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Trace methods in algebraic geometry

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Chapter 1

In the first part of Chapter 1 we introduce a main categorical tool of this work and prove some of its basic properties:

Proposition 0.1. *Let $\mathcal{E}$ be a symmetric monoidal $(\infty, 2)$-category (that is, a commutative algebra object in the ∞-category of $(\infty, 2)$-categories) and $X, Y \in \mathcal{E}$ be dualizable objects. Suppose we are given a (not necessary commutative) diagram*

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, bend right, "\varphi"'] & X \arrow[d, bend right, "\varphi"'] \arrow[dl, Rightarrow, "T"'] \\ Y \arrow[r, "F_Y"] \arrow[u, bend right, "\psi"'] & Y \arrow[u, bend right, "\psi"'] \end{tikzcd}$$

in $\mathcal{E}$, where φ is left adjoint to ψ and

$$\varphi \circ F_X \xrightarrow{T} F_Y \circ \varphi$$

is a 2-morphism in $\mathcal{E}$. Then there exist a natural morphism

$$\mathrm{tr}_{\mathcal{E}}(F_X) \xrightarrow{\mathrm{tr}_{\mathcal{E}}(\varphi, T)} \mathrm{tr}_{\mathcal{E}}(F_Y)$$

in the ∞-category $\mathrm{Hom}_{\mathcal{E}}(I, I)$ called a ***morphism of traces induced by*** *T. Moreover, $\mathrm{tr}_{\mathcal{E}}(-,-)$ is functorial with respect to the vertical compositions.*

We refer readers to [GR17a, Appendix] for a thorough discussion of the notion of $(\infty, 2)$-categories. In this work we will be most interested in the case $\mathcal{E} := 2\mathcal{C}\mathrm{at}_k$, a category of k-linear presentable stable categories and continuous k-linear functors, where k is some field. Note that the monoidal unit in $2\mathcal{C}\mathrm{at}_k$ is the category Vect_k (unbounded derived category of k-vector spaces) and $\mathrm{End}_{2\mathcal{C}\mathrm{at}_k}(\mathrm{Vect}_k) \simeq \mathrm{Vect}_k$. In particular for any dualizable k-linear category $\mathcal{C}$ equipped with an endofunctor G the trace $\mathrm{tr}_{2\mathcal{C}\mathrm{at}_k}(G)$ is naturally a complex of k-vector spaces, which we will sometimes denote by $HH(\mathcal{C}, G)$. In this setting we can define:

Construction 0.2. Given a dualizable k-linear category $\mathcal{C}$ equipped with an endofunctor $G\colon \mathcal{C} \to \mathcal{C}$ and a lax G-equivariant compact object (E, t), i.e. a compact object $E \in \mathcal{C}$ equipped with a map $t\colon E \to G(E)$, we can form a diagram

$$\begin{tikzcd} \mathrm{Vect}_k \arrow[r, "\mathrm{Id}_{\mathrm{Vect}_k}"] \arrow[d, bend right, "-\otimes E"'] & \mathrm{Vect}_k \arrow[d, bend right, "-\otimes E"'] \arrow[dl, Rightarrow, "T"'] \\ \mathcal{C} \arrow[r, "G"] \arrow[u, bend right] & \mathcal{C} \arrow[u, bend right, "\mathrm{Hom}_{\mathcal{C}}(E,-)"'] \end{tikzcd}$$

in $2\mathcal{C}\mathrm{at}_k$ with the 2-morphism T induced by the morphism t. The corresponding element $\mathrm{ch}(E, t) \in HH(\mathcal{C}, G)$ obtained via the formalism of traces is called a **categorical Chern character of** E.

With these notations the functoriality of traces can be restated in the following suggestive form:

Proposition 0.3 (Categorical trace formula)**.** *Let $\mathcal{C}, G$ and (E, t) be as above. Then given a functor $\Gamma\colon \mathcal{C} \to \mathrm{Vect}_k$ admitting a continuous right adjoint and a natural transformation $\varepsilon\colon \Gamma \circ G \to \Gamma$, we have*

$$\mathrm{tr}_{\mathrm{Vect}_k}\left(\Gamma(E) \xrightarrow{\Gamma(t)} \Gamma(G(E)) \xrightarrow{\varepsilon_E} \Gamma(E) \right) = \int_{\varepsilon} \mathrm{ch}(E, t),$$

where $\int_{\varepsilon}\colon HH(\mathcal{C}, G) \to k$ is the morphism of traces induced by ε.

