Compactness and Saturatedness in Definability Theorems

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In Model Theory, the results called Definability Criteria play a fundamental role. These are theorems that give necessary and sufficient conditions for a class of structures to be definable by a single formula or a set of formulas in the chosen language, or to be representable as the union of (finitely) axiomatizable classes (these four options are exhaustive, in some sense). Here we introduce, in a general setting that is applicable to a wide range of languages and types of structures, the notions of a compact and a saturated class of structures. In these terms, we formulate general Definability Criteria theorems. We also introduce the notions of a compatification operation on structures and a saturation operation on structures; the closure of a class of structures under these operations guarantee the compactness and saturatedness of the class, respectively. We formulate Definability Criteria in terms of closure under these operations.

Assume we are given a *logical system* $S = (\mathcal{M}, \models, \mathcal{L})$ that consists of a class of *structures* (or *models*) $\mathcal{M}$, a language $\mathcal{L}$ (usually a set, and sometimes a class, of *formulas*), and a *truth relation* $\models$ between models $M \in \mathcal{M}$ and formulas $A \in \mathcal{L}$, written as $M \models A$ and read as "the formula A is *true* in the model M" or "M is a model of A". The truth relation extends to classes of models $\mathbb{K} \subseteq \mathcal{M}$ and sets of formulas $\Gamma \subseteq \mathcal{L}$. Throughout this abstract, we assume that $\models$ *respects conjunction*, which means that $\mathcal{L}$ has a binary connective $\wedge$ such that $M \models A \wedge B$ iff $M \models A$ and $M \models B$. Often we also assume that $\models$ *respects negation*, which means that $\mathcal{L}$ has a unary connective $\neg$ such that $M \not\models A$ iff $M \models \neg A$.

We consider the following four *species* of classes of models. We call a class of models $\mathbb{K} \subseteq \mathcal{M}$ *finitely axiomatizable* (we write $\mathbb{K} \in \mathbb{L}$) if $\mathbb{K}$ is the class of all models of some formula $A \in \mathcal{L}$; *axiomatizable* (we write $\mathbb{K} \in \dot\cap\mathbb{L}$) if $\mathbb{K}$ is the class of all models of some set of formulas $\Gamma \in \mathcal{L}$, or equivalently, if $\mathbb{K}$ is the intersection of some finitely axiomatizable classes (hence the notation $\dot\cap$). Futhermore, we call $\mathbb{K}$ *co-axiomatizable* (in symbols, $\mathbb{K} \in \dot\cup\mathbb{L}$) if $\mathbb{K}$ is the union of some finitely axiomatizable classes; we call $\mathbb{K}$ *quasi-axiomatizable* ($\mathbb{K} \in \dot\cup\dot\cap\mathbb{L}$) if $\mathbb{K}$ is the union of some axiomatizable classes. We do not need any further species, even $\dot\cap\dot\cup\mathbb{L}$, due to the following lemma. For two models $M, N \in \mathcal{M}$, we write $M \sqsubseteq_{\mathcal{L}} N$ if, for all formulas $A \in \mathcal{L}$, $M \models A$ implies $N \models A$; $M \equiv_{\mathcal{L}} N$ if both $M \sqsubseteq_{\mathcal{L}} N$ and $N \sqsubseteq_{\mathcal{L}} M$.

Lemma 1. $\mathbb{K} \in \dot\cup\dot\cap\mathbb{L} \iff \mathbb{K} \in \dot\cap\dot\cup\mathbb{L} \iff \mathbb{K}$ *is closed under* $\sqsubseteq_{\mathcal{L}}$.
If $\models$ *respects negation, then this is equivalent to:* $\mathbb{K}$ *is closed under* $\equiv_{\mathcal{L}}$.

We want to obtain similar 'criteria' for the other three species. Here we do this for the case when $\models$ respects negation (only partial results are obtained in a more general case, and they are not discussed here). A set of formulas Γ is called *satisfiable* in $\mathbb{K}$ if $\exists M \in \mathbb{K}$: $M \models \Gamma$.

Definition 1 (Compact class)**.** A class of models $\mathbb{K}$ is called *compact* if, for any set of formulas $\Gamma \subseteq \mathcal{L}$, if every finite subset $\Delta \subseteq \Gamma$ is satisfiable in $\mathbb{K}$, then Γ is satisfiable in $\mathbb{K}$.

