ALGEBRAIC GEOMETRY. ORAL EXAM PROGRAM

I have used the following books: Hartshorne, *Algebraic Geometry*, Chapter II, §1 through 6, Kempf, *Algebraic varieties*, §1 though §5 and §7 (especially see §2 for the treatment of the dimension theory and §7 for valutive criterions), and Shafarevich *Basic Algebraic Geometry.* The proof (due to Kaplansky) of the Auslander-Bushsbaum isomorphism $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Cl}(X)$ for regular Noetherian schemes is borrowed from SGA, Exposé XI, Theorem 3.13.

1. Schemes, morphisms. Closed embeddings. Closed subschemes of an affine scheme.

2. Proj construction. Noetherian schemes, irreducible components, dimension. Prove that every Noetherian topological space is the union of finitely many irreducible components.

3. Fiber products. Morphisms of finite type. Finite morphisms. Prove that every finite morphism $f : X \to Y$ is closed and that, for every $y \in Y$, the set $f^{-1}(y)$ is finite.

4. The Noether Normalization Lemma (with a proof). Proof of Nullstellensatz.

5. Show that, for a finite surjective morphism $f : X \to Y$, one has that $\dim X = \dim Y$.

6. Show that $\dim \mathbb{A}^n_k = n$. Show that every scheme of finite type over k has finite dimension.

7. Let X be an integral scheme of finite type over field, $f \in \mathcal{O}(X)$. Show that every irreducible component of closed subset $\{x \in X | f(x) = 0\}$ has dimension $\dim X - 1$.

8. Proper and separated morphisms. Prove that $\mathbb{P}^n_{\mathbb{Z}}$ is proper over $\mathrm{Spec}\,\mathbb{Z}$. Is it true that every closed embedding is proper?

9. Normalization $f : Y \to X$ of an integral scheme X in a finite extension $K(X) \subset L$ of the field of rational functions. Prove, if X is of finite type over a field, then the normalization $f : Y \to X$ is a finite morphism.

10. Valutive criterions (with proofs). Prove that for a regular, connected scheme X of finite type over a field k, open subset $U \subset X$, and a morphism $f : U \to Y$ to a complete scheme Y, f extends to a larger open subset $U \subset W \subset X$ with $\mathrm{codim}(X - W) > 1$.

11. Quasi-coherent sheaves. Show that the property of an $\mathcal{O}_X$-module $\mathcal{F}$ of being quasi-coherent is local on X. Prove that the category of quasi-coherent sheaves on $\mathrm{Spec}\,A$ is equivalent to the category of A-modules.

12. Quasi-coherent sheaves on $\mathrm{Proj}\,A.$ as the Serre quotient of the category of graded $A.$-modules.

13. Show that for every projective morphism $f : X \to Y$, where Y is a scheme of finite type over a field, and a coherent sheaf $\mathcal{F}$ on X, the direct image $f_*\mathcal{F}$ is coherent. Is it true that, for every projective scheme X over a field k and $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$ the vector space $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is finite-dimensional?

14. The class group. Compute the class group of a smooth quadric over an algebraically closed field.

15. Prove that, for a regular Noetherian scheme X, one has that $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Cl}(X)$.