A parameter space of cubic Newton maps

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Cubic Newton maps with a parabolic fixed point

Consider the Newton's method applied to the entire maps of the form $(az^2+bz+c)\exp(dz+e)$. After an appropriate conjugation, we see that it is enough to represent any such a cubic Newton map with a single complex number $\lambda \neq 0$, and denote these maps by $f_\lambda(z) = z^2 \dfrac{z+\lambda-1}{(\lambda+1)z-1}$, and denote $\mathcal{F}$ the family of such a cubic Newton maps f_λ. We have $f'_\lambda(0) = f'_\lambda(\infty) = 0$ and $f'_\lambda(1) = 1$ at the three persistent fixed points. The Julia sets are connected by Shishikura's theorem.

In general, any cubic rational map with three fixed points two of which are superattracting and the third is multiple fixed point, then necessarily of multiplicity $+1$, can be conformally conjugated to the form f_λ.

The moduli space of $\mathcal{F}$ is double covered by the $\lambda-$parameter plane $\mathbb{C} \setminus \{0\}$, with identifications of λ and $-\lambda$, has a Riemann surface structure.

If $\lambda \neq \pm 1$ then there are two critical points that are not fixed by f_λ counted with multiplicities, and at least one of them always converges under the iterates of f_λ to the parabolic point at 1. We call the other critical point a "free" critical point.

Since fixed points are persistent it makes sense to consider the parameters $\lambda \in \mathbb{C} \setminus \{0\}$ such that the free critical point of f_λ belongs to basins of attracting cycles (including the fixed points 0 and ∞) or the parabolic basin of $z = 1$. These maps are J-stable maps. We call such an f_λ *stable*.

J-stable parameters form an open set in the $\lambda-$parameter plane and the connected components are called *stable components*.

Our main result is that every stable component is a topological disk with a unique center which is a cubic postcritically *minimal* Newton map, which was defined in [1, 2, 3].

In the main parabolic stable component, denoted H, charactered by maps f_λ for which the free critical point belongs to the immediate basin of 1, if there is no critical orbit relation between non fixed critical points then any two such maps are globally **quasiconformal conjugate**. If there is a critical orbit relation then the **quasiconformal conjugacy class** of f_λ consists of the single f_λ.