The problem with the previous proposition is that $HH(\mathcal{C}, G)$, $\mathrm{ch}(-,-)$, and $\int_{\varepsilon}$ are not very explicit in general. In the second part of the first chapter we address this problem by performing some trace computations in appropriate categories of correspondences, which can be then used by applying symmetric monoidal functors QCoh or ICoh.

Chapter 2

Conventions 0.4. For the rest of the document we will assume that k is a base field of characteristic zero.

The second chapter is devoted to the trace-based proof of the non-equivariant (Hodge cohomology valued) Grothendieck-Riemann-Roch theorem. First recall the statement:

Theorem ([BS58])**.** *Let $f\colon X \to Y$ be a morphism of smooth proper k-schemes. Then the diagram*

$$\begin{tikzcd} K_0(X) \arrow[r, "\mathrm{ch}(-)\cdot \mathrm{td}_X"] \arrow[d, "f_*"'] & \bigoplus_p H^p(X, \Omega^p_X) \arrow[d, "f_*"] \\ K_0(Y) \arrow[r, "\mathrm{ch}(-)\cdot \mathrm{td}_Y"'] & \bigoplus_p H^p(Y, \Omega^p_Y) \end{tikzcd}$$

commutes, where $\mathrm{ch}(-)$ is the classical Chern character and td_- is the Todd class of the tangent bundle. That is, for a perfect complex of quasi-coherent sheaves E on X we have

$$f_*(\mathrm{ch}(E)\,\mathrm{td}_X) = \mathrm{ch}(f_*(E))\,\mathrm{td}_Y\,. \tag{1}$$

Now the main idea is that one obtains an equality (1) as a corollary of functoriality of the morphisms of traces. More concretely, consider a commutative diagram

$$\begin{tikzcd} \mathrm{Vect}_k \arrow[r, "E\otimes -"] & \mathrm{QCoh}(X) \arrow[r, "f_*"] \arrow[d, "\sim"', "\otimes \mathcal{O}_X"] & \mathrm{QCoh}(Y) \arrow[d, "\sim"', "\otimes \mathcal{O}_Y"] \\ & \mathrm{ICoh}(X) \arrow[r, "f_*"'] & \mathrm{ICoh}(Y), \end{tikzcd}$$

where E is a perfect complex on X, and $\mathrm{QCoh}(X)$ and $\mathrm{ICoh}(X)$ are categories of quasi-coherent and ind-coherent sheaves on X respectively (see [GR17a, Chapter 3.1] and [GR17a, Chapter 4.1] for the definitions). Then by applying the functoriality of traces to the diagram above we obtain a commutative diagram of k-vector spaces:

$$\begin{tikzcd} k \arrow[r, "\mathrm{ch}(E,\mathrm{Id}_E)"'] \arrow[rr, bend left, "\mathrm{ch}(f_*(E),\mathrm{Id}_{f_*(E)})"] & HH(\mathrm{QCoh}(X)) \arrow[r] \arrow[d, "\sim"', "\mathrm{tr}_{2\mathcal{C}\mathrm{at}_k}(-\otimes\mathcal{O}_X)"] & HH(\mathrm{QCoh}(Y)) \arrow[d, "\sim"', "\mathrm{tr}_{2\mathcal{C}\mathrm{at}_k}(-\otimes\mathcal{O}_Y)"] \\ & HH(\mathrm{ICoh}(X)) \arrow[r, "\mathrm{tr}_{2\mathcal{C}\mathrm{at}_k}(f_*)"'] & HH(\mathrm{ICoh}(Y)). \end{tikzcd}$$

In order to deduce (1) it is then enough to identify the morphism of traces in the diagram above in the classical terms. The proof splits into 3 parts:

- Identify $HH_0(\mathrm{QCoh}(X))$ with $\bigoplus_p H^p(X, \Omega^p_X)$ and identify $\mathrm{ch}(E, \mathrm{Id}_E)$ under this isomorphism with the classical Chern character.
- Identify $HH_0(\mathrm{ICoh}(X))$ with $\bigoplus_p H^p(X, \Omega^p_X)$ and analogously for Y, and identify $\mathrm{tr}_{2\mathcal{C}\mathrm{at}_k}(f_*)$ with the push-forward in cohomology.
- Identify $\mathrm{tr}_{2\mathcal{C}\mathrm{at}_k}(- \otimes \mathcal{O}_X)$ with the multiplication with the Todd class td_X and analogously for Y.

