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Elliptic polylogarithms. General Theory and Applications

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Introduction

The concept of the elliptic curve was formulated at XIX century. Simultaneously the multiplicative group was recognised as locus of smooth points of special fibre of family of elliptic curves. Moreover many natural geometric objects on multiplicative group are regularized restrictions of objects on elliptic families. We apply this philosophy to so classical functions as polylogarithms. The last was subject of intensive study in the same XIX century.

We construct a bunch of functions on $\mathbb{H} \times \mathbb{C} \setminus \{\tau, \mathbb{Z}\tau \oplus Z\}$ by weighted average of the classical polylogarithms and apply to this divergent sum the ζ regularization procedure. The resulting analytic functions we call elliptic polylogarithms.

We study properties of these function in the same degree of understanding as general properties of classical polylogarithms are known. In particular we show that elliptic polylogarithms can be interpret as periods of some Hodge sheaf as the classical ones. This fact gives possibility to treat corresponding Hodge sheaf in pure cohomological technique, the last permits divine polylogarithms in more wide context, like, for example, other theories of cohomologies or/and tori of other dimensions. Moreover, the notion of elliptic polylogarithms clarifies some constructions in the theory of classical polylogarithms.The generalisation of polylogarithm became very useful tools for new more geometric proves of some known results like Klinderg-Ziegel theorem on values of zeta function or Manin-Drinfeld theorem on periods of Eisenstein series.

1 Elliptic Polylogarithms.[7]

1.1 Debye Polylogarithms.

Polylogarithms where first defined by Leonhard Euler as (analytic continuation of) the convergent in the unit disc series

$$Li_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n}.$$

They have been studied by some of the great mathematicians of the past – Abel, Lobachevsky, Kummer, and Ramanujan. In contrast with other Eulers higher transcendental functions, the polylogarithms were almost forgotten during long time. Recently they start to play an important role in the different brunches of mathematics and mathematical physics, f.e. in description of the cohomologies of linear groups or in theory of cluster varieties.

By technical reasons it is useful to operate with variant of polylogarithms known as Debye polylogarithms.

Definition 1.1 *n^{th} Debye polylogarithm $\Lambda_n(\xi)$ be the following multi-valued* *analytic function on* $\mathbf{C} \setminus \mathbf{Z}$*:*

$$\Lambda_n(\xi) = \int_{\xi}^{i\infty} \frac{t^{n-1}}{(n-1)!} \frac{d\dot{t}}{\exp(-2\pi i t) - 1}.$$

It is easy to check that

$$a) \qquad \Lambda_n(\xi) = \sum_{k=1}^{n} \frac{\xi^{n-k}}{(n-k)!} (-2\pi i)^{-k} Li_k(z);$$

$$b) \qquad Li_n(z) = (-2\pi i)^n \sum_{k=1}^{n} \frac{(-\xi)^{n-k}}{(n-k)!} \Lambda_k(\xi),$$

where $z = \exp(2\pi i \xi)$

It is useful to introduce generating function of these functions

$$\Lambda(\xi; K) = \sum_{n=1}^{\infty} \Lambda_n(\xi) K^{n-1} = \int_{\xi}^{i\infty} \frac{\exp(Kt)\, dt}{\exp(-2\pi i t) - 1}.$$

For $\Im(\xi) > 0$ one have canonical branch of this object, for $\Im(\xi) < 0$ fix two brunches $\Lambda^{\pm}$ by taking the path of integration crossing real axe at interval $(0, \pm 1)$.

Proposition 1.1 *The generating function* $\Lambda(\xi; K)$ *is covariant with respect to translations by integers and sign inversion:*

$$\Lambda(\xi + j; K) = \exp(jK)\Lambda(\xi; K), \ \textit{if} \ \Im(\xi) > 0$$

$$\Lambda^{+}(\xi + j; K) = \Lambda^{-}(-\xi; -K) + \frac{\exp(\xi K)}{K} - \frac{\exp(K)}{\exp(K) - 1}$$

Moreover, the push-forward of this function under multiplication by natural number coincides with itself.

$$\sum_{j=0}^{N-1} \exp\left(\frac{-jK}{N}\right) \Lambda(\xi + \tfrac{j}{N}; K) = \Lambda(N\xi; \tfrac{K}{N})$$

1.2 Definition and First Properties of Elliptic Polylogarithms.

Elliptic polylogarithms are multi-valued analytic functions on a punctured elliptic curve. We realize the elliptic curve as the quotient of $\mathbf{C}$ by lattice $L = \mathbf{Z}\tau \oplus \mathbf{Z}$. Hence these functions can be described as functions on $\mathbf{C} \setminus \mathbf{Z}\tau \oplus \mathbf{Z}$

Definition 1.2 *Elliptic (Debye) polylogarithm $\Lambda_{m,n}(\xi,\tau)$ with index (m, n) is multi-valued function on $(\mathbf{C}\times\mathbf{H})\setminus L$, defined by series*

$$\Lambda_{m,n}(\xi,\tau) = \frac{1}{m!}\left(\sum_{j=0}^{\infty} j^m \Lambda_n^+(\xi+j\tau) + (-1)^{m+n+1}\sum_{j=1}^{\infty} j^m\Lambda_n^-(-\xi+j\tau) + \sum_{k=0}^{n}\frac{\xi^{n-k}\tau^k}{(n-k)!k!}\frac{B_{m+k+1}}{m+k+1} + (-1)^{n+1}\frac{B_nB_{m+1}}{n!(m+1)}\right).$$

