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Syllabus
Applied Methods of Linear Algebra

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Meeting Minute # ___dated _____________ 20_

1. Course Description

Pre-requisites

It is assumed that the students have graduated either *Linear Algebra*, or *Linear algebra and Geometry*, or *Higher Mathematics* standard course. Some elementary programming skills (preferably, using Python) are also assumed.

Abstract

In the lecture course, we consider some topics of linear algebra beyond the standard first year course which are extremely important for applications. Mostly, these are applications to data analysis and machine learning, as well as to economics and statistics.
We begin with inversions of rectangle matrices, that is, we discuss pseudo-inverse matrices (and their connections to the linear regression model). Among others, we discuss iteration methods (and their using in models of random walk on a graph applied to Internet search such as PageRank algorithm), matrix decompositions (such as SVD) and methods of dimension decreasing (with their connection to some image compression algorithms), and the theory of matrix norms and perturbation theory (for error estimates in matrix computations). The course includes also symbolic methods in systems of algebraic equations, approximation problems, Chebyshev polynomials, matrix functions such as exponents etc.
We plan to invite some external lecturers who successfully apply linear algebra in their work. The students are also be invited to give their own talks on additional topics of applied or theoretical linear algebra.

2. Learning Objectives

The aim the course is to provide both theoretical background and practical experience of solutions of linear algebra problems which appear in computer science, data analysis, mathematical modelling, machine learning, and economical models. The course covers some topics of matrix analysis and numerical methods of linear algebra as well as some elements of functional analysis and mathematical statistics. We provide a number of useful algorithms which can be implemented and used by students. A number of of these algorithms are in the core of modern machine learning and data analysis.

3. Learning Outcomes

Upon completion of this course students would be able

- to find the pseudoinverse matrix and to use to find the minimal length solution of an indefinite linear system,
- to calculate and use SVD and full rank decompositions,
- to understand the least squares method and to find the least square solution of a system of linear equations,
- to use the linear regression model to make simple prognoses,
- to interpolate an unknown function by polynomials and by polynomial splines,
- to use Hermite polynomial to interpolate functions with known values of the derivatives,
- to use Bézier curve for elegant smoothing of a polygon,
- to find various norms of the same vectors and to find a vector norm which is the most appropriate for a give problem,
- to find matrix norms compatible with a given vector norm,
- to find a norm of given vector by using a unit circle associated to the norm only,
- to calculate the matrix norms induced by the most important vector norms,
- to evaluate matrix norms using the spectral radius and the singular radius of a matrix,
- to apply Chebyshev polynomials to various function interpolation problems,
- to use dot product in spaces of functions and orthogonal families of polynomials to for approximation problems,
- to find the condition number of a matrix and to use it to evaluate the error of a solution of a system of linear equations,
- to use the condition number to evaluate the error of an approximate calculation of a matrix inverse,
- to use the various iteration methods (including the Jacobi method and the Gauss-Seidel method) for solutions of (large) systems of linear equations,
- to evaluate the number of iterations needed to find a solution with a given precision,

- to apply the First and Second Gershgorin's theorems to evaluate the eigenvalues of a given matrix,
- to use advanced effective methods for finding the characteristic polynomials and the eigenvalues of a matrix,
- to use iterative methods for finding an eigenvector and an eigenvalue of a matrix, including the Perron eigenvector and the Perron-Frobenius eigenvalue of an indecomposable nonnegative matrix,
- to apply the iterative method of finding the Perron eigenvector in the PageRank algorithm,
- to use PageRank for influence ranking of a social network of a known configuration,
- to understand the linear productive model and to use the Leontiev inverse for evaluating the direct and indirect multi-sector transactions in the global economy,
- to calculate functions of square matrices, including matrix powers, exponents, polynomials, and logarithms using either the Jordan form or the Lagrange−Sylvester) polynomials.

4. Course Plan

The course will cover the following topics. Note that the starred topics are optional. Depending on the students' preferences, these topics might be proposed for the students' talks.

