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The volumes of arithmetic locally-symmetric spaces and their applications in the theory of automorphic forms

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1 Introduction

Consider X – a complex analytic manifold, and let Γ be the discrete group acting on X. Let the factorspace X/Γ have the finite volume. The action of the group Γ on the space X induces the action of the group Γ on the space of holomorphic functions on X.

Definition 1. *A holomorphic function f on the space X is called an automorphic form if for $x \in X$, $\gamma \in \Gamma$ $f(\gamma x) = j^{-r}(\gamma, x)f(x)$. We denote by $j^{-1}(\gamma, x)$ the factor of automorphy, i.e. non-zero holomorphic function, the number r is called the eight of the automorphic form.*

We denote by $A(\Gamma)$ the graded algebra of automorphic forms. It's very interesting to study the structure of the algebra $A(\Gamma)$, in particular, it's important to know for which groups Γ the algebra $A(\Gamma)$ is free.

The work in this direction started in the end of 19th century. The first example of the free algebra of automorphic forms belongs to Klein and Fricke [2]. They proved that the algebra of automorphic forms for the group $\Gamma = PSL_2(\mathbb{Z})$ is a polynomial algebra with two generators of weights 4 and 6. In this case the space X is the upper half-plane, i.e. the Cartan type IV domain of dimension 1.

The next examples of free algebras of automorphic forms appeared in 1960-s. In these examples the space X is the product of two upper-half planes, i.e. the Cartan IV domain of dimension 2. The group Γ is the augmented Hilbert Modular group $P\tilde{S}L_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of real quadratic field $K = \mathbb{Q}(\sqrt{d})$, and $P\tilde{S}L_2(\mathcal{O})$ is the group $PSL_2(\mathcal{O})$, augmented with the automorphism that changes the upper-half planes. In 1963 and 1965 respectively Gundlach proved the freeness of algebras of automorphic forms $A(\Gamma)$ for the fields $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{2})$. More precisely, it is proved in these articles that for $d = 5$ $A(\Gamma) \cong \mathbb{C}[G_2, G_6, G_{10}]$, and for $d = 2$ $A(\Gamma) \cong \mathbb{C}[G_2, G_4, G_6]$ where G_i is the generator of weight i. Besides it, there were a lot of works, describing the structure of the algebras $A(\Gamma)$ for the small values of d and explicitely writting out the generators and relations. For example, there are explicit descriptions of the rings of modular forms for the fields $\mathbb{Q}(\sqrt{6})$ in the paper [3], for the field $\mathbb{Q}(\sqrt{13})$ in the paper [5], for the fields $\mathbb{Q}(\sqrt{17})$ and $\mathbb{Q}(\sqrt{65})$ in [9] and [10] respectively, for $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{17})$ in [12].

In 1962 Igusa proved ([11]) that for the Siegel modular group of genus 2 $\Gamma = PSp_4(\mathbb{Z})$ the algebra $A(\Gamma)$ is free with generators of weights 4, 6, 10 and 12. The group Γ naturally acts on the Siegel upper half plane, which is the Cartan type IV domain of dimension 3.

For the long time there were no known examples of the free algebras of automorphic forms in dimensions $n > 3$. In 2010 E. B. Vinberg in his article [17] gave a series of examples of free algebras $A(\Gamma)$ in Cartan domains in dimensions 4, 5, 6, 7. In 2018 in the article [18] E. B. Vinberg expanded this series of examples by giving the examples of free algebras in dimensions 8, 9 and 10. More precisely, he proved that the algebras of automorphic forms $A(\Gamma)$ for the group $\Gamma = O^+_{2,n}(\mathbb{Z})$, naturally acting in the Cartan domain of type IV of dimension n, are free for $4 \le n \le 10$, and wrote down the weights of the generators for every n:

n	Weights
4	4, 6, 8, 10, 12
5	4, 6, 8, 10, 12, 18
6	4, 6, 8, 10, 12, 16, 18
7	4, 6, 8, 10, 12, 14, 16, 18
8	4, 6, 8, 10, 12, 12, 14, 16, 18
9	4, 6, 8, 10, 10, 12, 12, 14, 16, 18
10	4, 6, 8, 8, 10, 10, 12, 12, 14, 16, 18

It is proved in [14] that if the algebra $A(\Gamma)$ is free, then the group Γ is generated by reflections. It is possible only if the space X is the complex ball or the Cartan domain of type IV.