Remark 1.1 *The motivation of this definition is as follows. Naive weighted average*

$$\sum_{j\in\mathbf{Z}}\frac{j^m}{m!}\Lambda_n^+(\xi+j\tau)$$

diverges at $j\to-\infty$. Pass to the generating functions. Consider the series of generating functions

$$\underline{\Lambda}(\xi,\tau;X,Y) = \sum_{j\in\mathbf{Z}} e^{jX}\Lambda^+(\xi+j\tau;-Y)$$

It converges in the domain $\{0<\Re X,\quad 0<\Re(-Y\tau+X)<2\pi\Im\tau\}$ This function has first order poles at $X=0$ and $-Y\tau+X=0$. One have

$$\underline{\Lambda}(\xi,\tau;X,Y) = $$

$$= \sum_{m\geq 0, n\geq 1}\Lambda_{m,n}(\xi,\tau)(-Y)^{n-1}X^m + \frac{\exp(-Y\xi)}{(-Y)(-Y\tau+X)} + \frac{1}{(\exp Y-1)X}.$$

$\underline{\Lambda}(\xi,\tau;X,Y) = \sum_{m\geq 0 n>0}\Lambda_{m,n}(\xi,\tau)X^m(-Y)^{n-1}$

Remark 1.2 *A one-valued version of the elliptic polylogarithms was introduced by Bloch [B] in the case $m+n=3$ and by Zagier [Z2][Prop.2(ii),(iii)] for arbitrary m,n (see also Theorem 4.2).*

Theorem 1.1 *The modified generating function $\underline{\Lambda}(\xi,\tau;X,Y)$ is covariant with respect to translations by lattice $\mathbf{Z}\tau\oplus\mathbf{Z}n$.*

$$\underline{\Lambda}(\xi+1,\tau;X,Y) = \exp(-Y)\left(\underline{\Lambda}(\xi,\tau;X,Y) + \frac{1}{\exp(X)-1}\right) \text{ if } 0<\Im\xi<\Im\tau,$$

$$\underline{\Lambda}(\xi+\tau,\tau;X,Y) = \exp(-X)\underline{\Lambda}(\xi,\tau;X,Y)) \text{ if } 0<\Re\xi\Im\tau-\Im\xi\Re\tau<\Im\tau.$$

Moreover, the push-forward of this function under multiplication by natural number coincides with itself

$$\sum_{j=0}^{N-1}\exp\left(\frac{jY}{N}\right)\underline{\Lambda}(\xi+\tfrac{j}{N},\tau;X,Y) = \underline{\Lambda}(N\xi,N\tau;X,\tfrac{Y}{N})) \text{ if } 0<\Im\xi<\Im\tau.$$

Evidently $\Lambda_{0,1}(\xi,\tau) = \frac{1}{2\pi i}\log(\theta(\xi,\tau)/\eta(\tau))$,

$$\theta(\xi,\tau) = iq^{\frac{1}{8}}(z^{-\frac{1}{2}} - z^{\frac{1}{2}})\prod_{j=1}^{\infty}\left((1-q^j z)(1-q^j z^{-1})(1-q^j)\right), \eta(\tau) = q^{\frac{1}{24}}\prod_{j=1}^{\infty}(1-q^j)$$

.

1.3 Eisenstein - Kronecker series and elliptic polylogarithms

We recall a classical result of Kronecker [W] : Denote by L the lattice generated by 1 and τ. Any $\eta \in \mathbf{C}$ determines a character χ_η on L

$$\chi_\eta(\xi) = \exp(2\pi i\frac{\xi\overline{\eta} - \overline{\xi}\eta}{\tau - \overline{\tau}}).$$

Then [W, Z1] the Eisenstein-Kronecker series of weight 1 which is given by

$$K_1(\xi,\eta,1) = \sum_{w\in L}{}_e \frac{\chi_\eta(w)}{w+\xi}$$

(where $\sum_e$ denotes Eisenstein summation; see [W]) expresses as

$$K_1(\xi,\eta,1) \quad = \quad 2\pi i\exp\left(2\pi i\xi\frac{\eta-\overline{\eta}}{\tau-\overline{\tau}}\right)F(\xi,\eta,\tau), \tag{1}$$

$$F(\xi,\eta,\tau) = 1 - \frac{1}{1-z} - \frac{1}{1-w} - \sum_{m,n=1}^{\infty}(z^m w^n - z^{-m}w^{-n})q^{mn}$$
$$\Im\tau > \Im\xi > 0, \Im\tau > \Im\eta > 0$$

Moreover

$$F(\xi,\eta,\tau) = \frac{\theta'(0,\tau)\theta(\xi+\eta,\tau)}{\theta(\xi,\tau)\theta(\eta,\tau)}. \tag{2}$$

The transformation properties of $F(\xi,\eta,\tau)$ are very simple:

$$F(\xi+1,\eta,\tau) \quad = \quad F(\xi,\eta,\tau); \tag{3}$$
$$F(\xi+\tau,\eta,\tau) \quad = \quad \exp(-2\pi i\eta)F(\xi,\eta,\tau); \tag{4}$$
$$F(\frac{\xi}{c\tau+d},\frac{\eta}{c\tau+d},\frac{a\tau+b}{c\tau+d}) \quad = \quad (c\tau+d)\exp(2\pi i\frac{c\xi\eta}{c\tau+d})F(\xi,\eta,\tau). \tag{5}$$

$F(\xi,\eta,\tau)$ can be expressed as the exponential of the generating function of Eisenstein functions $E_k(\xi,\tau) = \sum_{w\in L}{}_e\frac{1}{(w+\xi)^k}$ and Eisenstein series $e_k(\tau) = \sum_{w\in L}{}'_e\frac{1}{(w)^k}$:

$$F(\xi,\eta,\tau) = \frac{1}{\eta}\exp(-\sum_{k=1}^{\infty}\frac{(-\eta)^i}{i}(E_k(\xi,\tau) - e_k(\tau))). \tag{6}$$

This statement is a simple corollary of Zagier's "Logarithmic Formula" for $F(\xi,\eta,\tau)$ [Z1, Section 3,Theorem(viii)] and the power series for E_n [W, Chapter III,formula(10)].

