Formal solution of $\hbar$– KP hierarchy

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**Tau-function**

Hereafter we work with the $\hbar-$ KP hierarchy from K. Takasaki and T. Takebe, *Integrable hierarchies and dispersionless limit*, Rev. Math.Phys. 7 (1995) 743-808. Tau-function of the hierarchy is a function $\tau = \tau(t)$ depended on the infinite set of time variables $t = \{t_1, t_2, \ldots\}$. There is also contains $\hbar$ as a parameter but we will not write it explicitly. Below we use the notation

$$\tau[z_1, \ldots, z_m](t) = e^{\hbar(D(z_1)+\ldots+D(z_m))} \tau,$$

where

$$D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k; \quad \partial_k = \partial / \partial t_k. \quad (1)$$
**Hirota relation**

Let $\hbar > 0$. A function $\tau$ is called $\tau$-function of $\hbar$– KP hierarchy if for any $z_1, z_2, z_3$ it satisfies $\hbar$–Hirota functional relation.

$$(z_1 - z_2)\tau[z_1, z_2]\tau[z_3] + (z_2 - z_3)\tau[z_2, z_3]\tau[z_1] + (z_3 - z_1)\tau[z_3, z_1]\tau[z_2] = 0. \quad (2)$$

For $\hbar = 1$ it gives the ordinary Hirota relation.

**Theorem**

1. A function $\tau$ is $\tau$-function of $\hbar$– KP hierarchy if and only if for any $z_1, z_2$

   $$\hbar \partial_1 \log \frac{\tau[z_1]}{\tau[z_2]} = (z_2 - z_1) \left( \frac{\tau[z_1, z_2]}{\tau[z_1] \tau[z_2]} - 1 \right). \quad (3)$$

2. A function $\tau$ is $\tau$-function of $\hbar$– KP hierarchy if and only if

   $$\prod_{1 \leq i < j \leq m} (z_j - z_i) \cdot \tau[z_1, \ldots, z_m]^{-m-1} = \det_{1 \leq j, k \leq m} \left( (z_j - \hbar \partial_1)^{k-1} \tau[z_j] \right) \quad (4)$$

for any $m \geq 2$ and any $z_1, \ldots, z_m$. 

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http://arxiv.org/1509.04472
Schur polynomials

For description of $\tau$-function we use Schur polynomials. An elementary Schur polynomial $h_k(t)$ depends from natural number $k$ and infinite number of variable $t = (t_1, t_2, \ldots)$. It is defined by the generating series

$$\exp\left(\sum_{k \geq 1} t_k z^k\right) = \sum_{k \geq 0} h_k(t)z^k.$$ 

An general Schur polynomial $s_\lambda(t)$ depends from a Young diagram $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ of degree $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ with a $\ell = \ell(\lambda) \geq 0$ rows of positive lengths $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$.

The general Schur polynomial is the determinant

$$s_\lambda(t) = \det_{i,j=1,\ldots,\ell(\lambda)} h_{\lambda_i-i+j}(t).$$
Formal tau-function

Let us introduce a deformation $\partial_k^\hbar$ of the $\partial_k = \partial/\partial t_k$

$$\partial_k^\hbar = \sum_{l=1}^{k} \frac{\hbar^{l-1}k}{l!} \sum_{l_1, \ldots, l_l \geq 1 \atop k_1 + \ldots + k_l = k} \frac{\partial_{k_1} \ldots \partial_{k_l}}{k_1 \ldots k_l} = \partial_k + \hbar \sum_{l=1}^{k-1} \frac{k\partial_l \partial_{k-l}}{2l(k - l)} + O(\hbar^2). \tag{5}$$

In the KP theory the first variable $t_1$ is distinguished. Thus we will consider $\tau$-function as evolution from a function $\tau(x, 0)$ of one variable by $\hbar$- KP flows

$$\tau(x, 0) \rightarrow \tau(x, t) = f(x)\hat{\tau}(x + t_1, t_2, t_3, \ldots).$$

A formal $\tau$-function of $\hbar$– KP hierarchy we mean a formal series for a function $\tau(x, t)$ of this type, that satisfy the Hirota equation by $t$ for any $x$.

Our first goal is a construction of formal $\tau$-functions of $\hbar$– KP hierarchy from $f$, $\tau(x; 0)$, and Cauchy-like data

$$\partial_k^\hbar \tau(x; t) \bigg|_{t=0}.$$
**Theorem**

Let $\hbar \neq 0$, and $f(x)$, $c_k(x)$, $k = 0, 2, 3, \ldots$, be arbitrary infinitely differentiable functions of $x$ (with $c_0(x)$ being not identically 0). Put $c_0(x) = c_0(x)$ and

$$
c_\lambda(x) = (c_0(x))^{1-\ell(\lambda)} \det_{1 \leq i, j \leq \ell(\lambda)} \left[ \sum_{k=0}^{j-1} (-\hbar)^k C_{j-1}^k \partial_x^k c_{\lambda,-i+j-k}(x) \right]$$

(6)

for Young diagram $\lambda \neq \emptyset$. Then the series

$$
\tau(x; t) = \sum_{\lambda} c_\lambda(x) s_\lambda(t/\hbar)
$$

(7)

is a formal solution to the $\hbar$-KP hierarchy, where

$$
\tau(x; 0) = c_0(x), \quad \partial_k^\hbar \tau(x; t) \bigg|_{t=0} = \frac{k}{\hbar} c_k(x), \quad k \geq 1
$$

and $c_1 = \partial_x c_0 - c_0 \partial_x \log f$. 
$\hbar$— KP hierarchy

For many applications in physics and mathematics one needs to deal with logarithm of the tau-function rather than with the tau-function itself. Let us put

$$F(x; t) = \hbar^2 \log \tau(x; t). \quad (8)$$

Then the Hirota equations on $\tau(x; t)$ go to $\hbar$— KP hierarchy on $F(x; t)$. That is

$$e^{\Delta(z_1)\Delta(z_2)F} = 1 - \frac{\Delta(z_1)\partial_x F - \Delta(z_2)\partial_x F}{z_1 - z_2}, \quad (9)$$

where

$$\Delta(z) = \frac{e^{\hbar D(z)} - 1}{\hbar}. \quad (10)$$

For $\hbar = 1$ this is the ordinary KP hierarchy.
For $\hbar = 0$ this is the dispersionless KP hierarchy.

