ON LINEARIZED L^2 -STABILITY OF FINITE DIFFERENCE SCHEMES FOR REGULARIZED 1D GAS DYNAMICS EQUATIONS

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The goal of our research is theoretical analysis of stability for a class of explicit difference schemes approximating gas dynamic equations. Namely, we are interested in the regularizations of Euler and Navier-Stokes systems proposed by M. Svaerd.

A new model for viscous and heat conducting flows of an ideal gas has recently been suggested by M. Svaerd

$$\partial_t \rho + \partial_x(\rho u) = \partial_x(\nu \partial_x \rho), \tag{1}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \partial_x(\nu \partial_x(\rho u)), \tag{2}$$

$$\partial_t E + \partial_x((E + p)u) = \partial_x(\nu \partial_x E), \tag{3}$$

where $\rho > 0$, u and $\varepsilon > 0$ are the density, velocity and internal energy of the gas (the basic sought functions); $p = (\gamma - 1)\rho\varepsilon$ is the pressure, $E = 0.5\rho u^2 + \rho\varepsilon$ is the total energy, ν is the diffusion coefficient. Note that for $\nu = 0$ we obtain the Euler system of equations.
In this report, the system of equations for both one-dimensional barotropic and heat conductive gas dynamics is considered.

Here we first derive theorems on criterions (necessary and sufficient conditions) of L^2-stability for finite-difference schemes discretizing equations (1)–(3) with the artificial viscosity of the form $\nu = \alpha \hat{c}_s h$, where c_s is the speed of sound and $\alpha > 0$ is a parameter, to solve the Euler system of equations. Both barotropic and the full system are covered for general background velocity.

The following is accomplished:

- A necessary specrtal (von Neumann) condition for L^2-stability is obtained
- Due to a special form of the regularization the von Neumann condition is also a sufficient condition.
- The criterions for explicit difference schemes discretizing system (1)–(3) with the physical viscosity as well as a combination of the physical and artifical viscosities are given as corollarie

Let ω_h be a uniform grid on $\mathbb{R}$ with nodes $x_k = kh, k \in \mathbb{Z}$ and the step $h = \frac{X}{N}$ and let ω_h^* be an auxiliary grid with nodes $x_{k-1/2} = (k - 1/2)h,\ k \in \mathbb{Z}$. Suppose that $\bar{\omega}^{\Delta t}$is a uniform grid in t on $[0, \infty)$ with nodes $t_m = m\Delta t,\ m \geq 0$ and a step $\Delta t > 0$. Define the grid operators

$$(sw)_{k-1/2} = \frac{w_k + w_{k-1}}{2},\ (\delta w)_{k-1/2} = \frac{w_k - w_{k-1}}{h},\ \mathring{\delta} w_k = \frac{w_{k+1} - w_{k-1}}{2h},$$

$$(\delta^* \mathbf{y})_k = \frac{y_{k+1/2} - y_{k-1/2}}{h},\ \delta_t v = \frac{v^+ - v}{\Delta t},\ v^{+,m} = v^{m+1}.$$

In barotropic case we consider only equations (1)-(2) with $p = p(\rho)$, where $p'(\rho) > 0$ for $\rho > 0$ (in particular, $p(\rho) = p_1\rho^\gamma$, $\gamma > 0, p_1 > 0$). We consider the following explicit two-level in time and three-point symmetric in space finite-difference scheme

$$\delta_t\rho + \delta^*[(s\rho)(su)] = \delta^*(\alpha\hat{c}_s h\delta\rho),$$
$$\delta_t(\rho u) + \delta^*[(s\rho)(su)^2] + \delta^*(sp) = \delta^*[\alpha\hat{c}_s h\delta(\rho u)],$$

where the basic sought functions ρ and u are defined on ω_h; p is the pressure, and $\hat{c}_s = \sqrt{p'(s\rho)}$ is the speed of sound.

We linearize the system about the constant solution$\rho = \rho_* + \rho_* \tilde{\rho}$, $u = u_* + c_* \tilde{u}$, где $u_* = Mc_*$, $c_* = \sqrt{p'(\rho)}$ is the background speed of sound and $|M|$ is the Mach number. Omitting the terms of second order of smallness with respect to the normalized perturbations $\tilde{\rho}, \tilde{u}, \tilde{\varepsilon}$ and tildes above them, we obtain the linearized scheme

$$\delta_t \rho + Mc_* \mathring{\delta} \rho + c_* \mathring{\delta} v = \alpha c_* h \delta^* \delta \rho, \tag{4}$$

$$\delta_t v + c_* \mathring{\delta} \rho + Mc_* \mathring{\delta} v = \alpha c_* h \delta^* \delta v. \tag{5}$$

