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Sustainable Debt Accumulation
in the Logistic Model of Global Leverage

The ILMA Workshop Presentation
December 5, 2019

National Research University Higher
School of Economics, Moscow
Literature

• Leverage and Financial Instability
  *Voprosy Economiki*, #9, 1-27, 2012 (in Russian)

• Logistic Model of Financial Leverage
  *HSE Economic Journal*, vol.17, #4, 585-616, 2013 (in Russian)

• Financial Assets Collateralization and Stochastic Leverage
  *HSE Economic Journal*, vol.18, #2, 183-215, 2014 (in Russian)

• Stochastic Leverage of the Global Financial System,

• Stochastic Logistic Model of the Global Financial Leverage,
  *The BE Journal of Theoretical Economics*, 2018, issue 1

• Does the Global Leverage Dynamic Gravitate to an Invariant?
  report at the Seminar of the Department of Theoretical Economics, NRU HSE, 2018

• Safe Debt Accumulation in the Logistic Model of Financial Leverage,
  submitted to the Journal of...
The Problem Formulation

1. This is the 5th report on the theme, so I shall concentrate only on the new problems and findings. There are three clusters of them:

a) Existence of the leverage values corresponding to safe debt and sound money;

b) Exploring differences between micro- and macro-debt. Transformation of debt redemption into its safe refinancing and rolling over;

c) Deleveraging as an empirical process and its modelling.
2. Basic Definitions

Debt is a contractual obligation, or a ”promise” of future payments;

Debt is reimbursed with money, and lenders deserve to be compensated at the face value of their loans;

The problem of money collateralization. The real bills doctrine of J. Law and A. Smith;

The sound money definition, \( M(t) / E(t) = 1; \)
Macrofinancial balance and ratios

3. The standard macro-balance of financial assets and liabilities.

No global bankruptcies are allowed, but crises of liquidity are possible.

Empirical balances are treated for the fixed time-to-maturity parameter $T$; the actual time $0 < t \leq T$

$$A(t,T) = [M(t) + B(t,T)] + E(t) = D(t,T) + E(t)$$

In terms of leverage assuming that money is sound, $M(t) / E(t) = 1$:

$$B_T(t) / E(t) = l(t) - 2 \quad \text{or} \quad D_T(t) = l(t) - 1$$
## Global assets, liabilities and some structural parameters

<table>
<thead>
<tr>
<th>Years</th>
<th>Assets, $tn</th>
<th>Debt, $tn</th>
<th>Money $tn</th>
<th>Bonds, $tn</th>
<th>Equity $tn</th>
<th>Leverage, $t</th>
<th>Rate of growth $N_{t+1}/$t</th>
<th>Spread $a_t$</th>
<th>Spread $c_t$</th>
<th>Parameter $b_t$</th>
<th>Purchasing power, $\theta$</th>
<th>Collateralization ratio, $\nu$</th>
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{Sources: World Bank, Institute of International Finance, Author’s estimations}
4. The basic “building block” of the fixed-income finance is a pure riskless zero-coupon bond:

\[ B(t, T) \exp[y(t, T)(T - t)] = M(T); \quad 0 \leq t \leq T \]

where \( B(t, T) \) is the debt contractual obligation; \( y(t, T) \) is the spot interest rate (yield-to-maturity); \( M(T, T) = M_B(T) \) is the debt principal or its “promised” money equivalent at \( t = T \).
The stylized debt reimbursement

5. The stylized process of the macro-debt reimbursement

\[ \lim_{T \to t} B(t, T) = B(t, t) = M_B(t) \]

where \( M_B(t) \) is the money equivalent of a loan. Hence money is a debt which is repaid instantaneously.

Banknote, digital money. Money is an instantaneous debt.

The actual process of the macro-debt reimbursement is a financial crisis hence it is a highly undesirable event.
Rates of return in the fixed-income finance

6. a) the spot interest rate: \( y(t, T) = -\ln B(t, T) / (T - t) \);

b) the forward rate of return: \( f(t, T) = -\partial \ln B(t, T) / \partial T \);

c) the instantaneous rate of return: \( r(t) = y(t, t) = -\partial \ln B(t, t) / \partial t \);

is a price of ultra-short credit like EONIA, SONIA, RUONIA

d) “Money does not beget money” (Aristotle):

\[ B(t, t) \exp[y(t, t)(t - t)] = B(t, t); \ 0 < t \leq T \]

\{Ineffectiveness of monetary stimuli within the QE policy\}
The existence of critical leverage values

7. There exist leverage values at which the debt is fully reimbursed with sound money:

\[ l(t) = 2; \frac{B_T(t)}{E(t)} = 0 \Rightarrow B_T(t) = 0 \approx D_T = M_B(t); \]
\[ l(t) = 3; \frac{B_T(t)}{E(t)} = 1 \Rightarrow B_T(t) = E(t) \approx B_T(t) = \hat{M}(t) = E(t); \]