For nonholomorphic Eisenstein series

$$e_{k,l}(\eta,\tau)=\sum_{w\in L}{}' {}_e\frac{\chi_\eta(w)}{w^k\overline{w}^l}$$

the modified generating function with respect to variables $\eta,\overline{\eta}$ has the form

$$K_0(\eta,\xi,1)=\sum_{w\in L} {}_e\frac{\chi_\xi(w)}{|w+\eta|^2}$$

$$\begin{aligned}\sum_{w\in L} {}_e\frac{\chi_\xi(w)}{|w+\eta|^2} &= \frac{1}{|\eta|^2}+\sum_{w\in L}{}' {}_e\sum_{k,l=0}^{\infty}(-\eta)^k(-\overline{\eta})^l\frac{\chi_\xi(w)}{w^{k+1}\overline{w}^{l+1}}\\ &= \frac{1}{|\eta|^2}+\sum_{k,l=0}^{\infty}(-\eta)^k(-\overline{\eta})^l\sum_{w\in L}{}' {}_e\frac{\chi_\xi(w)}{w^{k+1}\overline{w}^{l+1}}. \qquad (7)\end{aligned}$$

Theorem 1.2

$$a)\qquad \frac{\partial}{\partial\xi}\underline{\Lambda}(\xi,\tau;X,Y)=e^{-Y\xi}F(\xi,\frac{-Y\tau+X}{2\pi i},\tau); \qquad (8)$$

$$b)\qquad \frac{\partial}{\partial\tau}\underline{\Lambda}(\xi,\tau;X,Y)=e^{-Y\xi}\frac{\partial}{\partial X}F(\xi,\frac{-Y\tau+X}{2\pi i},\tau). \qquad (9)$$

Theorem 1.3 *The imaginary part of the shifted generated function of elliptic polylogarithms*

$$\exp\left(\frac{\xi-\overline{\xi}}{\tau-\overline{\tau}}X+\frac{\tau\overline{\xi}-\overline{\tau}\xi}{\tau-\overline{\tau}}Y\right)\underline{\Lambda}(\xi,\tau;X,Y)$$

is equal to $-\frac{\tau-\overline{\tau}}{(2\pi i)^2}K_0(\eta,\xi,1)$

Remark 1.3 *The statement of the previous Theorem means that the elliptic analogs of the Bloch-Wigner-Zagier polylogarithms are nonholomorphic Eisenstein series.*

1.4 Modular properties of elliptic polylogarithms

From the expression of the differential of generating function of elliptic polylogarithms in terms of the Kronecker function F and modular properties of the late function one can deduce that the generating function of elliptic polylogarithms is modular up to additive constant – em period in Ξ and τ power series in X and Y.

As the modular group intertwine action of lattice by shifts, these power series in X and Y can be calculated and occurred to have rational coefficients. We can formulated this property as

Theorem 1.4 *The modified generating function* $\underline{\Lambda}(\xi, \tau; X, Y)$ *is covariant with respect to action of modular group:*

$$\underline{\Lambda}(\tfrac{\xi}{c\tau+d}, \tfrac{a\tau+b}{c\tau+d}; aX+bY, cX+dY) = \underline{\Lambda}(\xi, \tau; X, Y) + C_M(X, Y),$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

C_M *is a Laurent series in* X *and* Y *with rational coefficients,*

$$C_{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}(X, Y) = \frac{1}{\exp(Y)-1}\left(\frac{1}{\exp(X+Y)-1} - \frac{1}{\exp(Y)-1}\right),$$

$$C_{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}(X, Y) = -\left(\frac{1}{\exp(X)-1}\right)\left(\frac{1}{\exp(Y)-1}\right)$$

Moreover, the push-forward of this function under isogeny coincides with itself up to addition of Laurent series in X *and* Y *with rational coefficients*

$$\sum_{\kappa\in\mathcal{K}} e^{<\kappa,(X,Y)>}\underline{\Lambda}(\xi+\kappa, \tau; X, Y) = \underline{\Lambda}(N\tfrac{\xi}{c\tau+d}, \tfrac{a\tau+b}{c\tau+d}; \tfrac{aX+bY}{N}, \tfrac{cX+dY}{N}) + C_M(X, Y),$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbf{Z}), N = \det M, \mathcal{K} = \mathrm{Ker}(\mathbf{Q}/\mathbf{Z})^2 \stackrel{M}{\to} (\mathbf{Q}/\mathbf{Z})^2,$$

for $\kappa = \begin{pmatrix} r \\ s \end{pmatrix}$ *we put* $<\kappa, (X, Y)>= rY - sX$, C_M *is a Laurent series in* X *and* Y *with rational coefficients,*

The second part of the Theorem follows from the first and decomposition of any isogeny in composition of modular transformation and just multiplication by natural number.

