SYLLABUS

Calculus-2

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1. Course description

a) Prerequisites.

High school algebra and trigonometry, basic concepts of calculus (e. g., sequences, limits and continuity, derivatives, integrals).

b) Abstract.

This course covers specific topics of advanced calculus, such as numeric and functional series, infinite products, Eulerian integrals, multiple integrals. The convergence and functional properties of power series are considered along with their applications to some problems of discrete mathematics involving the generating functions. Most of the developed concepts are illustrated with Matplotlib.

2. Learning objectives

After completing this course, students will understand the following concepts:

- Convergence and divergence of infinite series and infinite products; the rate of convergence.
- Pointwise and uniform convergence of the functional series; the functional properties of their sums.
- Representing functions by power series; Taylor series of the most common elementary functions.
- Generating functions and their applications for solving linear recurrence relations.
- Representing functions by trigonometric Fourier series.
- Integrals depending on a parameter; beta and gamma functions.
- Double and triple integrals; reduction to iterated integrals (Fubini's theorem).

- Change of variables in multiple integrals; polar, cylindrical and spherical coordinate systems.

3. Learning outcomes

After successful completion of the discipline the students should be able to:

- calculate sums of series using the methods of partial sums, power series, Fourier series;
- study given series and products for convergence;
- apply the Stirling's formula for establishing the asymptotics of specific sequences;
- examine a given functional sequence or series for uniform convergence on a given interval;
- determine the radius and the domain of convergence of power series;
- represent a given analytic function by convergent power series;
- solve linear difference equations of the first and the second order;
- find the Fourier series of a given function and justify its convergence;
- apply the properties of the Eulerian integrals for the calculation of specific integrals;
- compute double and triple integrals by means of Fubini's theorem, using a change of variables if necessary;
- calculate areas of regions, volumes of solids and surface areas.

Also students will develop skills of applying Python to some practical aspects of calculus. They will be able to:

- perform numerical calculations of series and integrals;
- make visualizations in order to achieve deeper understanding of a certain concept;
- find analytical and numerical solutions to the same problem and compare them to each other.

4. Course Plan

1. Infinite series.

Finite sums and products. Harmonic numbers. Convergent and divergent series. Examples of series: telescoping series, geometric series, decimal fractions, p-series, alternating series. Necessary condition for convergence. Integral test. Tail of a series.

2. Series of Nonnegative Terms. Convergence Tests.

Series of nonnegative terms. Comparison test. Limit comparison test. Ratio and Root tests, their relationship. Rate of convergence. Gauss test.

3. Alternating series. Absolute and conditional convergence.

Cauchy criterion. Alternating series test. Dirichlet/Abel tests. Absolute and conditional convergence. Sine and cosine sums. Conditionally convergent alternating and trigonometric series.

4. Products of series. Infinite Products

Rearrangement of series, Cauchy's and Riemann's theorems. Product of series, Cauchy products. Convergence and divergence of infinite products, reduction to series. Wallis product. Stirling's formula. Sine product formula.

5. Uniform convergence.

Sums of functions. Pointwise and uniform convergence of functional sequences and series, their relationship. Cauchy criterion for the uniform convergence. Tests for the uniform convergence of series: alternating series test, Weierstrass M-test, Dirichlet/Abel tests. Interchange of limits, continuity of a limit function. Term-by-term integration and differentiation of uniformly convergent series. Riemann zeta function.

6. Power series.

Examples of power series. Radius and interval of convergence of power series. Cauchy–Hadamard formula. Uniform convergence of power series. Term-by-term differentiation and integration of power series. Abel's theorem. Products of power series. Uniqueness of power series expansion. Taylor series of common functions. Binomial series. Analytic functions. Complex power series. Euler's formula.

7. Generating functions.

Examples of generating functions. Operations with generating functions. First-order and second-order difference equations. Fibonacci numbers. Convolutions, Catalan numbers. Exponential generating functions. Binomial convolutions, Bernoulli numbers. Tangent and cotangent power expansions.

8. **Fourier series.**

   Trigonometric series. Fourier coefficients and Fourier series. Parseval's identity. Piecewise functions. Riemann–Lebesgue lemma. Dirichlet kernel. Dini's conditions for convergence of Fourier series. Pointwise convergence of the Fourier series of a 2π-periodic piecewise continuously differentiable function. Applications of Fourier series.