Let $\mathbf{y} = \begin{pmatrix}\rho & v\end{pmatrix}^T$. Scheme (4)–(5) can be rewritten in the form:

$$\mathbf{y}^+ = \mathbf{y} - \beta \begin{pmatrix} M & 1 \\ 1 & M \end{pmatrix} \mathring{\delta}\mathbf{y} + \alpha\beta\delta^*\delta\mathbf{y}, \tag{6}$$

where $\beta = c_* \frac{\Delta t}{h}$ is the Courant number
Let H be the Hilbert space of $\mathbb{C}^n$-valued square summable vector functions defined on ω_h with the inner product
$\langle \mathbf{v}, \mathbf{y}\rangle_H = h \sum\limits_{k=-\infty}^{+\infty} \langle \mathbf{v}_k, \mathbf{y}_k\rangle_{\mathbb{C}^n}$. The question is under which conditions the following estimate holds

$$\sup_{m\geq 0} \|\mathbf{y}^m\|_H \leq \|\mathbf{y}^0\|_H \quad \forall \mathbf{y} \in H. \tag{7}$$

Following [4, 9] for explicit finite-difference schemes, we substitute a partial solution in the form $\mathbf{y}_k^m = e^{\mathbf{i}k\xi}\mathbf{v}^m(\xi)$, $k \in \mathbb{Z}$, $m \geq 0$, into (6), where $0 \leq \xi \leq 2\pi$ is a parameter, cancel out the common part $e^{\mathbf{i}k\xi}$ from all the terms of the equation, and write the result as $\mathbf{v}^{n+1}(\xi) = G(\xi)\mathbf{v}^n(\xi)$. This matrix G is called the *amplification matrix*. The von Neumann necessary condition has the form $|\lambda_i(G)| \leq 1$ for all i, where $\lambda_i(A)$ is the eigenvalue of matrix A.

Theorem 1

For scheme (4)–(5) *the von Neumann condition has the form*

$$\beta \le \min\left\{\frac{2\alpha}{(|M|+1)^2}, \frac{1}{2\alpha}\right\}. \tag{8}$$

The von Neumann condition is necessary and sufficient for estimate (7) *to hold due to the normality of the amplification matrix (see section 4.8 in [9])*

In the case of the full system of equations (1)–(3) we consider the following explicit two-level in time and three-point symmetric in space finite-difference scheme

$$\delta_t\rho + \delta^*[(s\rho)(sv)] = \delta^*[\alpha\,\hat{c}_s\,h\delta\rho],$$
$$\delta_t(\rho v) + \delta^*[(s\rho)(sv)^2] + \delta^*(sp) = \delta^*[\alpha\,\hat{c}_s\,h\delta(\rho v)],$$
$$\delta_t(E) + \delta^*[(E^{(0)} + sp)\cdot sv] = \delta^*[\alpha\,\hat{c}_s\,h\delta E],$$

where $\rho > 0$, v и $\varepsilon > 0$ are the basic sought functions defined on ω_h; $p = (\gamma-1)\rho\varepsilon$, $E = 0.5\rho u^2 + \rho\varepsilon$ and $\hat{c}_s = \sqrt{\gamma(\gamma-1)s\varepsilon}$ are the pressure, total energy and speed of sound, 59 respectively. Moreover, $E^{(0)} = 0.5(s\rho)(su)^2 + s(\rho\varepsilon)$.

We linearize the system about the constant solution $\rho = \rho_* + \rho_* \tilde{\rho}$, $v = v_* + \hat{c}_* \tilde{v}$, $\varepsilon = \varepsilon_* + \hat{\varepsilon}_* \tilde{\varepsilon}$, where $\rho > 0$, $\varepsilon > 0$, $v_* = Mc_*$, $\hat{c}_* = \frac{c_*}{\sqrt{\gamma}}$, $\hat{\varepsilon}_* = \sqrt{\gamma - 1}\varepsilon_*$ и $c_* = \sqrt{\gamma(\gamma-1)\varepsilon_*}$ is the background speed of sound, and $|M|$ is the Mach number. Omitting the terms of second order of smallness with respect to the normalized perturbations $\tilde{\rho}, \tilde{u}, \tilde{\varepsilon}$ and tildes above them, we obtain the linearized scheme

$$\delta_t \rho + Mc_* \mathring{\delta} \rho + \frac{c_*}{\sqrt{\gamma}} \mathring{\delta} v = \alpha c_* h \delta^* \delta \rho, \tag{9}$$

$$\delta_t v + \frac{c_*}{\sqrt{\gamma}} \mathring{\delta} \rho + Mc_* \mathring{\delta} v + \sqrt{\frac{\gamma - 1}{\gamma}} c_* \mathring{\delta} \varepsilon = \alpha c_* h \delta^* \delta v, \tag{10}$$

$$\delta_t \varepsilon + \sqrt{\frac{\gamma - 1}{\gamma}} c_* \mathring{\delta} v + Mc_* \mathring{\delta} \varepsilon = \alpha c_* h \delta^* \delta \varepsilon. \tag{11}$$