We will consider the Cartan domain of type IV in dimension $n = 2$. The distinctive feature of discrete reflection groups in Cartan type IV domains is their arithmetic nature. It allows to use modern technics of nivestigation of quadratic lattices over the algebraic number fields. We will need to compute the covolumes of some discrete groups/ The examples of computations of such covolumes can be found, for example, in [1], [6].

2 Basic definitons

Let A be the principal ideal ring. Quadratic A-module is a free A-module of finite rank, equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ with values in A. The module A^n, in which the scalar product is given by the Gram matrix S is denoted by (S). The quadratic module over the field is called a quadratic (vector) space, and over the ring $\mathbb{Z}$ – a lattice.

Let V be a quadratic space over the rationals. Then $V_{\mathbb{R}} = V \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ and $V_{\mathbb{C}} = V_{\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}$ – quadratic spaces over $\mathbb{R}$ and $\mathbb{C}$ respectively. Let the signature of the quadratic space $V_{\mathbb{R}}$ be $(2, n)$, $n \geq 2$.

We consider the domain

$$\tilde{D}_n = \{[z] \in PV_{\mathbb{C}} : (z, z) = 0, (z, \bar{z}) > 0\}$$

in the projective space $PV_{\mathbb{C}}$. This domain has two connected components, and we denote by D_n one of them. It's complex dimension is n. The domain D_n is a hermitian symmetric space of type IV (in Cartan's classification). The domain D_1 is biholomorphic to the upper half-plane H_+, D_2 – to the product of two upper half-planes.

Denote by $G = O(V_{\mathbb{R}})$ the orthogonal group of the quadratic space $V_{\mathbb{R}}$, and let G^+ be it's index 2 subgroup, preserving the domain D_n. The group G^+ acts in the domain D_n inefficiently with the kernel of inefficiency ± 1. It is well-known ([13]) that the group G^+ is a full group of holomorphic automorphisms of the domain D_n, acting transitively in D_n.

2.1 The arithmetic group $\Gamma \subset G^+$ that discretely acts on D_n

We choose a lattice L in the space V such that $\mathrm{rk}_{\mathbb{Z}}(L) = \dim_{\mathbb{Q}} V$. Denote by $O(L)$ the orthogonal group of the lattice $L \subset V$ and let $\Gamma = O(L) \cap G^+$. Such group Γ is called an arithmetic group. The group Γ acts discretely in the domain D_n (possibly, inefficiently). There always exists a normal finite index subgroup Γ_1 of the group Γ such that Γ_1 acts efficiently. It's well-known that the volume of the factor space D_n/Γ_1 (with respect to any G^+-invariant volume form on D_n) is finite. This volume is called the covolume of the group Γ_1 (with respect to the chosen volume form on D_n) and is denoted by $\mathrm{Covol}(\Gamma_1)$. The covolume of the group Γ is the number $\frac{\mathrm{Covol}(\Gamma_1)}{[\Gamma:\Gamma_1]}$ (we will denote it by $\mathrm{Covol}(\Gamma)$).

2.2 The algebra of Γ-automorphic forms $A(\Gamma)$

Denote by $D_n^\bullet$ the cone over the domain D_n in the space $V_{\mathbb{C}}$. From now on we suppose that if $n = 2$ the Witt index of the space V is less than 2.

Definition 2. *An automorphic form of weight k for the group Γ with the character $\chi : \Gamma \to \mathbb{C}^*$ is a holomorphic on $D_n^\bullet$ function f such that:*
1) $f(tz) = t^{-k} f(z)$, $t \in \mathbb{C}^*$
2) $f(g(z)) = \chi(g)f(z)$, $g \in \Gamma$.