The N-torsion point on the universal elliptic curve forms the modular curve of level N and the restriction of the differential of the modified generating function $\underline{\Lambda}(\xi, \tau; X, Y)$ of polylogarithms coincides with generating function of the Eisenstein series of this level. From this and rationality of theperiods of elliptic polylogarithms we get new prove of the following theorem of Yu.Manin and V.Drinfeld:

Theorem 1.5 *The periods of the Eisenstein series of any level and weight greater that 2 are rational.*

1.5 Hodge Theoretic interpretation of elliptic polylogarithms. Abstract cohomological approach to elliptic polylogarithms.[6]

The data $\{\mathbf{Q}X \oplus \mathbf{Q}Y \subset \mathbf{C}X \oplus \mathbf{C}Y \supset \mathbf{C}(X - \tau Y)\}$ is the Hodge structure on the first homology group $\mathcal{H}$ of elliptic curve. The action of generators $+1$

and $+\tau$ of the fundamental group of the curve determined as multiplication by $\exp(-Y)$ and $\exp(-Y)$ respectively on the ring $\mathbf{Q}[[X,Y]]$ of formal power series in two variables X and Y determines filtrated local (pro)system on the elliptic curve. The complexification of this local system can be equipped by holomorphic Hodge filtration by the rule $F^{-p} =< (X-\tau Y)^q Y^p \exp(-\xi Y) >_{\mathbf{C}}$ for all natural q and so produce Hodge sheaf $\mathcal{L}$ on the curve. Consider the extension $\mathcal{P}$ of the constant local system $\mathcal{H}$ (with generators X' and Y') by $\mathcal{L}$ over punctured curve defined by representation

$$+1 \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\exp(-Y)X}{\exp(X)-1} & \frac{\exp(-Y)Y}{\exp(X)-1} - \frac{1-\exp(-Y)}{X} & \exp(-Y) \end{pmatrix},$$

$$+\tau \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1-\exp(-X)}{X} & \exp(-X) \end{pmatrix},$$

so

$$\begin{pmatrix} X' \\ Y' \\ f(X,Y) \end{pmatrix} \stackrel{+1}{\to} \begin{pmatrix} X' + \frac{\exp(-Y)X}{\exp(X)-1} \\ Y' + \frac{\exp(-Y)Y}{\exp(X)-1} - \frac{1-\exp(-Y)}{X} \\ \exp(-Y)f(X,Y) \end{pmatrix}$$

$$\begin{pmatrix} X' \\ Y' \\ f(X,Y) \end{pmatrix} \stackrel{+\tau}{\to} \begin{pmatrix} X' \\ Y' - \frac{1-\exp(-X)}{X} \\ \exp(-Y)f(X,Y) \end{pmatrix}$$

Define filtration F^0 on $\mathcal{P}$ over punctured curve generated by $F^0(\mathcal{L})$ and $-Y'\tau + X + (-Y\tau + X)\underline{\Lambda}(\xi,\tau;X,Y) - \frac{\tau}{X}$. Note that the second sumond is a formal power series in X and Y, denominators of shape Y and $-Y\tau+X$ cancel. Then the properties of $\underline{\Lambda}(\xi,\tau;X,Y)$ yield property to be the Hodge sheaf for this fitration on local system. This Hodge sheaf is an extension of trivial sheaf $\mathcal{H}$ by $\mathcal{L}$ over punctured elliptic curve. We call this extension elliptic polylogarithms also. This construction can be generalized for any "reasonable" theory of cohomologies.

Consider an elliptic curve E with neutral element $\tilde{0}$. Denote by $\mathcal{H}$ the constant sheaf on E whose fibers are equal to the first homology of the curve. Denote by $\mathcal{L}$ the Poincare groupoid of paths from the origin $\tilde{0}$ to variable point. One has an *augmentation map* from $\mathcal{L}$ to trvial sheaf corresponding to passing to homology class of path.

Theorem 1.6 *There is an extension of $\mathcal{H}$ by $\mathcal{L}$ over punctured curve $E \setminus \tilde{0}$. Moreover such an extension is uniquely determined be extension of $\mathcal{H}$ by trivial sheaf by appling the augmentation map.*

In a prove we use the following speculations:

1) One can calculate the cohomology groups of E with coefficients in $\mathcal{L}$ and get that the top group is not trivial and is equal to top cohomology group with trivial coefficients.

2) For calculation cohomology groups of punctured curve $E \setminus \tilde{0}$ one can use the Mayer–Vietoris sequence for covering { formal neighborhoud of $\tilde{0}$,$E \setminus \tilde{0}$}.

2 Polylogarithic Forms on Tori.

The concept of elliptic polylogarithms can be generalised in more wide framework. In this section we present two examples.

2.1 Polylogarithms on Families of Smooth Metrised Tori.

We present pure topological analog of the construction in previous subsection see[1]. Let us consider d-dimensional torus $T = V/H$, H be a free abelian group of rank d, $V = H_{\mathbb{R}} = H \otimes_{\mathbb{Z}} \mathbb{R}$ Denote by $\tilde{0}$ the neutral element of $T =$ image of $0 \in V$. Then the fudnamental group $\pi_1(T, \tilde{0}$ is evidently equal to H. Denote by $R = \mathbb{Q}[H]$ the group algebra of H, put $I \subset R$ be the augmentation ideal, let $\check{R}$ be the unipotent completion of R.