Our strategy to perform the steps above is to restate trace computations in terms of the geometry of the derided loop space $\mathcal{L}X$. This approach is closely related to the work of Markarian [Mar08], but is more geometric in nature and gives somewhat more general results.

Remark 0.5. One nice feature of this proof is that it avoids Grothendieck's trick of factoring projective morphism into a composition of closed embedding followed by projection and instead works for all *proper* morphisms (in particular we don't use Chow's lemma to reduce a general case to a projective one).

Chern character

Notation 0.6. Let us denote the category of almost finite type derived k-schemes (see [GR17a, Chapter I.2, 3.5] for the definition) by $\mathcal{S}\mathrm{ch}_{\mathrm{aft}}$.

In order to understand the categorical Chern character we first recall some standard facts about quasi-coherent sheaves:

Theorem 0.7 ([GR17a, Chapter I.3, Proposition 3.4.2 and Proposition 2.2.2.]). *The functor*

$$\mathrm{QCoh}^*\colon \mathcal{S}\mathrm{ch}_{\mathrm{aft}}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_k$$

is symmetric monoidal and satisfies the base-change condition.

From computations in the category of correspondences one deduces:

Corollary 0.8. *Let X be a smooth k-scheme. Then*

1. *There is an equivalence*
$$HH(\mathrm{QCoh}(X)) \simeq \Gamma(X^{\mathrm{d\,Id}_X}, \mathcal{O}_{X^{\mathrm{d\,Id}_X}}),$$
where $X^{\mathrm{d\,Id}_X}$ is the derived fixed locus of the identity endomoprhism of X.

2. *Let E be a perfect complex of quasi-coherent sheaves on X equipped with an endomorphism t. Then under the equivalence above the categorical Chern character is equivalent to*
$$\mathrm{tr}_{\mathrm{QCoh}(X^{\mathrm{d\,Id}_X})}\left(i^*E \xrightarrow[\sim]{\alpha_E} i^*E \xrightarrow{i^*(t)} i^*E \right),$$
where $i\colon X^{\mathrm{d\,Id}_X} \to X$ is the canonical injection, and α_E is induced by the two-cell ensuring the commutativity of the diagram below
$$\begin{tikzcd} X^{\mathrm{d\,Id}_X} \arrow[r, "i"] \arrow[d, "i"] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

Key observation of the work is that the derived fixed locus $X^{\mathrm{d\,Id}_X}$ has rich geometry, which is intimately related to the trace constructions, and the ingredients of the Grothendieck-Riemann-Roch theorem can be naturally described in these terms. To see this, note first that $X^{\mathrm{d\,Id}_X}$, i.e. the pullback of the following diagram

$$\begin{tikzcd} \mathcal{L}X \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta"'] & X \times X, \end{tikzcd}$$

is equivalent to the derived loop space $\mathcal{L}X := \mathrm{Map}(S^1, X) \simeq X \times_{X \times X} X$.

The derived loop space has a natural structure of a group scheme over X. Moreover, since $i\colon \mathcal{L}X \to X$ is a nilpotent thickening, $\mathcal{L}X$ is a *formal* group scheme. Now, the formal deformation theory over arbitrary base of characteristic 0 was thoroughly studied in [GR17b]. We recall here the main results relevant to this work:

Theorem 0.9 ([GR17b, Chapter 7, Theorem 3.6.2 and Proposition 5.1.2]). *Let X be a smooth k-scheme and let $\widehat{\mathrm{Moduli}}$ be the category of the formal moduli problems over X (see [GR17b, Chapter 5, Definition 1.1.1]) and let $\mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ denote the category of formal groups over X. Then:*

1. *There is an equivalence of ∞-categories*
$$\mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X}) \xrightarrow[\sim]{\mathrm{Lie}_X} \mathrm{LAlg}(\mathrm{QCoh}(X)),$$
where $\mathrm{LAlg}(\mathrm{QCoh}(X))$ is the $(\infty,1)$-category of algebras in $\mathrm{QCoh}(X)$ over the Lie operad. Moreover, for a formal group $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}}_{/X})$ the underlying quasi-coherent sheaf of $\mathrm{Lie}_X(\widehat{G}) \in \mathrm{LAlg}(\mathrm{QCoh}(X))$ is equivalent to $\mathbb{T}_{\widehat{G}/X,e} := e^*\mathbb{T}_{\widehat{G}/X}$, where $X \xrightarrow{e} \widehat{G}$ is the identity section and $\mathbb{T}$ denotes tangent sheaf.