Let $\mathbf{y} = \begin{pmatrix} \rho & v & \varepsilon \end{pmatrix}^T$. Scheme (9)–(11) can be rewritten in the form:

$$\mathbf{y}^+ = \mathbf{y} - \beta \begin{pmatrix} M & \frac{1}{\sqrt{\gamma}} & 0 \\ \frac{1}{\sqrt{\gamma}} & M & \sqrt{\frac{\gamma-1}{\gamma}} \\ 0 & \sqrt{\frac{\gamma-1}{\gamma}} & M \end{pmatrix} \mathring{\delta}\mathbf{y} + \alpha\beta\delta^*\delta\mathbf{y}. \qquad (12)$$

Theorem 2

For scheme (9)–(11)the von Neumann condition has form (8). The von Neumann condition is necessary and sufficient for estimate (7) to hold due to the normality of the amplification matrix.

Remarks I

1.Changing the speed of sound $\hat{c}_s$ in the artificial diffusion coefficient from schemes (4)–(5) as well as (9)–(11) by the absolute value of speed of gas $|su|$ gives us the condition

$$\beta \leq \min\left\{\frac{2\alpha|M|}{(|M|+1)^2}, \frac{1}{2\alpha|M|}\right\},$$

the right-hand side of which behaves like $O(|M|^{-1})$ as $|M| \to \infty$.
2. If we want to consider the physical viscosity as presented in equations (1)–(3),we make the change $\nu_* = \alpha c_* h$, and condition (8) can be rewritten as

$$\Delta t \leq \min\left\{\frac{2\nu_*}{(|M|+1)^2 c_*^2}, \frac{h^2}{2\nu_*}\right\}.$$

Remarks II

3. If we want to consider the combination of physical and artificial viscosities by changing the diffusion coefficient in the schemes from ν to $\nu_* + \hat{k} c_* h$, condition (8) can be rewritten as

$$\Delta t \leq \min \left\{ \frac{2(\nu_* + \hat{k} c_* h)}{(|M|+1)^2 c_*^2}, \frac{h^2}{2(\nu_* + \hat{k} h c_*)} \right\}.$$

Monolithic parabolic regularization presented in [6] is a simpler regularization than in [10] and has the following form in 1D:

$$\partial_t \rho + \partial_x(\rho u) = \varepsilon \partial_x^2 \rho,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x \rho = \varepsilon \partial_x^2(\rho u),$$
$$\partial_t E + \partial_x((E + p)u) = \varepsilon \partial_x^2 E,$$

where $\varepsilon > 0$ is a small parameter.
Obviously, for this regularization the linearized barotropic and full schemes take forms (4)–(5) and (9)–(11) with $\varepsilon = \alpha c_* h$.
This means that Theorems 1 and 2 also hold for this regularization

The barotropic 1D variant of the simplified quasi-hydrodynamical regularization proposed in [5] has the following form:

$$\partial_t \rho + \partial_x(\rho(u-w)) = 0,$$
$$\partial_t(\rho u) + \partial_x(\rho(u-w)v) + \partial_x p = \partial_x \Pi_{NS},$$

where $w = \frac{\tau}{\rho}\partial_x p$, $\Pi_{NS} = \mu \partial_x v$ and $\mu = \alpha_s \tau \rho p'(\rho)$.
In barotropic case the scheme linearized on a constant solution can be written in the form

$$\mathbf{y}^+ = \mathbf{y} - c_* \begin{pmatrix} M & 1 \\ 1 & M \end{pmatrix} \mathring{\delta}\mathbf{y} + \tau_* c_*^2 \begin{pmatrix} 1 & 0 \\ 0 & \alpha_s \end{pmatrix} \delta^* \delta \mathbf{y}, \tag{13}$$

where $\mathbf{y} = \begin{pmatrix} \rho & v \end{pmatrix}^T$.
If $\alpha_s = 1$, then the linearized barotropic scheme written in matrix form is the same as (6). Thus, for scheme (13) the von Neumann condition has form (8). This means that Theorems 1 and 2 also hold for this regularization in this particular case.
In the case of the full system, for this regularization Theorems 1 and 2 hold only for special relations between parameters. More general case requires the use of technique from [12].

- The criterion (necessary and sufficient condition) for L^2-stability is deduced for special regularizations of gas dynamics equations for arbitrary Mach number.
- The criterion can be made to behave as $O(|M|^{-1})$ as $|M| \to \infty$.

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Tänan teid tähelepanu eest! Thank you for your attention!