Sublattice $H' \subset H$ of finite index determined isogenic torus $T' = V/H'$, $H/H' \to T' \overset{\phi}{\to} T$. One have canonical embeddings $\mathbb{Q}[H'] \to \mathbb{Q}[H]$ and $\check{R}' \to \check{R}$

The completed regular representation H on $\check{R}$ determines local system $\mathcal{L}$ on T. One has canonical push forward map $\phi_* \mathcal{L}' \to \mathcal{L}$.

The cohomology groups of T with coefficients in $\mathcal{L}$ are equal to cohomology groups of H with coefficients in regular representation R, so they are trivial except the top one, the last group is just $\mathbb{Q}$.

The Mayer–Vietoris sequence for covering { formal neighborhood of $\tilde{0}$,$T \setminus \tilde{0}$} shows that $H^{<d-1}(T \setminus \tilde{0}, \mathcal{L} = 0$ and $H^{d-1}(T \setminus \tilde{0}, \mathcal{L}) = I$.

One has natural map $H_{\mathbb{Q}} \to I$, hence we have distinguished $\mathrm{SL}(H)$ invariant class in $H^{d-1}(T \setminus \tilde{0}, \mathrm{Hom}(H, \mathcal{L}))$. This class is compatible with push forward under isogenies of the torus.

We realize this class in the following way. Consider the contractible domain B_H of all positive definite matrics on V. The group $\mathrm{SL}(H)$ acts on $B \times (T \setminus \tilde{0})$ and on the local system $\mathrm{Hom}(H, \mathcal{L})$ lifted from the second factor. The cohomological class under considereation can be represented by closed $\mathrm{Hom}(H, \mathcal{L})_{\mathbb{R}}$-valued differential $(d-1)$ form P on the quotient $(B \times (T \setminus \tilde{0}))/\mathrm{SL}(H)$ in the sense of orbifold.

This form is sum of two sumonds. The first is multiplication by $\mathcal{L}$- valued (d-1)-form g. This form is the Green function of the differential $d: \ d(g) = \delta_{\tilde{0}} - \mathrm{vol}$, where vol denotes fiberwise volume form. Hence g can be constructed as Fourier series over $H \setminus 0$. The second has rather elementary shape. This form is covariant with respect to push forward under isogenies of the torus.

As the class is defined for rational coefficients, the periods of the constructed above form are rational.

2.2 Rationality of Values of Zeta Values for Totally Real Fields.[1]

We apply the construction from the previous section to the sublattices of integer algebraic numbers of totally real field.

Let $\mathcal{O}$ be maximal totaly real order of rank d. Then $\mathcal{O} \otimes \mathbb{R} = \mathbb{R}^{\oplus d}$ is coordinate vector space. Consider the domain B_{diag} of positive determined diagonal matrices of volume 1 on $\mathcal{O} \otimes \mathbb{R}$. The group $\mathcal{O}^*$ of units acts on B_{diag}; the quotient $B_{\text{diag}}/\mathcal{O}^*$ is torus of dimension $d-1$.The group $\mathcal{O}^*$ of units is embedded into $\mathrm{SL}(\mathcal{O})$. Hence $B_{\text{diag}}/\mathcal{O}^*$ can be treated as compact $d-1$ cycle Z_{diag} in $B_{\mathcal{O}}/\mathrm{SL}(\mathcal{O})$.

Let $\mathfrak{a}$ be an ideal of $\mathcal{O}$. It will be the lattice under consideration. The polynomial function (Norm^k on $\mathfrak{a}$ determines the map $\check{R} \to \mathbb{Q}$, so one have a map $\tilde{N}^{(k)} : \mathrm{Hom}(H, \mathcal{L}) \to \mathrm{Hom}(\mathfrak{a}, \mathfrak{a})$ induced by $\mathfrak{a}$ valued function $\mathrm{Norm}^k \mathrm{Id}_{\mathfrak{a}}$ on $\mathfrak{a}$. So we have differential form $\tilde{z}_k = \mathrm{tr}(\tilde{N}^{(k)}(P))$ on $(B \times (T \setminus \tilde{0}))/\mathrm{SL}(H)$. For $k > 0$ this forms smoothly continue to the zero section $\tilde{0}$. Denote by z_k the restriction to the zero section $\tilde{0}$ of the differental form $\tilde{z}_k$.

Theorem 2.1 *The integral of the $(d-1)$ form z_k over $(d-1)$ cycle Z_{diag} is rationally proportional to value of partial ζ-function of ideal class of ideals $\mathfrak{a}$ at the point $1-k$.*

Combining this result with rationality of class of P we get

Theorem 2.2 *(Kliberger-Siegel) For totaly real field values of partial ζ-function of ideal class at negative integers is rational.*

For noninteger algebraic number $\alpha \in \mathcal{O} \otimes \mathbb{Q} \setminus \mathcal{O}$ one can consider a compact $d-1$ cycle $Z_{\alpha,\text{diag}}$ in $(B \times (T \setminus \tilde{0}))/\mathrm{SL}(H)$ defined as image of $(B_{\text{diag}}, \alpha)$ The form z_k over $Z_{\alpha,\text{diag}}$ are related to the values of the partial zeta functions of ray class.

By considering reasonable integer structure on the objects above on can deduce some congruence properties of values of zeta values.

2.3 Polylogarithms on Abelian Varieties.[4]

The notion of Polylogarithms on Abelian varieties or, more generally, on mixed Shimura varieties was introduced by Joerg Wildeshause. We present corresponding differential forms.

