**National Research University «Higher School of Economics»**
**"Algebra and Geometry" – Course syllabus**
**HSE and University of London**
**Parallel Degree Program in Management and Digital Innovation**

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**Syllabus**

# Algebra and Geometry

(6 ECTS)

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## 1. Course description

### a) Pre-requisites

To learn the course, students should have the following knowledge and skills:

- knowledge of algebra and geometry at the level of the school curriculum;
- ability to navigate the Internet resources and to know the basics of bibliography;
- sufficient proficiency in English.

### b) Abstract

In the process of studying the discipline, students will become familiar with theoretical foundations and basic methods of solving tasks on the following topics

- Systems of linear equations. Row operations and Gaussian elimination. Vectors and Matrices. Linear spaces. Homogeneous systems and null space.
- Matrix inversion and determinants. Leontief input-output analysis.
- Complex numbers and their properties.
- Eigenvalues and eigenvectors. Diagonalization of matrices.
- Sequences, series and difference equations. Coupled first-order difference equations. Their applications in economics and finance
- Inner product and orthogonality. Lines in $\mathbf{R}^2$, planes and lines in $\mathbf{R}^3$, lines and hyperplanes in $\mathbf{R}^n$.
- Orthogonal diagonalisation. Quadratic forms and conic sections
- Direct sum and projections. Fitting function to data: least squares approximation.

## 2. Learning Objectives

The course aims to provide students with understanding of key concepts and methods of algebra and geometry for understanding other practical courses, related to data analysis and programming.

## 3. Learning outcomes

Upon completion of the course, students should:

**Be aware of:**

- the basics of linear algebra and geometry;
- key concepts and approaches to their study.

**Be able to:**

- formalize the problem from subject area,
- choose the adequate methods of solutions,
- perform essential calculations and to interpret the results;

**Learn how to:**

- be able to formalize the problem from subject area,
- choose the proper methods of solutions,
- find and to interpret the results obtained.

## 4. Course plan

**Theme 1. System of linear equations and matrices.**

Introduction to Systems of Linear Equations. Gaussian Elimination. Matrices and Matrix Operations. Algebraic Properties of Matrices. Powers of the matrix. Transpose matrix. Inverse matrix. Method for finding $A^{-1}$ with row operations. Symmetric matrices. Vectors in $\mathbf{R}^n$. Inner product.

**Theme 2. Vector spaces and Homogeneous systems.**

Real Vector Spaces and Subspaces. Linear Independence and Dependence of vectors. Coordinates and Basis. Dimension. Solution Spaces of Homogeneous Systems. Change of Basis. Row Space, Column Space, and Null Space. Rank, Nullity and the Fundamental Matrix Spaces.

**Theme 3. Determinants and inverse matrix.**

Determinants of matrices. Finding determinants by Cofactor Expansion. Evaluating Determinants by Row Reduction. Properties of Determinants. Cramer's Rule. Nondegenerate matrix and existence of inverse. Adjoint matrix. Using adjoint matrix to find inverse matrix. Leontief input-output analysis.

**Theme 4. Complex numbers.**

Complex numbers. Complex conjugate. Algebra of complex numbers. The complex plane. The polar form of a complex number. The modulus and the argument of a complex numbers. Complex vector spaces and complex matrices.

**Theme 5. Eigenvalues and Eigenvectors.**

Eigenvalues and Eigenvectors. Diagonalization of a square matrix. Eigenvalues and Eigenvectors of Matrix Powers. Determinants and eigenvalues. Similar matrices. Finding the power of a matrix using diagonalization.

**Theme 6. Difference equations.**

Sequences and progressions. Compound interest. Frequent compounding
Series and financial applications. First-order difference equations and their solution. Long-term behavior. The cobweb model. Second-order difference equations. Behavior of solutions. Economic applications.

**Theme 7. Euclidean vector spaces. Lines, planes and hyperplanes.**

Inner product and orthogonality. Euclidean vector spaces. Lines in $\mathbf{R}^2$, planes and lines in $\mathbf{R}^3$, lines and hyperplanes in $\mathbf{R}^n$. Geometry of linear systems.

**Theme 8. Quadratic forms and conic sections.**

Orthogonal diagonalization of symmetric matrices. Quadratic forms. Quadratic forms and conic sections. Circle, ellipse or hyperbola.

**Theme 9. Direct sum and projections. Fitting function to data.**

The direct sum of two subspaces. The orthogonal complement of a subspace. Orthogonal complements of null spaces and ranges. Projections. Orthogonal projections. Orthogonal projection onto the range of a matrix. Minimizing the distance to a subspace. Fitting functions to data: least squares approximation.

## 5. Reading list

### a) Required

1. Anthony, M. Linear algebra: concepts and methods / M. Anthony, M. Harvey. – Cambridge [etc.]: Cambridge University Press, 2012. – 516 c.
2. Anton, H. Elementary linear algebra: with supplement applications / H. Anton, C. Rorres. – 10th ed. – Hoboken: John Wiley & Sons (Asia), 2011. – 777 c.

### b) Optional

1. Lay, D. C. Linear algebra and its applications / D. C. Lay. – 4th ed. – Boston [etc.]: Addison-Wesley: Pearson, 2012. – 559 c.
2. Anthony, M. Mathematics for economics and finance: methods and modelling / M. Anthony, N. Biggs. – 19th ed. – Cambridge [etc.]: Cambridge University Press, 2012. – 394 c.
3. Simon, C. P. Mathematics for economists / C. P. Simon, L. Blume. – New York: W.W.Norton & Company, 1994. – 930 c.

