ALGEBRAIC GEOMETRY. HW4. (DUE OCTOBER, 4)

The fourth lecture covered the notions of noetherian scheme, irreducible components, dimension of a scheme, finite morphisms and morphisms of finite type. We also proved that, for a finite surjective morphism $\phi : X \to Y$ of noetherian schemes, one has that $\dim X = \dim Y$.

1. Let $X \to \operatorname{Spec} k$ be a scheme of finite type. Assume that $\dim X = 0$. Prove that X is affine and that $\mathcal{O}_X(X)$ is a finite-dimensional algebra over k. Conversely, for any finite-dimensional algebra A over k, the scheme $\operatorname{Spec} A$ has dimension 0.

2. A morphism $X \to \operatorname{Spec} k$ is said to be étale iff X (viewed as scheme over k) is a disjoint union of spectra of fields, each of which is a finite separable extension of k.
(a) Let $k \subset K$ be any field extension. Show that $X \to \operatorname{Spec} k$ is étale if and only if its base change $X \times_{\operatorname{Spec} k} \operatorname{Spec} K \to \operatorname{Spec} K$ is étale.
(b) Let $X \to \operatorname{Spec} k$ be a scheme over k. Then the set $X(k^{sep})$, where k^{sep} is a separable closure of k, is equipped with an action of the Galois group $\operatorname{Gal}(k^{sep}/k)$: if $\phi : \operatorname{Spec} k^{sep} \to X$ is a point of $X(k^{sep})$ and $\sigma \in \operatorname{Gal}(k^{sep}/k)$, then $\sigma(\phi) = \phi \circ \sigma_*$, where σ_* is the automorphism of $\operatorname{Spec} k^{sep}$ induced by σ. Show that this construction provides an equivalence of categories between the category of finite étale schemes over $\operatorname{Spec} k$ and the category of finite sets with a *continuous* action[1] of the Galois group $\operatorname{Gal}(k^{sep}/k)$. (This is a very usefull reformulation of the main theorem of the Galois theory.)

3. Show that the embedding of the category of finite étale schemes over $\operatorname{Spec} k$ into the the category of all schemes of finite type over k admits a left adjoint $X \mapsto \underline{\pi}_0(X)$. (Here is a way to think about $\underline{\pi}_0(X)$. Consider the scheme $X_{k^{sep}} := X \times_{\operatorname{Spec} k} \operatorname{Spec} k^{sep}$. The Galois group $\operatorname{Gal}(k^{sep}/k)$ acts on $X_{k^{sep}}$ inducing an action on the set of connected components of $X_{k^{sep}}$. By 2 (b) this defines a finite étale scheme over $\operatorname{Spec} k$. This is $\underline{\pi}_0(X)$.)

4. A group scheme G over k is said to be étale if the morphism $G \to \operatorname{Spec} k$ is étale. Show that the category of finite étale group schemes over k is equivalent to the category of finite groups equipped with a continuous action of $\operatorname{Gal}(k^{sep}/k)$ (by group automorphisms).

[1]Recall that the topology on $\operatorname{Gal}(k^{sep}/k)$ is defined by taking as a base of open neighborhoods of the neutral element $Id \in \operatorname{Gal}(k^{sep}/k)$ subgroups of the form $\operatorname{Gal}(k^{sep}/k') \subset \operatorname{Gal}(k^{sep}/k)$, where $k' \supset k$ is a finite separable extension. That is an action of the Galois group $\operatorname{Gal}(k^{sep}/k)$ on a finite set S is continuous if and only if the stabilizer subgroup of every point of S has the form $\operatorname{Gal}(k^{sep}/k')$, for some finite separable extension $k' \supset k$.