Remark 2.1. *The choice of suitable non-zero section of the tautological bundle over the domain D_n gives a bijection between homogeneous Γ-invariant holomorphic functions in the cone $D_n^\bullet$ and functions f, holomorphic on D_n, such that$f(g(z)) = a(g, z)\chi(g)f(z)$, where $g \in \Gamma$, $a(g, z)$ is the factor of automorphy for Γ.*

For an arithmetic group Γ it is well-known that Γ-automorphic forms with trivial character of all non-zero weights form a finitely-generated graded algebra which we will denote by $A(\Gamma)$.

3 Results

Suppose that $\dim_{\mathbb{Q}} V = 4$ (i.e. $n = 2$), the space V is isotropic, but doesn't contain 2-dimensional isotropic subspaces (i.e. the Witt index of the space V is 1).

We choose a natural square-free number $d > 1$ an consider as L the lattice $L_d = U \oplus B_d$, where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_d = \begin{cases} \begin{pmatrix} 2 & 1 \\ 1 & \frac{1-d}{2} \end{pmatrix}, \text{for } d \equiv 1 \pmod 4; \\ \begin{pmatrix} 2 & 0 \\ 0 & -2d \end{pmatrix}, \text{for } d \equiv 2, 3 \pmod 4. \end{cases}$$

The group Γ, corresponding to the lattice L_d is denoted by Γ_d. The choice of these series of arithmetic subgroups is not accidental. Recall that the real quadratic field $\mathcal{K} = \mathbb{Q}(\sqrt{d})$ with the ring of integers $\mathcal{O}_d$ goes together with the extended Hilbert modular group $P\tilde{S}L(2, \mathcal{O}_d) = Gal(\mathcal{K}/\mathbb{Q}) \ltimes PSL(2, \mathcal{O}_d)$, naturally acting in D_2 as discrete arithmetic group. The group $P\tilde{S}L(2, \mathcal{O}_d)$ is embedded in the group Γ_d ([16]). It's well-known that the group Γ_d is a maximal discrete subgroup in the group G^+, containing the group $P\tilde{S}L(2, \mathcal{O}_d)$ ([4]).

The main result of the paper is the following theorem ([19], [20]).

Theorem 1. *Let $\Gamma' \subset \Gamma_d$ be a subgroup of finite index in the group Γ_d, containing* $-\mathrm{Id}$. *If the algebra $A(\Gamma')$ is free then $d \in \{2, 3, 5, 6, 13, 21\}$.*

Corollary 1. *The algebra $A(\Gamma_d)$ is free only iff $d = 2$, 3 or 5.*

The proof of corollary 1. It is showed in [15], in particular, that for the algebra $A(\Gamma)$ to be free it is a necessary condition that the stabiliser Γ_v in the group Γ of each isotropic vector $v \in V$ is generated by reflections, mirrors of which contain this vector. But for the group Γ_d it is not so if $d \in \{6,\ 13,\ 21\}$.

Consider the cases $d = 2$ and $d = 5$. The group $P\tilde{S}L(2, \mathcal{O}_d)$ is embedded into the group Γ_d, but doesn't contain $-\mathrm{Id}$. Adjusting $-\mathrm{Id}$ to the subgroup $P\tilde{S}L(2, \mathcal{O}_d)$ we get the whole group Γ_d. It follows from the fact that the fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ have one class and their fundamental unit has the norm -1, so the group $P\tilde{S}L(2, \mathcal{O}_d)$ is a maximal arithmetic group acting in the product of two upper half planes ([16]). But it is proved in the papers[8] and [7] that the algebra of automorphic forms of even weight for the groups $P\tilde{S}L(2, \mathcal{O}_d)$ is free for $d = 2$ and $d = 5$ respectively.

The case $d = 3$ is proved in the paper [8]. $\square$

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