2. *For $\widehat{G} \in \mathrm{Grp}(\widehat{\mathrm{Moduli}_{/X}})$ there is an equivalence of ∞-categories*

$$\mathrm{Rep}_{\widehat{G}}(\mathrm{QCoh}(X)) \xrightarrow[\sim]{} \mathrm{Mod}_{\mathrm{Lie}_X(\widehat{G})}(\mathrm{QCoh}(X)).$$

3. *Let $\widehat{G}$ be a formal group over X such that the underlying complex of the corresponding Lie algebra $\mathfrak{g} := \mathrm{Lie}_X(\widehat{G})$ lies in $\mathrm{Coh}^{<0}(X)$. Then there is a canonical equivalence (as formal moduli problems over X)*

$$\exp_{\widehat{G}} \colon \mathbb{V}(\mathfrak{g}) \xrightarrow{\sim} \widehat{G},$$

*where $\mathbb{V}(\mathfrak{g})$ is the so-called **vector scheme** defined as $\mathbb{V}(\mathfrak{g}) := \mathrm{Spec}_{/X}(\mathrm{Sym}(\mathfrak{g}^\vee))$.*

Since $\mathrm{Lie}_X(\mathcal{L}X) \simeq \mathbb{T}_X[-1]$ as a first corollary we immediately obtain a description of $HH(\mathrm{QCoh}(X))$:

Corollary 0.10 (Hochschild-Kostant-Rosenberg)**.** *Let X be a smooth scheme over a field k of characteristic 0. Then the restriction along $\exp_{\mathcal{L}X}$ induces an equivalence*

$$\mathrm{HKR} \colon \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) \simeq \bigoplus_{p=0}^{\dim X} \Gamma(X, \Omega^p_X[p]).$$

In order to understand $\mathrm{ch}(E,t)$ more concretely, note that the formal classifying space $\widehat{B}_{/X}$ of $\mathcal{L}X$ is equivalent to $(X \times X)_{\widehat{\Delta}}$. Now, since any quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ is a pullback of $q_2^* E \in \mathrm{QCoh}((X \times X)_{\widehat{\Delta}})$ along the diagonal morphism $\Delta \colon X \to (X \times X)_{\widehat{\Delta}}$, we see that $\mathcal{F}$ acquires a canonical $\mathcal{L}X$-action. We prove:

Proposition 0.11. *Let X be as above and let E be a perfect complex of quasi-coherent sheaves on X. Then the morphism α_E from Corollary 0.8 is given by the canonical $\mathcal{L}X$-action on E.*

To understand $\mathcal{L}X$-action more concretely, we will use the second part of Theorem 0.9 above. To this end we introduce the following definition:

Definition 0.12. The cohomology class $\mathrm{At}(E) \in H^1(X, \Omega^1_X \otimes \mathcal{E}\mathrm{nd}(E))$ dual to the canonical action map $\mathbb{T}_X[-1] \to \mathcal{E}\mathrm{nd}(E)$ is called the **Atiyah class of** E.

Remark 0.13. If k is a field of complex numbers $\mathbb{C}$ and E is a vector bundle this definition gives the same answer as the original one given by Atiyah in [Ati57]. Namely, choose a smooth connection ∇ on E and let $F_\nabla \in \mathcal{E}\mathrm{nd}(E) \otimes \Omega^2_X$ be the corresponding curvature form. Then the curvature F_∇ splits into a sum $F^{2,0}_\nabla + F^{1,1}_\nabla$. One can prove that $F^{1,1}_\nabla$ is a representative of $\mathrm{At}(E)$.