An abelian variety over $\mathbb{C}$ can be described as follows. Let H be a free abelian group of even rank $d = 2n$. Decompose $H_{\mathbb{C}}$ into direct sum $H_{\mathbb{C}} = H^{(0,-1)} \oplus H^{(-1,0)}$ such that $\overline{H^{(0,-1)}} = H^{(-1,0)}$. Then $A_H = H_{\mathbb{C}}/(H + H^{0,-1})$ is a torus with complex structure. So, as C^∞ variety A can be described as in first subsection of this Section with $V = H^{(-1,0)}$. We will use notation from this subsection.

We s construct differntial form P_{Hodge} on family of Abelian varities over compatible with Hodge decomposition:

$$(P'_{\text{Hodge}}, P''_{\text{Hodge}}) \in \text{Hom}(H^{0,-1}, \mathcal{L}) \otimes \mathcal{A}^{d,d-1} \oplus \text{Hom}(H^{-1,0}, \mathcal{L}) \otimes \mathcal{A}^{d-1,d},$$

such that $\overline{P'_{\text{Hodge}}} = P''_{\text{Hodge}}$ and $\text{Hodge}' - P''_{\text{Hodge}} = d\,G$ for some $(d-1, d-1)$ form G.

This construction produce a bunch of Siegel forms with rational periods.

3 Elliptic 1-2-3-Logarithms

3.1 The Group of Abstract Elliptic Onelogarithms.[5]

The definition of abstract elliptic onelogarithms is based on the Deligne construction of pairing of invertible sheaves on a curve and just by the definition they form an abelian group which is extension of the symmetric square of the Jacobian of the elliptic curve by the group of nonzero elements of the field.

Definition 3.1 *The Deligne pairing $[L_1, L_2]$ of invertible sheaves L_1 and L_2 on a curve X over field k is a k^*-torsor determined in the following way: any rational sections s_i of L_i with non-itersecting divisors determines an element $< s_1, s_2 > \in [L_1, L_2]$ and for any rational function f $< fs_1, s_2 >= f((s_1)) < s_1, s_2 >$ and familiar for second .*

Theorem 3.1 *The pairing $[L_1, L_2]$ is bilinear symmetric.*

It is possible to check that for generic sections s_i of L_i one can determines an element $< s_1, s_2 >$ in the torsor $[L_1, L_2]$ twisted be relevant power of punctured tangent spaces at points of points of intersection of support of divisors (s_i) of sections s_i. For elliptic curve tangent spaces at points are canonically isomorphic by action of shifts to tangent space T at neutral element, as elliptic curve is abelian group.

For a point a of elliptic curve X denote by L_a the invertible sheaf $\mathcal{O}(a - \tilde{0})$ where $\tilde{0}$ denotes the neutral element of the elliptic curve. Denote by s_a the canonical meromorphic section of this sheaf.

The "square" $< s_a, s_a >$ of the canonical section s_a belongs to $[L_a, L_a] \otimes T_0^2$, where T_0 denotes punctured tangent space at neutral element.Twentieth power T_0^{12} of T_0 is trivialized be discriminant Δ.

Naively the abstract elliptic onelogarithm is logarithm of abstract theta function, these means nothing more that we change the operation sign from "times" to "plus".

Definition 3.2 *The "exponential" of the abstract elliptic onelogarithm (= abstract theta function) is defined as inverse of the normalized square root of $< s_a, s_a >$:*

$$\theta(a) = \exp(Li_1^{ell}(a)) = \Delta^{\frac{1}{12}} < s_a, s_a >^{-\frac{1}{2}} \in [L_a, L_a]^{-\otimes \frac{1}{2}}$$

By bilinearity of pairing $[L_1, L_2]$ the tensor product $\bigotimes [L_a, L_a]^{\otimes n_a}$ is trivial torsor if $\sum n_a a^2 = 0$. Hence, for the group A of k-points of X one have a map the kernel of the map $\mathbb{Q}[A] \to \mathrm{Sym}^2 A \otimes \mathbb{Q}$ to k^*.

Proposition 3.1 *For any finitely generated commutative group A the kernel of the map $\mathbb{Q}[A] \to \mathrm{Sym}^2 A \otimes \mathbb{Q}$ from the group algebra to the symmetric square $[a] \to a^2$ is generated by elements of the shape $[a + b] + [a - b] - 2[a] - 2[b]$*

Proposition 3.2 *For points a and b of elliptic curve $[a+b]+[a-b]-2[a]-2[b] \in \mathbb{Q}[A]$ maps to $\wp(a) - \wp(b) \in k^*$.*

Denote the kernel of this map by $\mathcal{R}_1$

Definition 3.3 *The group of abstract elliptic onelogarithms is the quotient $\mathbf{\Theta}_1(X) = \mathbb{Q}[A]/\mathcal{R}_1$.*

3.2 Elliptic Dilogarithm and Values of L-fuctions of Elliptic Curves.[5]

Recall that for a field F the second K-group $K_2(F) = \bigwedge^2 F^*/(a \wedge (1 - a))$, for local field F with residue field k the tame symbol is $\partial K_2(F) \to k^*, \tau(f \wedge g) = (-1)^{\mathrm{ord}(f)\mathrm{ord}(g)} g^{\mathrm{ord}(f)}/f^{\mathrm{ord}(g)}$; for a curve X its second K-group consist of elements of second K-group of the function field with trivial tame symbols at all points.