Topic 1. Pseudoinverse matrix, the least squares method, and linear regression

The pseudoinverse (aka Moore-Penrose inverse) matrix, its definitions, main properties, and methods of computations. Full rank decomposition, calculation of the pseudoinverse using full rank decomposition. Calculation of the pseudoinverse using the singular value decomposition (SVD).

The idea of least squares method. Solution of a linear problem by the least squares method using the pseudoinverse. Formula of general solution of a linear system using the pseudoinverse matrix. The linear regression problem and its solution in terms of the least squares method. Examples of practical problems solution (such as weather forecasts and market price forecasts).

Topic 2. Polynomial interpolation.

The polynomial interpolation problem. Lagrange interpolation polynomial and Vandermonde determinant. Hermite interpolation, Hermite polynomial. Uniqueness of the solution of the Hermite interpolation problem. Quadratic and cubic splines. Bézier curves. Bézier splines and their applications in desing.

Topic 3. Metrics and norms.

Metric spaces. Triangles and balls in metric spaces. Finite metric spaces and graphs. Normed vector spaces, p-norms. Norms and dot products. A parallelogram criterion for a normed vector space to be a Eucledian space. Equivalence of all norms in a finite-dimensional real vector space. Minkovski's theorem about unit balls in finite-dimensional real vector spaces.

Matrix norms, their compatibility with vector norms. Frobenius norm. Spectral radius as a lower bound for matrix norms. Matrix norm induced by a vector norm. Examples: the matrix norms induced by the 1-norm and the infinity-norm. Singular radius as a matrix norm induced by the Euclidean vector norm.

Topic 4. Chebyshev polynomials and polynomial approximation.

Chebyshev polynomials of the first kind. Recurrent definition and main properties, trigonometric definition. Chebyshev polynomials of the second kind. Relations between Chebyshev polynomials of the first and the second kinds. Explicit expressions for the Chebyshev polynomials.

Some important norms and dot products in functional spaces. Polynomials approximating the zero function. Chebyshev theorem about polynomials minimizing the infinity-norm. Polynomials minimizing the 1-norm (Zolotarev's theorem).

Orthogonal systems of polynomials. Orthogonality properties of the Chebyshev polynomials. Approximation of functions by polynomials.

Topic 5. Elements of perturbation theory and evaluation of errors.

The condition number of a matrix, its properties. Bounds for the conditional number in terms of the eigenvalues. The conditional number with respect to the Eucledian norm and singular values.

Error bounds of approximate solutions of systems of linear equations in terms of the errors of the evaluation of the right-hand side and the matrix of the system. Examples of approximate solutions of linear systems and error evaluations. Approximate inverse matrix and evaluation of the error of the approximation.

Topic 6. Iterative methods and systems of linear equations.

The basic iteration method for systems of linear equations. Convergence, norms and spectral radius. Changing the form of the system for convergence of the basic iteration method. Gauss–Seidel method. Jacobi method.

Topic 7. The eigenvalues problem and the Gershgorin's theorems.

Location and perturbation of eigenvalues, the first and the second Gershgorin's theorems.

Cyclic cells and the Frobenius normal form of a matrix. Krylov's method of finding the minimal polynomial of a matrix. Danilevski's method. Interpolation method of finding the characteristic polynomial of a matrix. Iterative methods for eigenvalues.

Topic 8. Nonnegative matrices and PageRank.

Nonnegative and positive matrices. Irreducible nonnegative matrices and strongly connected graphs. Properties of powers and of the inverse of the irreducible nonnegative matrices.

The Perron-Frobenius theorem. Linear productive model, productive matrices, and the Leontiev inverse. The foundation of the Input-Output analysis in macroeconomics. Brief overview of some linear algebra problems appearing in modern Input-Output analysis.

Random walk on a graph. PageRank algorithm and the Internet search. PageRank and social influence indices.

Topic 9. Functions of matrices.