## 6. Grading System

The *final grade* can be obtained by rounding the score S obtained by the following formula:

$$S=0{,}18*C1+0{,}18*C2+0{,}28*W+0{,}36*E,$$

where

- E is a mark for the final exam on the course, held at the end of the second module (its duration is 120 minutes).
- C1 and C2 are grades for the first (held at the end of the first module or at the beginning of the second one) and the second (held at the middle of December) control works. The duration of both works is 90 minutes. Control works are not allowed to be rewritten at extra time.
- W is a score obtained for the regular quizzes held at seminars, homeworks and seminar activity. W is calculated as average of all marks obtained for quizzes, homeworks (each of them is 10 mark max) and seminar activity.

## 7. Examination Type

**Examples of control tasks**

1. Solve the system of linear equations $\begin{cases} x_1 + 3x_2 + 6x_3 = 40, \\ 3x_1 + 10x_2 + 19x_3 = 130, \\ 2x_1 + 3x_2 + 10x_3 = 55. \end{cases}$

2. Find the general solution of the system of linear equations $\begin{cases} x_1 + x_2 + x_3 + x_4 = 4, \\ -x_1 - 2x_3 = -5, \\ -2x_1 - 3x_2 - 2x_4 = -1. \end{cases}$

3. Find the matrix $\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}^7$.

4. Find the determinant of the matrix $\begin{pmatrix} 5 & 0 & 0 & 5 \\ 3 & 0 & 1 & 2 \\ 5 & 4 & 3 & 1 \\ 2 & 0 & 0 & 2 \end{pmatrix}$.

5. From the rows system of matrix $A = \begin{pmatrix} 1 & -5 & 7 & -7 \\ -5 & 26 & -36 & 36 \\ -3 & 1 & -7 & 7 \\ 2 & 1 & 3 & -3 \end{pmatrix}$ select the maximal linearly independent subsystem and express the remaining rows as a linear combination of the selected ones.

6. Find the inverse matrix $A^{-1}$ if $A = \begin{pmatrix} 2 & -5 & 3 \\ -5 & 4 & 3 \\ 3 & -2 & 3 \end{pmatrix}$ and check the condition $A \cdot A^{-1} = E$ is valid.

7. Consider an economy with three industries, $i_1$: water, $i_2$: electricity and $i_3$: gas inter-linked so that the corresponding consumption matrix is $C = \begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0 & 0 & 0.1 \end{pmatrix}$. Each week the external demands for water, electricity and gas are, respectively, $d_1 = 40000$, $d_2 = 100000$, $d_3 = 72000$ (units measured in dollars). (a) How much water, electricity and gas is needed to produce a unit of electricity? (b) What should be the weekly production of each industry in order to satisfy all demands exactly?

8. Solve the matrix equation $\begin{pmatrix} 2 & -4 \\ 3 & 5 \end{pmatrix} X = \begin{pmatrix} 2 & 1 & 0 \\ 5 & -3 & 2 \end{pmatrix}$.

9. Consider the complex numbers $z = \sqrt{3} - i$, $w = 1 + i$ and $q = \dfrac{(\sqrt{3} - i)^6}{(1+i)^{10}}$. Plot $z$ and $w$ as points in the complex plane. Express them in exponential form and hence evaluate $q$. Express $q$ in the form $a + ib$.

10. Find eigenvalues and eigenvectors of the matrix $C = \begin{pmatrix} 2 & 10 \\ -5 & 13 \end{pmatrix}$.

11. Find the Cartesian equation of the plane passing through the point $A(1,-1,4)$ which is orthogonal to the line $\frac{x+4}{-3} = \frac{y}{-2} = \frac{z+1}{20}$.

12. Find the Cartesian equation of the median $AM$ of the triangle $ABC$ with vertices $A(-5,3,1)$. $B(10,9,4)$, $C(-20,-15,6)$.

13. Find the Cartesian equation of the plane, all points of which are equidistant from the points $A(-5,2,-2)$ и $B(-15,-10,-14)$.

14. Express the quadratic form $9x^2 + 4xy + 6y^2$ as $\vec{u}^T A \vec{u}$, where $A$ is a symmetric $2 \times 2$ matrix and $\vec{u} = (x, y)$, and find the eigenvalues of $A$. Deduce whether the quadratic form is positive definite or otherwise, and determine what type of conic section is given by the equation $9x^2 + 4xy + 6y^2 = 10$. Orthogonally diagonalize the matrix $A$ and use this information to sketch the curve $9x^2 + 4xy + 6y^2 = 10$ in the $xy$-plane.

15. Suppose that vector space $V = \mathbf{R}^3$ with the standard inner product and assume that subspace $S = \mathrm{Lin}\{\vec{u}, \vec{v}\}$ where $\vec{u} = (1;2;-1)$ and $\vec{v} = (1;0;1)$. Describe $S^{\perp}$.

16. Quantities $X$ and $Y$ are related by a rule of the form $Y = \frac{a}{X} + b$ for some constants $a$ and $b$. Use the following data to estimate $a$ and $b$ by the least squares method:

| $X$ | 1/5 | 1/4 | 1/3 | 1/2 | 1 |
|---|---|---|---|---|---|
| $Y$ | 4 | 3 | 2 | 2 | 1 |

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## 8. Methods of instruction

**Lectures** are mostly used to present a new material and theoretical basement to students in the systematic way. They form a theoretical basis for seminars, homeworks, quizzes and the final exam.

**Seminars** are intended to demonstrate how lecture materials can be applied to solve practical problems and to help students complete a related homework assignment.

**Homeworks** are the key part of the course as they give students the possibility to develop the methods and skills obtained during lectures and seminars.