For compact complex curve X we can correspond to $f \wedge g$ the expression $\beta(f \wedge g) = \log|g|(\partial - \overline{\partial}) \log|f| - \log|f|(\partial - \overline{\partial}) \log|g|$. It is smooth form on the complement to support of divisors of functions and is a current on the compact curve as singularities at support of divisors of functions are integrable. The differential of the current $\beta(f \wedge g)$ is compatible with tame symbol: $d\,\beta(f \wedge g) = \sum_{p \in X} 2\pi i \log|\tau_p(f \wedge g)|\delta_p$. Hence the regulator maps K_2 of the curve to closed currents. The cohomology class of of this current is called the regulator. Usually we recognize it by integrating against regular forms.

In his keystone article S.Bloch studied regulator for elliptic curve. He decompose elliptic functions into product of theta function and apply to logarithms of theta functions classical Kronecker limit formula. As result he got values of nonholomorphic Eisenstein series (= the elliptic dilogarithm) at differences of points of divisors of functions.

It is known that for elliptic curve E over $\mathbb{Q}$ this regulator is equal up to rational factor to the value of the L-function of this elliptic curve at $s = 2$

In the spirit of ideology of Zagier conjecture it will be natural to recognize the conditions on the arguments of this function.

Theorem 3.2 *For any elliptic curve E over $\mathbb{Q}$ the values $L(E, 2)$ of its L-function at 2 up to π and rational is equal to value of singlevalued elliptic dilogarithm at divisor $P = \sum n_i P_i$ such that $\sum n_i P_i^3 = 0$. Morover, this divisor is subject of some similar qubic conditions involving local heights. Also the divisor satisfies some intagrability conditions for prime numbers at which the curve has split multiplicative reduction.*

The approach to proof is based on the notion of abstract theta functions introduced in previous subsection. By decomposition of elliptic functions into product of abstract theta functions we present reduced tame symbol of an element of K_2 of the functional field as combination of elements of the shape $[a-b]\otimes(a-b)$ in $\mathbf{\Theta}_1(X)\otimes \mathrm{Jac}(X)$, where a and b denotes points of the divisors of functions. So, vanishing of the tame symbol is cubic condition.

3.3 Elliptic Threelogarithm at CM-point and Classical Dilogarithm at Algebraic Numbers.[3]

This section, as the previous one, lies in the stream of the Zagier approach to the values of L-functions. We shall discuss values of the Dedeckind ζ-function of the following classical field. Consider imaginary quadratic field. Its abelian extensions are generated by values of elliptic and modular functions at CM points. For Hilbert class field of imaginary field values of its Dedeckind ζ-function can be expressed in terms of nonholomorphic Eisenstein series. Values of ζ-function at $s=2$ by (proved for $s=2$) Zagier conjecture should be expressed in terms of classical dilogarithms of some algebraic numbers. Hence, it is natural to expect that values of nonholomorphic Eisenstein series at CM point(which is for $s=2$ the "middle" component of elliptic threelogarithm) are related with values of classical dilogarithm. We prove the following very explicit result:

Theorem 3.3 *The value of the "middle" component of elliptic threelogarithm at CM point up to elementary factor is equal to sum of Bloch-Wigner dilogarithms of cross-ratio of values of the Weierstraß at torsion points.*

The proof is inspired by Goncharov explicit formulas for Bloch-Beilinson regulator for symbolic part of K-theory. In sequence, Goncharov present integral formula for BW-dilogarithm. The region of integration is just complex projective line and the integrand is application of regulator formulas to onelogarithms.

The main hint of our proof of the Theorem is the following. We present similar expression for elliptic threelogarithm as integral of regulator formula of elliptic onelogarithms over square of the elliptic curve. We also introduce some auxiliary function, elliptic $(1,1)$-logarithm as similar integral over elliptic curve.

The elliptic $(1,1)$-logarithm is related with BW-dilogarithm via double covering of projective line by elliptic curve. At the other hand, CM-curve can be embedded to its square as the graph of complex multiplication, hence integral over individual curve reduces to integral over its square.

4 Differential Forms on Powers of Elliptic Curves.[2]

In this section we discuss some results which are not so close to the subject of dissertation, we use the differential of the generating function of elliptic polylogarithms.

4.1 Cohomology Groups of Configuration Space.

Fix a natural number N. Denote by $\mathcal{X}$ its N-th power X^N.

For $n < N$ consider n distinct points $\tilde{\mathbf{z}} = \{\tilde{z}_1, \cdots \tilde{z}_n\}$ on X coming from points $\mathbf{z} = \{z_1, \cdots z_n\}$ on $\mathbb{C}$.

Definition 4.1 *The* discriminantal arrangement $\tilde{C}_{\mathbf{z}}$ *is the arrangement of divisors* H_{jk} *– the* jk *diagonals and* H_{ja} *of the shape* $X^{j-1} \times \tilde{z}_j \times X^{N-j}$

The complement to this arrangement is the configuration space $\tilde{M}_{\tilde{\mathbf{z}}}$ of N points on $X \setminus \bigcap \tilde{z}_a$.

The fundamental group of $\mathcal{X}$ is equal to $\bigoplus_{j=1}^{N} (\mathbb{Z}\tau \oplus \mathbb{Z})$ as direct sum of fundamental groups $\mathbb{Z}\tau \oplus \mathbb{Z}$ of elliptic curve X.