Power series of matrices and convergence. Using the Jordan normal form to calculate an analytical function of a matrix. Calculation of a value of an analytical function of a matrix using the Hermite (or the Lagrange−Sylvester) polynomials.

Topic 10. Low rank approximation and dimensionality reduction.

A problem of approximation of a matrix by a matrix of bounded rank. Using SVD for the approximation of a matrix by a matrix of bounded rank and minimal norm in the case of a unitary invariant norm. Examples with image compression and handwritten digits recognition.

Topic 11. * Linear algebra and optimization.

Linear programming model. Examples. The dual problem. Methods of solutions of the linear programming problem, simplex-method.

Topic 12. *Linear codes.

Error-correcting codes. Linear codes.

Topic 13. *Numerical methods of the solutions of the restricted (non-negiative) quadratic programming problem.

Topic 14. *Effective algorithms and parallel computations in linear algebra. Quick matrix multiplication.

Topic 15. *Tropical algebra.

Tropical (max-plus) algebra. Tropical systems of linear equations, linear algebra over the tropical semiring. Applications to the scheduling theory and to random walks on graphs.

Topic 16. *Algebraic equations and symbolic computations.

Algebraic equations and polynomial ideals. Admissible ordering (i.e., multiplicative well-ordering) of monomials on commuting variables. Groebner bases. Groebner reduction with respect to list of polynomials. Critical pairs and s-polynomials. Buchberger algorithm. Algorithmic elimination theory for systems of algebraic equations. Examples of solutions of systems with finite and infinite sets of solutions. Applications to robotics, geometrical problems, puzzles.

Students are encouraged to prepare their own talks on additional topics of applied and theoretical linear algebra. Ideally, a talk includes both theoretical part and some examples of applications of the theory to practical problems.

A sample of topics for the students' talk is given below.

1. Singular value decomposition (SVD) and image compression.

Problem. To develop a software for image compression using SVD (see, e.g., [1]).

It is assumed that the software finds SVD for rectangular matrices of high size and to test it for the matrices of small size. Then the software should be applied to photo compression. One can use a standard library function to convert a phot image to a matrix.

What would happens if, in contrast to the standard algorithm, one vanishes the *largest* singular values?

References.

1. R. Horn and C. Jonson. *Matrix analysis*. 2nd edition. Cambridge Univ. Press, 2013
2. Demmel, James W. *Applied numerical linear algebra*. Siam, 1997.
3. http://demonstrations.wolfram.com/ImageCompressionViaTheSingularValueDecomposition/

2. Using of the singular value decomposition for face recognition

Problem. To develop a software for face recognition on a photo image using SVD (see, e.g., [1]).

It is assumed that the software finds SVD for rectangular matrices of high size and to test it for the matrices of small size. Then the software should be applied to image recognition (see, e.g., [2]). One can use a standard library function to convert a phot image to a matrix.

What would happens if we change one of the terms in SVD?

References.

1. R. Horn and C. Jonson. *Matrix analysis*. 2nd edition. Cambridge Univ. Press, 2013
2. Demmel, James W. *Applied numerical linear algebra*. Siam, 1997.
3. http://link.springer.com/chapter/10.1007%2F978-1-4020-6264-3_26
4. Tian Y, Tan T, Wang Y, Fang Y: Do singular values contain adequate information for face recognition? Pattern Recogn. 2003, 36:649–655.

3. Linear algebra and the Internet search

Problem. To develop a software that realize the open version of the most known algorithm of the internet page ranking. You can decide the degree of ``toyness'' of the software.

References.

K Bryan, T Leise The $25,000,000,000 eigenvector: The linear algebra behind Google. - Siam Review, 2006 – SIAM

5. Quadratic optimization

The quadratic optimization problem (aka the quadratic programming problem) is a generalized version of the least squares approach. In this problem, one needs to minimize a quadratic function (generalized ``squared length of the error vector'') with additional restrictions having the form of linear equalities and linear inequalities.

There are several methods of solution of the quadratic optimization problem. It is proposed to study and elements such methods and to compare their applications to an actual problem of economic statistics, that is, to construction and balancing of the input-output tables of a national economy.