Using this we obtain the following Chern-Weil style description of $\mathrm{ch}(E,t)$:

Proposition 0.14. *Let X be a smooth k-scheme. Then under the HKR-equivalence (Corollary 0.10) we have*

$$\mathrm{ch}(E,t) = \mathrm{tr}_{\mathrm{QCoh}(X)}(e^{\mathrm{At}(E)} \circ t).$$

Finally, if X is proper, essentially by the splitting principle, we can relate this to the classical Chern character:

Proposition 0.15. *Let X be a smooth proper k-scheme and let E be a perfect complex of quasi-coherent sheaves on X equipped with an endomorphism t. Then $\mathrm{ch}(E,t) \in \bigoplus_p H^p(X, \Omega^p_X)$ uniquely characterized by the following properties:*

- $\mathrm{ch}(E,t)$ *is additive in cofiber sequences and commutes with pullbacks.*
- $\mathrm{ch}(E, \lambda t) = \lambda\, \mathrm{ch}(E,t)$ *for any $\lambda \in k$.*
- $\mathrm{ch}(E, \mathrm{Id}_E)$ *coincides with the classical Chern character $\mathrm{ch}(E)$ of E, defined via the splitting principle.*

The pushforward

We now discuss how one can describe the morphism of traces

$$\bigoplus_{p=0} H^p(X, \Omega_X^p) \simeq HH_0(\mathrm{QCoh}(X)) \xrightarrow{\mathrm{tr}_{2\mathfrak{Cat}_k}(f_*, \mathrm{Id}_{f_*})} HH_0(\mathrm{QCoh}(Y)) \simeq \bigoplus_{p=0} H^p(Y, \Omega_X^Y)$$

obtained by applying 0.1 to the commutative diagram

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[r, "\mathrm{Id}_{\mathrm{QCoh}(X)}"] \arrow[d, bend right, "f_*"'] & \mathrm{QCoh}(X) \arrow[d, bend right, "f_*"'] \\
\mathrm{QCoh}(Y) \arrow[r, "\mathrm{Id}_{\mathrm{QCoh}(Y)}"'] \arrow[u, bend right, "f^!"'] & \mathrm{QCoh}(Y) \arrow[u, bend right, "f^!"']
\end{tikzcd}$$

in more familiar terms. Note that due to the post factum knowledge that the answer should involve the Todd class, one should not expect to obtain this description in a purely formal way.

To circumvent this difficulty, recall that there is another important ∞-category one can associate to X, namely, the ∞-category $\mathrm{ICoh}(X) \in 2\mathfrak{Cat}_k$ of ind-coherent sheaves on X (see [GR17a, Part II], and [Gai13]). For a smooth classical scheme X as a plain category the category of ind-coherent sheaves $\mathrm{ICoh}(X)$ is equivalent to the category of quasi-coherent sheaves $\mathrm{QCoh}(X)$, but has a different symmetric monoidal structure. Similar to quasi-coherent sheaves, ind-coherent sheaves admit many nice properties:

Theorem 0.16 ([GR17a, Chapter II.1, Proposition 6.3.4 and Chapter II.2, Theorem 4.2.5])**.** *The functor*

$$\mathrm{ICoh}^! \colon \mathcal{S}\mathrm{ch}_{\mathrm{aft}}^{\mathrm{op}} \longrightarrow 2\mathfrak{Cat}_k$$

is symmetric monoidal and satisfies the base-change condition with respect to the proper morphisms.

It follows:

Corollary 0.17. *Let X be a proper k-scheme. There is a natural equivalence*

$$HH(\mathrm{ICoh}(X)) \simeq \Gamma(\mathcal{L}X, \omega_{\mathcal{L}X}).$$

Moreover, if additionally X is smooth, then

$$\Gamma(\mathcal{L}X, \omega_{\mathcal{L}X}) \overset{\mathrm{HKR}^\vee}{\simeq} \left(\bigoplus_p H^p(X, \Omega_X^p[p]) \right)^\vee.$$

By formal manipulation in the category of correspondences we then prove:

Proposition 0.18. *Let $f \colon X \to Y$ be a morphism between proper k-schemes. Then the corresponding morphism of traces*

$$\Gamma(\mathcal{L}X, \omega_{\mathcal{L}X}) \simeq HH(\mathrm{ICoh}(X)) \xrightarrow{\mathrm{tr}(f_*, \mathrm{Id}_{f_*})} HH(\mathrm{ICoh}(X)) \simeq \Gamma(\mathcal{L}Y, \omega_{\mathcal{L}Y})$$