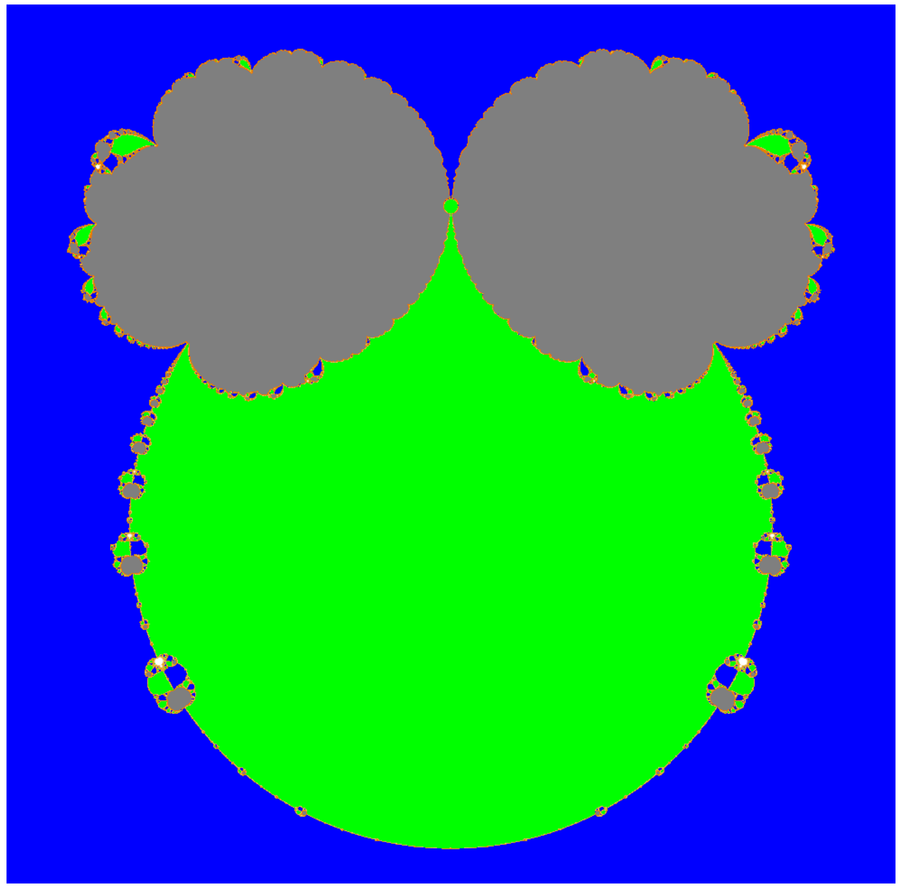

Figure 1: The λ-parameter plane of cubic Newton maps f_λ in the coordinate $\xi \mapsto \frac{8-i2\sqrt{2}\xi}{1+i2\sqrt{2}\xi}$ to highlight the stable regions. Grey areas – the free critical point belongs to the parabolic basin. Blue and green areas – the free critical point belongs to the basins of the two superattracting fixed points. The image is in the window $|\operatorname{Re}\xi| \leq 2.2$ and $-3 \leq \operatorname{Im}\xi \leq 1.35$. The centers of the two gray areas are at -1 and 1. The singular point $\lambda = 0$ corresponds to $\xi = -i2\sqrt{2}$, the bottom of the green region. The $\lambda = \infty$ corresponds to $\xi = i/\sqrt{8}$, the meeting point of the two gray regions and the top of the green region.

For the stable components parametrized by maps f_λ for which its free critical point belongs to the immediate basin of one of the two superattracting fixed points, it is well known that the value of the Bötcher map of the superattracting fixed point evaluated at the orbit point of the free critical point first visits the immediate basin gives a surjective conformal map from the stable component to the unit disk and parametrizes these stable components.

Blaschke products and Model space

Denote $\beta_a(z) = \frac{1-\bar{a}}{1-a} \cdot \frac{z-a}{1-\bar{a}z}$ the unique automorphism of $\mathbb{D}$ sending $a \in \mathbb{D}$ to the origin and normalized to fix 1. For complex constants a_1, a_2, a_3 in $\mathbb{D}$, we define a cubic Blaschke product by $B(z) = \beta_{a_1}(z) \cdot \beta_{a_2}(z) \cdot \beta_{a_3}(z)$ normalized to fix 1.

Heins proved that for a set of critical points in $\mathbb{D}$ there is a Blaschke product of degree d that has critical points exactly at these points. The Blaschke product is unique up to post-composition by an automorphism of $\mathbb{D}$.

Denote $\mathcal{B}_3$ the model space consisting of cubic parabolic Blaschke products B with critical points at 0 and w in $\mathbb{D}$ such that $B(1) = 1$, $B'(1) = 1$, and $B''(1) = 0$. It is natural to parametrize $\mathcal{B}_3$ by the location of the critical point w of B.

For the Blaschke product B of $\mathcal{B}_3$ with the critical points $\{0, w\}$ we can consider $\tilde{B}(z) = B \circ \beta_c(z)$ for $c = \frac{w(1+\sqrt{1-|w|^2})-|w|^2}{(w-2)(1+\sqrt{1-|w|^2})+|w|^2}$. This Blaschke product $\tilde{B}$ has symmetrical critical points at $\{c, -c\}$. Denote $\tilde{\mathcal{B}}_3$ the model space of such a $\tilde{B}$.

Proposition 1 (Normal forms for the model spaces $\mathcal{B}_3$ and $\tilde{\mathcal{B}}_3$)**.** *Every cubic parabolic Blaschke product in the model space $\mathcal{B}_3$ has a normal form given by $B_b(z) = \beta_a(z^2 \cdot \beta_b(z))$, where $a = \frac{(1-\bar{b})\left(3(1-b)+b(1-\bar{b})\right)}{(1-b)\left(-6+3b+8\bar{b}-3|b|^2-\bar{b}^2(3-b)\right)}$, parametrized by $b \in \mathbb{D}$. Such a B_b has critical points at the origin and at $w = \frac{b}{4|b|^2}\left(3+|b|^2-\sqrt{(3+|b|^2)^2-16|b|^2}\right)$ (if $b = 0$ then let $w = 0$). For such a B_b and its critical point w, let $c = \frac{w(1+\sqrt{1-|w|^2})-|w|^2}{(w-2)(1+\sqrt{1-|w|^2})+|w|^2}$ and $\tilde{B}_b(z) = B_b \circ \beta_c(z)$ then $|c| < 1$ and $\tilde{B}_b$ is a new Blaschke product with symmetrical critical points at $\{c, -c\}$. The critical point $w = w(b)$ (and $c = c(b)$) of B_b (of $\tilde{B}_b$) depends real analytically and bijectively on $b \in \mathbb{D}$. The location of the critical point w (and c) also parametrizes $\mathcal{B}_3$ (respectively $\tilde{\mathcal{B}}_3$), which is complex analytic.*

The moduli spaces $\mathcal{M}$ and $\tilde{\mathcal{M}}$ have orbifold structures considered at the complex analytic coordinates w and c respectively.