Theorem 2. *Assume that* $\models$ *respects negation, and the class of all models* $\mathcal{M}$ *is compact. Then:*
(a) $\mathbb{K}$ *is axiomatizable* $\iff \mathbb{K}$ *is closed under* $\equiv_{\mathcal{L}}$ *and compact;*
(b) $\mathbb{K}$ *is co-axiomatizable* $\iff \overline{\mathbb{K}} = \mathcal{M} \setminus \mathbb{K}$ *is closed under* $\equiv_{\mathcal{L}}$ *and compact;*
(c) $\mathbb{K}$ *is finitely axiomatizable* $\iff$ *both* $\mathbb{K}$ *and* $\overline{\mathbb{K}}$ *are closed under* $\equiv_{\mathcal{L}}$ *and compact.*

The results obtained so far can be summarized in the following, rather symmetric, table:

Criterion for...		Both	$\mathbb{K}$ is...	$\overline{\mathbb{K}}$ is...
$\mathbb{K} \in \dot\cup\dot\cap\mathbb{L}$	($\mathbb{K}$ is quasi-axiomatizable)	$\equiv_{\mathcal{L}}$		
$\mathbb{K} \in \dot\cup\mathbb{L}$	($\mathbb{K}$ is co-axiomatizable)	$\equiv_{\mathcal{L}}$		compact
$\mathbb{K} \in \dot\cap\mathbb{L}$	($\mathbb{K}$ is axiomatizable)	$\equiv_{\mathcal{L}}$	compact	
$\mathbb{K} \in \mathbb{L}$	($\mathbb{K}$ is finitely axiomatizable)	$\equiv_{\mathcal{L}}$	compact	compact

Next we want to characterize compactness in terms of closure under operations on models.

Definition 3. An operation on models $*$ is called a *compactification operation* in the logical system S if 1) it is truth preserving: if every M_i in $\vec{M}$ is a model of A then $*(\vec{M})$ is a model of A; 2) every class of models $\mathbb{K}$ closed under $*$ is compact.

Examples of compactification operations: the *ultraproduct* of FO structures [1] (which is in fact a family of operations, for each set of indices and each ultrafilter over it); the ultraproduct of (pointed) Kripke models [2]; the *ultra-union* of pointed Kripke models introduced by Venema [3]. Clearly, if a compactification operation is available, then the class of all models $\mathcal{M}$ is compact.

Theorem 4. *The same as Theorem 2, but with "compact" replaced by "closed under $*$".*

Now we want to get rid of the 'linguistic' relation $\equiv_{\mathcal{L}}$ from the Criteria. In many logical systems, there is a 'structural' *similarity* relation $\sim$ between models that does not depend on the language $\mathcal{L}$, and $M \sim N$ implies $M \equiv_{\mathcal{L}} N$. Examples are the *isomorphism* and *partial isomorphism* between FO structures; the *bisimulation* between (pointed) Kripke models for various modal languages. In addition, we often have a subclass of the so-called *saturated* models $\mathcal{M}^{\circ}$ on which the 'linguistic' relation $\equiv_{\mathcal{L}}$ coincides with the 'structural' relation $\sim$. In the FO logic, these are *ω-saturated* models; in modal logic, the so-called *modally saturated* (pointed) Kripke models.

Definition 5. We call a class of models $\mathbb{K}$ *saturated* if, for every model $M \in \mathbb{K}$, there is a saturated model $M' \in \mathbb{K} \cap \mathcal{M}^{\circ}$ such that $M \equiv_{\mathcal{L}} M'$.

Lemma 2. *Let $\mathbb{K}$ be a saturated class of models. Then for any models $M, N \in \mathbb{K}$ such that $M \equiv_{\mathcal{L}} N$, there are (saturated) models $M', N' \in \mathbb{K}$ such that $M \equiv_{\mathcal{L}} M'$, $N \equiv_{\mathcal{L}} N'$, and $M' \sim N'$.*

Theorem 6. *Assume that $\models$ respects negation. Then we have the following criteria:*

	Both	$\mathbb{K}$ is...	$\overline{\mathbb{K}}$ is...
$\mathbb{K} \in \dot{\cup}\dot{\cap}\mathbb{L}$	$\sim$	saturated	saturated
$\mathbb{K} \in \dot{\cup}\mathbb{L}$	$\sim$	saturated	compact saturated
$\mathbb{K} \in \dot{\cap}\mathbb{L}$	$\sim$	compact saturated	saturated
$\mathbb{K} \in \mathbb{L}$	$\sim$	compact saturated	compact saturated

Definition 7. A unary operation on models $\sharp$ is called a *saturation operation* in the logical system S if 1) it is truth-invariant: $M \equiv_{\mathcal{L}} M^{\sharp}$; 2) every class $\mathbb{K}$ closed under $\sharp$ is saturated.

Examples of saturation operations: the *ultrapower* of a FO structure or of a (pointed) Kripke model (provided that the signature is at most countable and the ultrafilter is countably incomplete); the *ultrafilter extenstion* of a (pointed) Kripke model; the *canonical model* of the theory of a Kripke model.

Theorem 8. *The same as Theorem 6, but with "saturated" replaced by "closed under $\sharp$".*

This gives many Definability Criteria known in literature. It often happens that closure under $*$ (e.g. ultraproducts) implies the closure under $\sharp$ (ultrapowers). For this reason, in the known criteria, there is only one closure condition, not two.

References

[1] C.Chang, H.Keisler, *Model Theory*, Studies in Logic and the Foundations of Mathematics, vol. 73, Elsevier, 1973.

[2] P.Blackburn, M. de Rijke, Y. Venema, *Modal Logic*, Cambridge University Press, 2001.

[3] Y.Venema. Model definability, purely modal. In *JFAK: Essays dedicated to Johan van Benthem on the occasion of his 50th birthday*, 1999.