Some references.

Hadley G. *Nonlinear and Dynamic Programming.* Addison-Wesley, Massachussetts, 1964

For modern methods of convex optimization, see

Nesterov Y. *Lectures on convex optimization.* Springer, 2018

For examples of problems from economics, see http://www.wiod.org/publications/papers/wiod2.pdf , Subsection 2.7 and 2.8

6. Multidimensional arrays, tensor train decomposition, and big data analysis

Problem. To illustrate using of the tensor train decompositions and their analogues in the big data analysis.

References.
http://epubs.siam.org/doi/abs/10.1137/090752286

V. Oseledets and E. E. Tyrtyshnikov. TT-Cross approximation for multidimensional arrays. INM RAS Preprint, 2009-05.
http://people.csail.mit.edu/moitra/docs/bookex.pdf

7. Solutions of systems of algebraic equations and inequalities

Modern symbolic methods of solutions of systems of algebraic equations and inequalities are based on Groebner bases and Cylindrical algebraic decomposition. It is assume that the main methods of these theories (or at least one of them) will be illustrated in the talk. See Chapters 11 and 12 in [1] as well as [2] and [3].

References.

1. Basu, Saugata; Pollack, Richard; Roy, Marie-Françoise “Algorithms in real algebraic geometry.” Second edition. Algorithms and Computation in Mathematics, 10. Springer-Verlag, Berlin, 2006. x+662 pp. ISBN 978-3-540-33098-1; 3-540-33098-4 ; авторский вариант см.https://perso.univ-rennes1.fr/marie-francoise.roy/bpr-ed2-posted3.pdf
2. D. Cox, J. Little, and D. O’Shea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Springer Science & Business Media, 2013
3. Mats Jirstrand."Cylindrical Algebraic Decomposition – an Introduction" http://www.diva-portal.org/smash/get/diva2:315832/FULLTEXT02

8. Tropical algebra and scheduling

Problem. To give a description of main concepts of tropical linear algebra and to illustrate its applications to scheduling.

References.

1. P. Butkovic. *Max-Linear Systems: Theory and Algorithms*. Springer, 2010
2. Francois Louis Baccelli, Guy Cohen, Geert Jan Olsder, Jean-Pierre Quadrat. *Synchronization and Linearity: An Algebra for Discrete Event Systems*. John Wiley&Sons, 1993. P. Butkovic. Max-Linear Systems: Theory and Algorithms. Springer, 2010.Francois Louis Baccelli, Guy Cohen, Geert Jan Olsder, Jean-Pierre Quadrat, "Synchronization and Linearity: An Algebra for Discrete Event Systems", John Wiley&Sons, 1993 , pp. 514

5. Reading List

a) Required

Olver, P.J., and Shakiban, C. *Applied linear algebra*. 2nd edition. Springer, 2018

R. Horn and C. Jonson. *Matrix analysis*. 2nd edition. Cambridge Univ. Press, 2013

F. Aleskerov, H. Ersel, and D. Piontkovski. *Linear algebra for economists.* Springer, 2011

b) Optional

G. Birkhoff and T. Bartee. *Modern applied algebra*. McGraw-Hill, NY, 1970

Bryan, K. and Leise, T., 2006. *The $25,000,000,000 eigenvector: The linear algebra behind Google*. SIAM review, *48*(3), pp.569-581

P. Butkovic. *Max-Linear Systems: Theory and Algorithms.* Springer, 2010

D. Cox, J. Little, and D. O’Shea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Springer Science & Business Media, 2013

6. Grading System

The final mark is calculated by the formula

Total = max(10, (IntermadiateTest + FinalExam)/2 +BonusScore),

where IntermadiateTest means the mark for the written test after the 1st module, FinalExam means the mark for the final exam, and BonusScore means the additional score for the students who either prepare a special talk or demonstrate enormous activity during the classes. The talks are evaluated according to the scientific and practical value of the talk, the quality of the presentation, and the number of students participated in the project.