induced by the diagram

$$\begin{tikzcd}
\mathrm{ICoh}(X) \arrow[r, "\mathrm{Id}_{\mathrm{ICoh}(X)}"] \arrow[d, bend right, "f_*"'] & \mathrm{ICoh}(X) \arrow[d, bend right, "f_*"'] \\
\mathrm{ICoh}(Y) \arrow[r, "\mathrm{Id}_{\mathrm{ICoh}(Y)}"'] \arrow[u, bend right, "f^!"'] & \mathrm{ICoh}(Y) \arrow[u, bend right, "f^!"']
\end{tikzcd}$$

in $2\mathfrak{Cat}_k$ coincides with the morphism, obtained by applying the global section functor $\Gamma(\mathcal{L}Y, -) \colon \mathrm{ICoh}(\mathcal{L}Y) \to \mathrm{Vect}_k$ to the counit $\mathcal{L}f_*\omega_{\mathcal{L}X} \to \omega_{\mathcal{L}Y}$ of adjunction $\mathcal{L}f_* \dashv \mathcal{L}f^!$.

In particular, if X and Y are smooth, then under the HRK-equivalence the induced morphism

$$\left(\bigoplus_{p=0}^{\dim X} H^p(X,\Omega_X^p)\right)^\vee \longrightarrow \left(\bigoplus_{p=0}^{\dim X} H^p(Y,\Omega_Y^p)\right)^\vee$$

is given under the Poincaré duality by the pushforward in homology.

The Todd class

It is left to understand the morphism of traces

$$\Gamma(\mathcal{L}X,\mathcal{O}_{\mathcal{L}X}) \xrightarrow[\sim]{\mathrm{tr}_{2\mathcal{C}\mathrm{at}_k}(-\otimes\mathcal{O}_X,\mathrm{Id}_{-\otimes\mathcal{O}_X})} \Gamma(\mathcal{L}X,\omega_{\mathcal{L}X}) \tag{2}$$

induced by the diagram

$$\begin{tikzcd}
\mathrm{QCoh}(X) \arrow[r, "\mathrm{Id}_{\mathrm{QCoh}(X)}"] \arrow[d, shift right, "-\otimes\mathcal{O}_X"'] & \mathrm{QCoh}(X) \arrow[d, shift right, "-\otimes\mathcal{O}_X"'] \\
\mathrm{ICoh}(X) \arrow[r, "\mathrm{Id}_{\mathrm{ICoh}(X)}"'] \arrow[u, shift right, "-\otimes\omega_X"'] & \mathrm{ICoh}(X) \arrow[u, shift right, "-\otimes\omega_X"']
\end{tikzcd}$$

in $2\mathcal{C}\mathrm{at}_k$. To this end we introduce the following notion:

Definition 0.19. Let Z be a derived almost finite type scheme. An **orientation** on Z is a choice of an equivalence $u\colon \mathcal{O}_Z \simeq \omega_Z$.

As an example, using Serre duality one can obtain a **Serre** orientation $\mathcal{O}_{\mathcal{L}X} \overset{S}{\simeq} \omega_{\mathcal{L}X}$ on $\mathcal{L}X$, characterized by the property that the induced map on the space of global sections

$$\bigoplus_{p=0}^{\dim X} H^p(X,\Omega_X^p) \xrightarrow[\sim]{\mathrm{HKR}} \Gamma(\mathcal{L}X,\mathcal{O}_{\mathcal{L}X}) \xrightarrow[\sim]{S} \Gamma(\mathcal{L}X,\omega_{\mathcal{L}X}) \xrightarrow[\sim]{\mathrm{HKR}^\vee} \left(\bigoplus_{p=0}^{\dim X} H^p(X,\Omega_X^p)\right)^\vee$$

sends a form η to the functional $\int_X - \wedge\, \eta$, i.e. coincides with the usual Poincaré duality. Note that using the isomorphism $\mathcal{O}_{\mathcal{L}X} \overset{S}{\simeq} \omega_{\mathcal{L}X}$ any other orientation corresponds to a unique class in $\bigoplus_p H^p(X,\Omega_X^p)$.

