3.4. It is assumed that the function  is measurable and has a locally integrable second degree; including the proposed technique applicable to the potentials of quantum harmonic () and quantum anharmonic (,) oscillators. We also impose on the following condition: the Hamiltonian of equation (1) on the space of all finite smooth functions  is self-adjoint in essential in ; this is the case, for example, when the function is non-negative.

3.5. At present, a relatively small number of situations are known in which it is possible to express explicitly the solution of a differential equation with variable coefficients in terms of these coefficients. Moreover, often the formulas for the solution, obtained under some restrictions on the coefficients, turn out to be unsuitable under other, weaker restrictions. This is usually expressed in the fact that the previously convergent series and integrals are divergent. Usually, these difficulties are tried to be overcome with the help of regularizations, which make it possible to use the old formulas, understanding them in a new way. In this report, regularizations are not used in this sense, but formulas of a qualitatively new form are proved, giving the solution to the Cauchy problem for the Schrödinger equation with a nonnegative potential growing arbitrarily fast at infinity.

3.6. A method for solving the Cauchy problem for a linear parabolic equation of an evolution type with partial derivatives and variable coefficients is proposed. The method is applied to the second-order equation (heat conduction equation) with a one-dimensional spatial coordinate and consists in applying the Chernoff approximation procedure to a specially constructed family of shift operators. The uniform convergence of approximations to the exact solution is proved. The expression for the solution is interpreted as a Feynman formula with a singular integral kernel.

3.7. The Cauchy problem for a parabolic equation is posed, where the argument x belongs to an infinite-dimensional separable Hilbert space H, and a differential operator L with variable coefficients, containing derivatives of the second, first, and zero orders, plays the role. The second derivative is included L in the form of Volterra-Gross laplacian, which is constructed from a linear operator  with a finite trace like this: if f it is a numerical function on H, then by definition . The solution using the Chernov theorem is written in the form of a limit of multiple integrals of unboundedly increasing multiplicity, while integration in a Hilbert space is carried out by a Gaussian measure with a correlation operator equal to the product of the operator and the function dependent on the coefficients of the equation.

3.8. The Cauchy problem for the one-dimensional Schrödinger equation with derivatives of arbitrarily high order and variable coefficients was considered. A particular case of this equation (when the potential is quadratic, and the coefficients of the derivatives are constant) can be interpreted as the Schrödinger equation in the momentum representation. A formula is found that expresses the solution of the Cauchy problem in terms of (variable) coefficients of the equation and the initial condition. The formula is based on a specially constructed mix of shift operators in the formula belonging to I. D. Remizov , which allows one to construct a -semigroup according  to the family of self-adjoint operators related to the Chernoff self-adjoint operator H.

Direction 4) the development of methods for studying the qualitative behavior of foliations that are consistent with geometric structures. The development of computational topology and its applications

4.1. The structure of holonomy groupoids of pseudo-Riemannian foliations of arbitrary codimension on A-dimensional pseudo-Riemannian manifolds is investigated. Necessary and sufficient conditions are obtained for the foliation on a pseudo-Riemannian manifold to be pseudo-Riemannian. A description of the structure of layers of induced foliations on holonomy groupoids is given. The specificity of the holonomy groupoids of transversely complete pseudo-Riemannian foliations is elucidated.

4.2. For a complete Cartan foliation , two algebraic invariants are introduced  and , which we call structural Lie algebras. If the transversal Cartan geometry of a foliation is effective, then . We prove that equality to zero  guarantees that the group of all basic automorphisms of a foliation  admits a unique structure of a finite-dimensional Lie group. In particular, we obtain sufficient conditions for the indicated group to be discrete. We obtain some exact (that is, the best possible) estimates of the dimension of this group depending on the transversal Cartan geometry and the topology of the layers. We construct several examples of basic automorphism groups of complete Cartan foliations.

4.3. We study foliations that admit Weyl geometry modeled on pseudo-Riemannian manifolds of arbitrary signature as a transversal structure. Various interpretations of the holonomy groups of the layers of such foliations are given. The criterion of pseudo-Riemannian Weyl foliation is proved. Sufficient conditions for the existence of an attractor for a foliation are obtained . Moreover, it is proved that if the Waile foliation  is complete, then the condition obtained guarantees the existence of a global attractor of this foliation.

4.4. For two-dimensional compact polyhedra with a given Euclidean cell decomposition, which are pseudo-manifolds with boundary, new efficient algorithms have been developed for calculating the bases of the groups of absolute and relative homology modulo 2.

4.5. Numerical schemes were obtained for solving problems of continuum mechanics by the finite element method. Earlier, a method for accelerating computations was developed, consisting in the use of a simplicial mesh inscribed in the original cubic cell partitioning of a three-dimensional body. This paper shows that the obstacle to the construction of this construction is described in terms of homology groups modulo two. The main goal of the work is to develop a method to eliminate this obstacle. Achieving the goal is based on efficient algorithms for calculating the bases of homology groups that are dual with respect to the shape of the intersection.

4.6. The relations between the dynamics of divergence-free vector fields on an orientable three-dimensional smooth manifold M and the dynamics of Hamiltonian systems are found.