Hence, a collection $\mathbf{w} = \{w_i\}$ of N complex numbers determines a one-dimensional local system $\mathcal{L}_{\mathbf{w}}$ corresponding to representation $\rho : \{(l_j\tau + m_j)\} \to \exp(2\pi i \sum l_j w_j)$

Definition 4.2 *The collection* $\mathbf{w} = \{w_i\}$ *is* discriminantal convenient *if for any subset* $I \subset \{1, \cdots, N\}$ *the sum* $\sum_{j\in I} w_j$ *is not in* $\mathbb{Z} \oplus \mathbb{Z}\tau$.

Theorem 4.1 *For discriminantal convenient collection* $\mathbf{w} = \{w_i\}$ *the cohomology group* $H^j(\tilde{M}_{\tilde{\mathbf{z}}}, \mathcal{L}_{\mathbf{w}})$ *vanishes for* $0 \leq j < N$ *and coincide with correspondent cohomology group* $H^N(M_{\mathbf{z}}, \mathbb{C})$ *of affine configuration space* $M_{\mathbf{z}}$ *for* $j = N$.

A prove is based on consideration of the spectral sequence of complement to divisor. Its degeneration in second term follows from explicit construction of the required differential forms with logarithmic singularities. This forms can be determined as product of functions $K_1(*, *, 1)$ from Subsection 1.3the first arguments of these functions are either equations of diagonals or equations of points z_j, the second arguments are partial sums of w_j.

4.2 The Elliptic Arrangements

Fix a natural number N. Recall that $\mathcal{X}$ denotes N-th power X^N of the curve X. For $k < N$ any integer $N \times k$-matrix Q of rank k determines a surjective map from $\mathcal{X} = \mathcal{X^N}$ onto X^k.

Definition 4.3 *The elliptic* $N-k$*-plane is preimage of some point of* X^k*, hence any hyperplane* H_α *can be define by some primitive integer raw of length* $N + 1$.

The collection of l hyperplanes can be determined be $(N + 1) \times l$ matrix, the raws of this matrix will be equations of hyperplanes;

Definition 4.4 *Hyperplanes intersect transversally if the rank of this matrix equals* l.

A collection $\mathbf{w} = \{w_i\}$ of N complex numbers determines a one-dimensional local system $\mathcal{L}_{\mathbf{w}}$ corresponding to representation $\rho : \{(l_j\tau + m_j)\} \to \exp(2\pi i \sum l_j w_j)$,

Definition 4.5 *A collection is* convenient *for transversal arrangement* $A\mathbf{t} = \mathbf{p}$ *of* N *if for some integer solution* $\mathbf{l} = \{l_1 \cdots l_N\}$ *of equation* $\mathbf{l}A = 0$ *the sum* $\sum l_j w_j$ *is not in the lattice* $\mathbb{Z} \oplus \mathbb{Z}\tau$.

Definition 4.6 *A collection is* convenient *for arbitrary arrangement if is* convenient *for all transversal subarrangements.*

For convenient collection we have combinatorial description of the cohomology groups of the complement to arrangement with coefficients in local system:

Theorem 4.2 *For convenient collection we have the following description of the cohomology groups of the complement to arrangement with coefficients in local system. All cohomology group except the N-th vanishes. The N-th cohomology group can be realized as direct sum of local Orlik-Solomon algebras for intersections of zeroth dimension. Moreover, the representing this classes differential forms with logarithmic singularities can be written explicitly as in the previous theorem*

5 List of author's publications

1. Beilinson Alexander; Kings Guido; Levin Andrey Topological polylogarithms and p-adic interpolation of L-values of totally real fields. Math. Ann. 371 (2018), no. 3-4, 1449 –1495.

2. Levin Andrey; Varchenko Alexander Cohomology of the complement to an elliptic arrangement. Configuration spaces, 373–388, CRM Series, 14, Ed. Norm., Pisa, 2012.

3. Levin Andrey Kronecker double series and the dilogarithm. Number theory and algebraic geometry, 177–201, London Math. Soc. Lecture Note Ser., 303, Cambridge Univ. Press, Cambridge, 2003.

4. Levin Andrey Polylogarithmic currents on abelian varieties. Regulators in analysis, geometry and number theory, 207–229, Progr. Math., 171, Birkhäuser Boston, Boston, MA, 2000.

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6. Levin Andrey Elliptic polylogarithms: an analytic theory. Compositio Math. 106 (1997), no. 3, 267–282.

7. Beĭlinson A.; Levin A. The elliptic polylogarithm. Motives (Seattle, WA, 1991), 123–190, Proc. Sympos. Pure Math., 55, Part 2, Amer. Math. Soc., Providence

6 Main Results

We formulate the main constructions and statements of this habilitation thesis.

I introduce new collection of functions – Elliptic Polylogarithms which generalize simultaneously either classical Euler polylogarithms or modular forms. The construction is based on the zeta-regularization of polynomial divergent sums.

I establish relation of this new objects with well known Eisenstein series, more precisely, the Eisenstein series are derivatives and single-valued version of the elliptic polylogarithms.

I investigate modular behavior of elliptic polylogarithms.

I realize these properties as conditions on matrix coefficients of Hodge sheaves extension.

We construct such extension for wide class of theories of cohomologies.

We present pure algebraic construction of elliptic onelogarithm in terms of Deligne pairing.

We deduce conditions on the arguments of elliptic dilogarithm for getting value of L-function of this elliptic curve at $s = 2$.

I present a collection of algebraic numbers in ray class field of imaginary quadratic field such that combination of values of Bloch-Wigner dilogarithm at them equals to middle component of elliptic threelogarithm at CM-point.

We calculate top cohomology groups of the complement to elliptic arrangement for generic system of coefficients

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