A less known orientation on $\mathcal{L}X$ can be constructed as follows: for any endomorphism $X \xrightarrow{g} X$ the series of equivalences

$$\mathcal{O}_{X^g} \simeq i^*\omega_X \otimes i^*\omega_X^{-1} \simeq i^*\omega_X \otimes i^*\omega_{X/X\times X} \simeq i^*\omega_X \otimes \omega_{X^g/X} \simeq i^!\omega_X \simeq \omega_{X^g}$$

induces an orientation on X^g which we will further call **canonical**. By formal but tedious manipulations in the category of correspondences one can show:

Lemma 0.20. *The morphism of traces (2) can be obtained by applying the global sections functor to the canonical orientation on $\mathcal{L}X = X^{\mathrm{Id}_X}$.*

The final goal will be to prove that under the Serre orientation the canonical orientation corresponds to the classical Todd class td_X. In order to prove this, we first give an interpretation of any multiplicative characteristic class in terms of the canonical action of $\mathbb{T}_X[-1] \in \mathrm{LAlg}(\mathrm{QCoh}(X))$. Namely for a Lie algebra $\mathfrak{g} \in \mathrm{LAlg}(\mathrm{QCoh}(X))$ such that $\mathfrak{g} \in \mathrm{QCoh}(X)^{<0}$ let $\mathbb{V}(\mathfrak{g}) := \mathrm{Spec}_{/X}\,\mathrm{Sym}(\mathfrak{g}^\vee)$ be the corresponding vector scheme with the canonical map $i\colon \mathbb{V}(\mathfrak{g}) \to X$. Then for a representation $\mathfrak{g} \xrightarrow{\rho} \mathcal{E}\mathrm{nd}_{\mathrm{QCoh}(X)}(E)$ and a formal power series $f \in k[[t]]$ one can construct a **formal $\mathcal{E}\mathrm{nd}(E)$-valued function** $f(\rho) \in \mathcal{E}\mathrm{nd}_{\mathrm{QCoh}(\mathbb{V}(\mathfrak{g}))}(i^*E)$ informally given by sending $x \in \mathfrak{g}$ to $f(\rho(x))$. We prove:

Proposition 0.21. *The multiplicative characteristic class $c^f(E)$ constructed from f via the splitting principle is equal to $\det(f(\mathrm{ad}_{\mathbb{T}_X[-1]}))$.*

As a corollary, we get
$$\mathrm{td}_X = \mathrm{td}_X(\mathbb{T}_X) = \mathrm{td}_X(\mathbb{T}_X[-1])^{-1} = \frac{1-e^{-\mathrm{ad}_{\mathbb{T}_X[-1]}}}{\mathrm{ad}_{\mathbb{T}_X[-1]}}.$$
The rest of the proof is motivated by the following observation:

Example 0.22. Let G be a real Lie group with the corresponding Lie algebra $\mathfrak{g}$. In a small enough neighborhood of 0 we then have two trivializations of $\mathbb{T}_G$ induced by the group structure on G and the ablian group structure on $\mathfrak{g}$ (vie the exponential map $\exp_G\colon \mathfrak{g}\to G$). For $x\in\mathfrak{g}$ (close enough to 0) the change of trivialization isomorphism
$$\mathfrak{g} \xrightarrow[\sim]{+x} \mathbb{T}_{\mathfrak{g},x} \xrightarrow[\sim]{(d\exp_G)_x} \mathbb{T}_{G,\exp(x)} \xrightarrow[\sim]{L_{\exp_G(x)^{-1}*}} \mathfrak{g}\ ,$$
(where $L_g\colon G\to G$ denotes the left translation by g map) is given by $(1-e^{-\mathrm{ad}(x)})/\,\mathrm{ad}(x)$.

Motivated by this example we prove that the formal function $f(\mathrm{ad}_{\mathbb{T}_X[-1]})$ for $f(t)=(1-e^{-t})/t$ coincides with the change of trivializations morphism $d\exp_{\mathcal{L}X}\colon i^*\mathbb{T}_X\simeq i^*\mathbb{T}_X$. This question makes sense for arbitrary Lie algebra in any presentably symmetric monoidal k-linear category. Using high functoriality of the morphism in question one then reduces the statement to the category of k-vector spaces and free discrete Lie algebras, where this is easy to prove by hands.

Finally, to relate this story to the morphism of traces, we prove that the determinant of the trivialization $\mathbb{T}_{\mathcal{L}X}\simeq i^*\mathbb{T}_X$ defined by its groups structure is equal to the canonical orientation and that the determinant of the analogues trivialization using abelian group structure coincides with the Serre orientation.

