ALGEBRAIC GEOMETRY. HW1. (DUE SEPTEMBER, 13)

The first lecture covered (a part of) §2 of Chapter II of Hartshorne's textbook. I recommend to read this section (up to the construction of $Proj$ of a graded ring: this important construction will be discussed in class later). In addition, please, do the following problems (some of them will be discussed at the problem session next Friday).

1. Let $(X, \mathcal{O}_X) = \operatorname{Spec} A$ be an affine scheme, $f \in A$. Show that the locally ringed space $(D(f), \mathcal{O}_{X|D(f)})$ is also an affine scheme.

2. Let $(X, \mathcal{O}_X)$ be a scheme, $U \subset X$ open subset. Show that $(U, \mathcal{O}_{X|U})$ is a scheme (called *an open subscheme of* X).

3. Show that, for $n > 1$, the scheme $\mathbb{A}^n_k - 0$ is not affine.

4. A scheme $(X, \mathcal{O}_X)$ is called reduced if, for every open $U \subset X$, the ring $\mathcal{O}_X(U)$ is reduced (that is has no nilpotent elements).

(a) Show that $(X, \mathcal{O}_X)$ is reduced if and only if, for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is reduced.

(b) Show that the embedding of the category of reduced schemes into the category of all schemes has a right adjoint which takes a scheme $(X, \mathcal{O}_X)$ to a scheme $(X, (\mathcal{O}_X)_{red})$ with the same underlying topological space.

5. Let $(X, \mathcal{O}_X)$ be a scheme and $x \in X$ a point. The residue field $k(x)$ is defined to be the quotient of the stalk $\mathcal{O}_{X,x}$ by its unique maximal ideal. What are the residue fields of points of $\mathbb{A}^1_{\mathbb{F}_p}$? How many points $x \in \mathbb{A}^1_{\mathbb{F}_p}$ with given degree $[k(x) : \mathbb{F}_p]$ are there?

6. A topological space is said to be quasi-compact if every open cover has a finite subcover.

Show that, for any ring A, the topological space $\operatorname{Spec} A$ is quasi-compact.

7. Let M be a module over a commutative ring A. Assume that, for every maximal ideal $m \subset A$, the localization $M_m = M \otimes_A A_m$ is 0. Show that $M = 0$.