9. **Integrals depending on a parameter.**

   Proper integrals depending on a parameter, their properties. Complete elliptic integrals. Convergence tests of the improper integrals. Improper integrals depending on a parameter (IIDP). Uniform convergence, Weierstrass M-test, Dirichlet/Abel tests. Properties of IIDP. Dirichlet integral.

10. **Eulerian integrals.**

    Beta and gamma functions, their properties and relationship. Gauss representation formula. Euler's reflection formula. Euler–Poisson integral. Digamma function.

11. **Double integrals.**

    Riemann sums. Double integrals over rectangles. Lower and upper Darboux sums. Darboux criterion. Properties of double integrals. Fubini's theorem, reduction to iterated integrals. Double integrals over general regions. Change of variables in double integrals. Polar coordinate system.

12. **Triple integrals. Applications of double and triple integrals.**

    Fubini's theorem, reduction to iterated integrals. Change of variables in triple integrals. Cylindrical and spherical coordinate systems. Calculating areas of domains, volumes of solids, areas of surfaces.

13. **Improper integrals. Multiple integrals.**

    Exhaustions. Improper double integrals. Volumes of the standard simplex and an n-ball. Gaussian integral.

5. Reading List

a) Required.

- Stewart J. Calculus. Early Transcendentals. 6th edition. Thomson Brooks/Cole, 2008.

- Avner Friedman. Advanced Calculus. Dover edition, 2007.

b) Optional.

- Vladimir A. Zorich. Mathematical Analysis I, II. Springer, 2004.
- Виноградова И. А., Олехник С. Н., Садовничий В. А. Математический анализ в задачах и упражнениях. – М.: МЦНМО, 2018.
- Ronald R. Graham, Donald E. Knuth, Oren Patashnik. Concrete Mathematics. 2nd edition. Addison-Wesley, 1994.

6. Grading System

The academic term is divided into 2 modules. At the end of the first module the students pass a written test (T). At the end of the second module the students pass a written exam (E). During the term students must also complete weekly home assignments. A randomly chosen problem from a homework can be given as a small test in classes at the beginning of the next seminar. Quizzes are held regularly in classes at the end of the lectures.

Regular activity grade (R) consists of grades for quizzes and tests passed at class and weekly home assignments. Problems solved by students at the desk in class are also taken into account. The grade of each problem is announced before this problem is given to students.

Some specific more complicated problems are marked as «bonus». Students can pass the solutions of bonus problems to the teacher in written form before the end of the module.

Here is the rule that defines the final grade of the term. At the first stage we compute the quantity

$$A = \min\{0.4 \cdot R + 0.2 \cdot T + 0.2 \cdot E + 0.2 \cdot J + 0.2 \cdot B, 10\},$$

where R is the mark for the regular activity, T is the test mark, E is the exam mark, J is the mark for the Jupyter notebook tasks, and B is the total amount of bonus points. The ranges of these marks are the following:

$$0 \leqslant R, J \leqslant 10, \quad 0 \leqslant T, E \leqslant 12, \quad 0 \leqslant B \leqslant 12.$$

At the second stage, the final grade is computed by rounding the quantity A off to an integer, according to the standard mathematical rule: if the fractional part of the grade lies within the interval $[0,\ 0.5)$, then A is rounded off downward (to the greatest integer less than or equal to A), if the fractional part of the grade lies within the interval [0.5, 1), then A is rounded off upward. Only grade A is rounded off; any other grades are not rounded. The final grade obtained by students at the end of the second module is the final course grade.

7. Examination Type

Both the test and the exam consist of a selection of problems similar to those from the seminar/homework exercise lists. Students solve the problems in written form during 2 academic hours and pass the solutions to the teacher.

8. Methods of Instruction

There will be 15 theory lectures, of 2 academic hours each, during which conceptual ideas are explained.

Each lecture is followed by a seminar of 2 academic hours, in which students solve exercises that deepen the understanding of the materials and train problem solving skills.

Each week a list of 10–15 exercises is provided. 5–10 exercises from this list are given for homework. These homeworks need to be submitted before the next lecture.

Lecture notes, exercise lists, and all practical information is maintained on the wikipage of the course: http://wiki.cs.hse.ru.