Chapter 3

In the last chapter we deduce an equivariant Grothendieck-Riemann-Roch theorem. Our strategy is to reduce the equivariant statement to the non-equivariant one. In order to achieve this we prove the following formality criterion interesting in its own right:

Theorem 0.23. *Let X be a smooth proper scheme over k equipped with an endomorphism g such that the classical fixed locus X^g is smooth. Then the following two statements are equivalent:*

- *The induced map $1-dg_{|\mathcal{N}^\vee_{X^g/X}}$ is invertible, where $\mathcal{N}^\vee_{X^g/X}$ is the conormal bundle of X^g in X.*
- *The natural map $\mathcal{L}X^g\to X^{\mathrm{d}g}$, induced by the g-equivariant inclusion $X^g\hookrightarrow X$, is an equivalence.*

Remark 0.24. A similar formality result for automorphisms g of finite order (in which case $1-dg_{|\mathcal{N}^\vee_{X^g/X}}$ is automatically invertible) was proved in [ACH19, Corollary 1.12] by other methods.

Remark 0.25. This theorem is partially motivated by a similar result in the equivariant cohomology from [AB84].

In particular the pullback along $j\colon \mathcal{L}X \xrightarrow{\sim} X^{\mathrm{d}g}$ identifies
$$HH(\mathrm{QCoh}(X),g^*)\simeq \Gamma(X^{\mathrm{d}g},\mathcal{O}_{X^{\mathrm{d}g}})\simeq \Gamma(\mathcal{L}X^g,\mathcal{O}_{\mathcal{L}X^g})\simeq \bigoplus_p \Gamma(X^g,\Omega^p_{X^g}[p]).$$
Combining this with Proposition 0.14 we obtain a concrete description of the Chern character $\mathrm{ch}(E,t)$.

To state our main application we need to introduce the following definition:

Definition 0.26. Let X,g be as above. Define **equivariant Euler class** $e_g\in\Gamma(\mathcal{L}X^g,\mathcal{O}_{\mathcal{L}X^g})$ to be $j^*j_*(1)$. Unwinding the definition, we find that
$$e_g = \mathrm{ch}\left(\mathrm{Sym}(\mathcal{N}^\vee_{X^g/X}[1]),\mathrm{Sym}(dg_{|\mathcal{N}^\vee_{X^g/X}}[1])\right).$$
In particular $e_0=\det(1-dg_{|\mathcal{N}^\vee_{X^g/X}})$ and hence e_g is invertible by our assumption on X,g.

Then by the functoriality of traces and the non-equivariant case it is not hard to deduce:

Theorem 0.27 (Equivariant Grothendieck-Riemann-Roch). *Let*

$$\begin{tikzcd} X \arrow[loop left, "g_X"] \arrow[r, "f"] & Y \arrow[loop right, "g_Y"] \end{tikzcd}$$

be an equivariant morphism between smooth proper k-schemes such that

- *Reduced fixed loci X^{g_X} and Y^{g_Y} are smooth.*
- *The induced morphisms on conormal bundles $1-(dg_X)_{|\mathcal{N}^\vee_{g_X}}$ and $1-(dg_Y)_{|\mathcal{N}^\vee_{g_Y}}$ are invertible.*

Then for a lax g_X-equivariant perfect sheaf E on X there is an equality

$$(f^g)_*\left(\mathrm{ch}(E,t)\frac{\mathrm{td}_{X^{g_X}}}{e_{g_X}}\right) = \mathrm{ch}(f_*(E,t))\frac{\mathrm{td}_{Y^{g_Y}}}{e_{g_Y}} \quad \in \quad \bigoplus_p H^{p,p}(Y^{g_Y}),$$

where $X^{g_X} \xrightarrow{f^g} Y^{g_Y}$ is the induced map on reduced fixed loci.

If endomorphisms g_X and g_Y are equal to identity, then $e_{g_X} = 1$ and $e_{g_Y} = 1$ and we recover the usual Grothendieck-Riemann-Roch theorem. In the other extreme case, when Y is a point and the graph of g_X intersects with the diagonal in $X \times X$ transversely, one recovers the holomorphic Atiyah-Bott formula.

Approbation

The results of the dissertation were presented at:

1. A talk "Higher traces and fixed point theorems", on the seminar "Introduction to automorphic forms" (Skoltech), November 13, 2019.
2. A talk "Derived loop space and Riemann-Roch like theorems", on the seminar "Algebro-geometric methods in integrable systems and quantum physics" (MIPT), December, 19, 2019.

Publications

Main results of the dissertation are published in [KP18] and [Pri19].

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