7. Examination Type

There be a written test after the first module and a final written exam. The test after the first module could be replaced by the homework with a list of individual problems covering the most topics of the first module. In this homework, using of a (created by the author of the solution) is allowed. In the final exam, no computer instruments but a simple calculator can be used.

Sample examination problems are given below.

1. Using the Lagrange polynomial formula, find a polynomial f of degree at most 3 such that $f(-4) = -53, f(-2) = -9, f(2) = 31, f(1) = 12$.

2. Find a full rank decomposition and the pseudoinverse of the matrix $\begin{pmatrix} -4 & -3 & -7 \\ 12 & 9 & 0 \\ -6 & -3 & 5 \\ 16 & 12 & -14 \end{pmatrix}$.

3. Among all lest squares solutions of the system of equations given below, find a vector of minimal length:

$$\begin{cases} -3y - 3z = 0 \\ 8x + 5y - z - 4t = 5 \\ -3x - 3y + z + t = -1 \\ 16x + 4y - 8z - 4t = 11 \end{cases}$$

4. For the polynomial $x\text{^}3 - 2x\text{^}2 + x + 1$, find the best approximation by a polynomial of degree at most two with respect to the 1-norm on the interval $[1, 3]$.

5. For the function $cos(x) + x\text{^}2 + x + 1$, find an approximation by a polynomial of degree at most two which is the best with respect to the norm induced by the dot product $< g(x), f(x) > = \frac{2}{\pi}\int_{-1}^{1} \frac{g(x)f(x)}{\sqrt{1-x^2}}$.

6. Find the singular value decomposition of the following matrix. Use it to find a full rank decomposition of the matrix:

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

7. Find the value $f(A)$ of the function $f(l) = ln(l + 2)$, where A = $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

8. Either find a matrix norm for 2 by 2 matrices such that the sequence of norms of matrices A^n converges or prove that such a norm does not exist, where A=$\begin{pmatrix} -0.85 & -1 \\ 0 & 0.85 \end{pmatrix}$.

9. Find all q such that the equation $18x^2 + 24xz + 4y^2q^2 - 16y^2 + 9qy^2 + 12yqz + 4qz^2 + 8z^2 = 1$ defines a unit circle with respect to some norm on the 3-space. For each such q, find the value of such a norm on the vector $(1,1,1)$.

10. Find an approximate solution of the following system of equations and give bound for the approximation error with respect to the norms $|\cdot|_1$, $|\cdot|_2$, $|\cdot|_\infty$:

$$\begin{cases} 2(1+\varepsilon_1)x + (3+\varepsilon_2)y = 5 + \varepsilon_3, \\ -3x + (2+\varepsilon_1)y = -1 + \varepsilon_4, \end{cases}$$

where the unknown numbers ε_j satisfy the inequalities $|\varepsilon_j| < 0.05$ for $j = 1,\ldots,4$.

11. Do there exist a non-symmetric real matrix A such that the condition numbers with respect to the Euclidean norm satisfy the equality $\varkappa_2(A^2) = (\varkappa_2(A))^2$?

8. Methods of Instruction

The course is provided by a lecturer, an instructor, and teaching assistants.

The instructor provides problem solution sessions, encourage students to implement the algorithms described in the lectures. During the office hours, the instructor can discuss possible course project topics, references and internet sources for them, and gives comments on the drafts of slides of the students' talks.

The teaching assistants in their office hours discuss the topics of elementary linear algebra (for students which have some lacunae in their knowledge of elementary linear algebra which is assumed to be a pre-requisite) and discuss the methods and algorithms of the solutions of standard problems. Also, the teaching assistants develop a software to generate individual lists of homework problems and collect the solutions of these problems.

9. Special Equipment and Software Support

It is assumed that the students have their own laptops or have a possibility to use computer classes to implement and test the algorithms described in the lectures. A preferable software is a Python interpreter with a programming environment (e.g., Jupyter Notebook or PyCharm). No additional special equipment is required.