Hence \[ 2 \leq l \leq 3 \]
8. The feasibility of leverage values located in the above the interval. The lower boundary is the (nontrivial) solution to the well-known equation:

\[ l = l / (l - 1) \quad \text{or} \quad l^2 - 2l = 0. \]

The upper boundary could be reformulated in terms of the long run no-arbitrage relations between aggregates of borrowers and creditors.
Leverage and the Collateral Ratio

Assets-to-Loans

Leverage
Assets-to-Loans

Leverage
The balanced financial market condition

9. Macrofinancial balances of levels and flows

\[ A(t) = D(t) + E(t) \quad \text{and} \quad dA(t) = dD(t) + dE(t) \]

are equivalently represented by their convolution:

\[ \mu l(t) = r(l-1) + \rho \]

where \( \mu \equiv \text{ROA} \); \( \rho \equiv \text{ROE} \); \( r = r_s = r_L \). It is the balanced financial market, BFM, condition which is always true in the short run. In the model financial spreads are subject to

\[ \frac{c}{a} = \frac{a}{b} \quad \text{where parameter} \quad b = a^2 / c . \]
Some global rates of return in 1999-2017
10. Natura abhorret vacuum. In the long run aggregates of borrowers and creditors are mutually adjusted so as

\[ \rho^e(l) = r + (\mu - r)l; \]
\[ \mu^e(l) = r + (\rho - r)l^{-1} \]

and \( \rho^e(l) = \mu^e(l) \) or \( l^2 - a/b = 0 \).

The root of the latter defines the anchor leverage \( l_N = (a/b)^{0.5} \).
The safe debt reimbursement in the long run

11. If the macro-debt is reimbursed in the long run then

$$l^2 - 2l = l^2 - a/b$$

with the positive root \( \hat{l} = a/2b \). If \( \hat{l} \geq 3 \) the safe debt interval is

$$2 \leq l_N \leq a/2b$$
The logistic leverage dynamic

12. The leverage deterministic dynamic is

\[ dl(t) = l(t)[a - bl(t)]dt; \quad l(0) = l_0 \]

with solution:

\[ l(t) = K \left\{1 + \left(\frac{K}{l_0} - 1\right) \exp[-at]\right\}^{-1} \]

where \( K = a / b \) is the nontrivial steady state of the logistic equation.

The transition (the pass-through) function:

\[ f(l) = al - bl^2 \]

measures the debt refinancing and its ability to be rolled over. Its maximum takes place at \( \hat{l} = a / 2b \) where money is sound:

\[ f'(l) = a - 2bl = 0 \]
The logistic leverage trajectories
The micro- vs macro-debt

13. Now, the second strand of the study: micro- vs macro-debt.

Any particular micro-debt has to be repaid in the finite time. It is safe, by definition, being paid off in full.
{Russian rates for “non-callable credits” in 90ties were in vogue due to disordered economic transition}

Contrary to that, the macro-debt is a perpetual aggregate of promises; separate tranches have to be repaid, and their simultaneous reimbursement is tantamount to a financial crisis.

The proof of the macro-debt safety is in its ability to be refinanced and rolled over. If every market participant is convinced in the safety of her/his money the debt as a whole is safely rolled over.
The self-negation mechanism

14. The mechanism of self-negation: when everybody knows that her/his money is callable on short notice nobody would claim it back without particular personal reasons.

Debt reimbursement is transformed into its safe rolling over at the leverage value $\hat{t} = a / 2b$. Thus, the problem seems to be solved; it is, but not completely because the macrofinancial system is unstable (semi-stable) at the safe debt roll-over point.
The self-negation mechanism

- Leverage Change
- Collateralization Line
- Debt LongTerm Balance
- Collateralized Debt
- Anchor Leverage
15. Let us return to the empirical data in 1999-2017. The asset/equity elasticity in 1999-2017 is 1.25, but the leverage process is very different: it is, in fact, a process of deleveraging.

16. Deleveraging: debts outstanding are netted and compressed; {mutual indebtedness offsetting in the Soviet economy, and a lucrative multibillion business now, especially on markets of derivatives}

central banks purchases of large chunks of public debt; {BoJ holds 43 percent of government debt};

world equity markets tend to be overvaluated for many years {August, 2019 correction and the YC inversion on the major markets}
Two phases of deleveraging in 1999-2017
The process of deleveraging

17. The standard logistic equation is modified into the following model:

\[ dl(t) = [(a - \delta)l(t) - bl^2(t)]dt \]

with pass-through function: \( f(l, \delta) = al - bl^2 - \delta l \), and stable steady state \( K(\delta) = K(1 - \delta / a) \).

The maximal effect of deleveraging, \( \max_\delta [\delta K(\delta)] \) takes place at \( \tilde{\delta} = 0.5a \), that is, precisely, at the safe debt roll-over point, \( \tilde{l} = a / 2b \).