Proposition 2. *The moduli space $\mathcal{M}$ consisting of conformal conjugacy classes of maps of the model space $\mathcal{B}_3$ is an orbifold $\mathbb{D}/\Gamma$, a topological open disk, with an elliptic point of order 2 at the origin and a group action is by $\Gamma = \{\mathrm{id}, w \mapsto -w\frac{1-\bar{w}}{1-w}\}$. The moduli space $\tilde{\mathcal{M}}$ consisting of conformal conjugacy classes of maps of the model space $\tilde{\mathcal{B}}_3$ is $\mathbb{D}/\tilde{\Gamma}$ with an orbifold structure with an elliptic point of order 2 at the origin, where the group action is by $\tilde{\Gamma} = \{\mathrm{id}, c \mapsto -c\}$. Moreover, the action of Γ is conjugate with that of $\tilde{\Gamma}$ by $c(w) = \frac{w(1+\sqrt{1-|w|^2})-|w|^2}{(w-2)(1+\sqrt{1-|w|^2})+|w|^2}$.*

For every cubic Newton map $f_\lambda \in H$ corresponds a conformal class in the moduli space $\mathcal{M}$, and thus H can be identified with $\mathcal{M}$, which is a topological open disk.

Theorem 3. *The main parabolic component H is simply connected with its unique center. The quasi-conformal conjugacy classes of maps in H are of two types: type-I, a single class, which is topologically an infinitely punctured disk, and its punctures are of type-II. The central puncture, the center of H, is the unique cubic postcritically minimal Newton map f_λ for $\lambda = 2\sqrt{2}\,i$, except this, the set of type-II punctures are in one-to-one correspondence with the set of all cubic postcritically non-minimal Newton maps in H. Moreover, the boundary ∂H is a Jordan curve.*

The main result in summary is the following.

Theorem 4. *Every stable component of the parameter plane of $\mathcal{F}$ is an open topological diskl with its unique center that is a cubic postcritically minimal Newton map. There exists a bijective map from the space of Haïssinsky equivalent classes of centers (hyperbolic postcritically finite maps) of hyperbolic components of the space of standard cubic Newton maps of cubic polynomials to the centers of stable components of this space of parabolic cubic Newton maps. This bijection preserves the dynamics on the corresponding Julia sets and is obtained by parabolic surgery.*

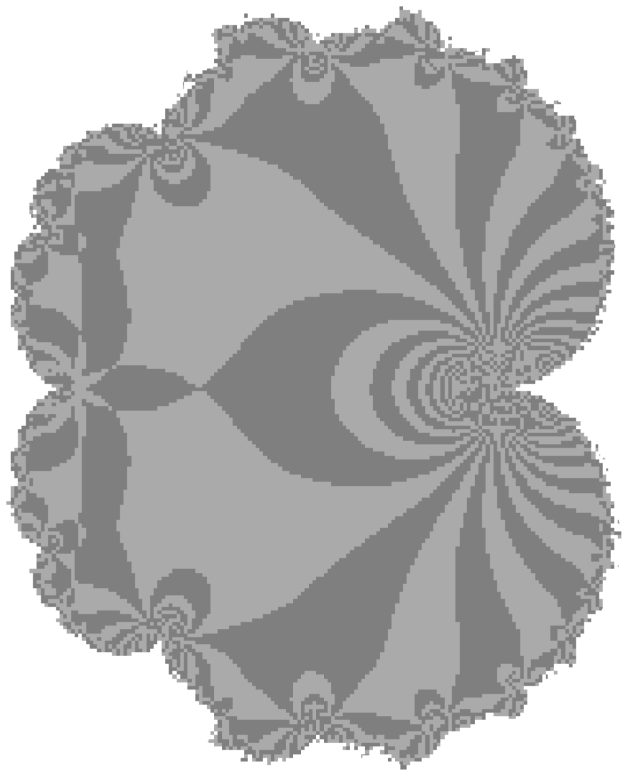

Figure 2: The main parabolic component H with a checkerboard structure in a suitable parametrization

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