The solution of \( K_{17}(\delta) = K_{12}(1 - \delta / a) \) defines \( a_{17} = 0.0029; \delta = 0.0004 \) hence the prolong and uncertain travel between different steady states.
Deleveraging in 2002-2007 reconstruction

Mathematical functions showing the relationship between leverage and functions over time from 2002 to 2008.
The global leverage regression
The stochastic logistic diffusion

18. The new factor governing the process appears in the logistic stochastic diffusion:

\[ dl(t) = l(t)[a - bl(t)]dt + \sigma l(t)dZ(t) \]

where \( \sigma = 0.16 \) is the empirical volatility; and \( Z(t) = \int_0^t dZ(u) \) is a standard Brownian motion with independent and self-similar (the Hurst exponent, \( H = 0.5 \)) increments \( dZ(t) \).

Parameter \( a \) was used *per se* or including the deleveraging intensity \( \delta \).

Due to high empirical volatility, \( \sigma = 0.16 \), the future leverage realizations dramatically decreased: from \( l_{17} = 4.14 \) to \( l^* = 0.416 \), that is, to values smaller than one.
Realizations of stochastic leverage
A random leverage equivalent representations

19. It is known that a random process $x(t)$ with coefficients of drift and noise given by the general Itô processes $P[x(t),t]$ and $\sqrt{Q[x(t),t]}$ can be equivalently represented either by its SDE:

$$dx(t) = P[x(t),t]dt + \sqrt{Q[x(t),t]}dZ(t),$$

or by the Kolmogorov-Fokker-Plank equation:

$$\frac{\partial}{\partial t} p[x(t),t] = -\frac{\partial}{\partial l} \{P[x(t),y]p[l(t),t]\} + \frac{1}{2} \frac{\partial^2}{\partial l^2} \{Q[x(t),t]p[x(t),t]\}$$

where $p[x(t),t]$ is the “transition” probability density function. Boundaries and initial conditions of the KPF equation are specified by the actual process under consideration.
The KFP stationary equation and its solution

20. A non-trivial solution to the ordinary differential KFP equation:

\[- \frac{d}{dl}[l(a - bl)p(l)] + \frac{1}{2} \frac{d^2}{dl^2}[\sigma^2 l^2 p(l)] = 0\]

is the stationary pdf, \(p(l)\), of a random leverage process

\[p(l; \alpha, \beta) = [\beta^\alpha / \Gamma(\alpha)] l^{\alpha-1} \exp[-\beta l].\]

It defines the gamma distribution with parameters \(\alpha = (2a/\sigma^2) - 1\) and \(\beta = 2b/\sigma^2\) (the parameter of scale \(1/\beta\) is used in Mathematica 10).

The normalization constant \(\beta^\alpha / \Gamma(\alpha)\) is found from a gamma function:

\[\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp[-x] dx.\]
Figure IX
Three forms of a gamma distribution

- Gamma Distribution, $a < \sigma^2 < 2a$
- Exponential Distribution, $\sigma^2 = a$
- Gamma Distribution, $\sigma^2 < a$
Moments of the stationary gamma distribution

22. The PDF of a gamma distribution is skewed to the right and its expectation:

\[ \langle L \rangle = \frac{\alpha}{\beta} = K - \frac{\sigma^2}{2b} \]

is larger than its mode (the most probable long-term leverage):

\[ Mo = \frac{\alpha - 1}{\beta} = K - \frac{\sigma^2}{b}; \quad \alpha \geq 1. \]
The random leverage convergence criterion

23. If the mode is the same as the anchor leverage $l_N$ the above formula could be transformed into:

$$\left(\frac{a}{b}\right)^{0.5} = \frac{a - \sigma^2_N}{b} \quad \text{or} \quad \frac{a}{2b} = \frac{a - \sigma^2_N}{b}$$

where $\sigma^2_N$ is the implied variance of the gamma distributed stationary leverage. It is an indicator of the long-term default risk. For the positive parameters $\{a, b\}$ the implied variance $\sigma^2_N$:

$$\sigma^2_N = a - \sqrt{ab} \quad \text{or} \quad \sigma^2_N = 0.5a .$$
The long-term leverage scenarios
Conclusions

24. The global leverage evolution is modelled as a generalized process of parametric deleveraging; the safe debt is rolled-over within particular interval of leverage values. The model outlined alternatives: either deleveraging is kept under control and debt is safely accumulated, or devastating consequences of a bursting equity bubble are inevitable. Without excessive global equity valuation the first alternative is realized at $\tilde{\delta} = 0.5a$ ; its stochastic analogue $\sigma_N^2 \leq 0.5a$ provides convergence to the unimodal gamma distribution.

The sustained deleveraging could be accomplished due to coordinated efforts of financial regulators and central banks.