Our goal is a formula, expressing any solution of $\hbar$— KP hierarchy by 

*Cauchy-like data* $f_k(x) = \partial^\hbar_k F(x; t_1, t_2, \ldots) \bigg|_{t=0}$. 


The $\hbar$-KP hierarchy in terms of $\partial_k^\hbar$

Define some combinatorial constants $\tilde{P}_{ij}(s_1, \ldots, s_m)$ as the number of sequences of positive integers $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_m)$ such that $i_1 + \ldots + i_m = i$, $j_1 + \ldots + j_m = j$ and $s_k = i_k + j_k - 1$. Put

$$P_{ij}(s_1, \ldots, s_m) = \frac{(-1)^{m+1} ij}{m s_1 \ldots s_m} \tilde{P}_{ij}(s_1, \ldots, s_m).$$

Theorem

The $\hbar$-KP hierarchy is equivalent to the system of equations

$$\partial_i^\hbar \partial_j^\hbar F = \sum_{m \geq 1} \sum_{s_1, \ldots, s_m \geq 1 \atop s_1 + \ldots + s_m = i + j - m} P_{ij}(s_1, \ldots, s_m) \partial \partial_{s_1}^\hbar F \ldots \partial \partial_{s_m}^\hbar F,$$

for the function $F = F(x; t)$, where $\partial = \partial_1$. 
Combinatorial constants

Let $K_l(l^1, \ldots, l^r)$ be the number of partitions of a set of $l$ elements into ordered groups of $l^1, \ldots, l^r > 0$ elements.

Define the constants $P_{i_1 \ldots i_k}^{\vec{n}} \left( \begin{array}{c} s_1 \ldots s_m \\ l_1 \ldots l_m \end{array} \right)$ from integer positive $m, \{i_r\}, \{s_r\}, \{l_r\}$ by the following recurrence relations:

1) $P_{i_1, i_2}^{\vec{n}} \left( \begin{array}{c} s_1 \ldots s_m \\ 1 \ldots 1 \end{array} \right) = P_{i_1 i_2}(s_1, \ldots, s_m)$ and $P_{i_1, i_2}^{\vec{n}} \left( \begin{array}{c} s_1 \ldots s_m \\ l_1 \ldots l_m \end{array} \right) = 0$, if $\prod_{j=1}^{m} l_j > 1$.

2) $P_{i_1 \ldots i_r}^{\vec{n}} \left( \begin{array}{c} x_1 \ldots x_v \\ y_1 \ldots y_v \end{array} \right) = \sum P_{i_1 \ldots i_r-1}^{\vec{n}} \left( \begin{array}{c} s_1 \ldots s_m \\ l_1 \ldots l_m \end{array} \right) \frac{\vec{n}^{\nu(k_1, \ldots, k_m)-1} i_r}{[k_1 \ldots k_m]} \times \left( \begin{array}{c} x_1 \ldots x_v \\ y_1 \ldots y_v \end{array} \right) - \prod_{n=1}^{m} P_{s_n k_n}^{\vec{n}}(s_1^1 \ldots s_{n_1}^1) \ldots K_{l_m}^{n_m} P_{s_n k_n}^{\vec{n}}(s_1^1 \ldots s_{n_m}^1),$

where $\nu(k_1, \ldots, k_n)$ is the number of positive numbers between $k_i$ and $[k_1, \ldots, k_n] = \prod_{i=1}^{n} \max\{k_i, 1\}$.
The summation is carried over all sets of integer numbers such that

\[ (x_1 \ldots x_v) = (s^1_1, \ldots s^1_{n_1}, s^2_1, \ldots, s^2_{n_2}, \ldots, s^m_1, \ldots, s^m_{n_m}), \quad s_i = \sum_{j=1}^{n_i} s^i_j, \]

\[ (y_1 \ldots y_v) = (l^1_{n_1} + 1, \ldots, l^1_{n_1} + 1, l^2_{n_2} + 1, \ldots, l^2_{n_2} + 1, \ldots, l^m_{n_m} + 1, \ldots, l^m_{n_m} + 1), \]

\[ l_i = \sum_{j=1}^{n_i} l^i_j; \sum_{i=1}^{m} (s_i + l_i) = \sum_{j=1}^{r-1} i_j, \quad \sum_{j=1}^{m} k_i = i_r, \quad \sum_{i=1}^{n_j} s^j_i = k_j + s_j. \]

**Theorem**

The $\hbar$-KP hierarchy is equivalent to the system of equations for $r \geq 2$

\[ \partial^{\hbar}_{i_1} \ldots \partial^{\hbar}_{i_r} F = \sum_{m \geq 1} \sum_{s_1 + l_1 + \ldots + s_m + l_m = i_1 + \ldots + i_r}^{r-1} P^{\hbar}_{i_1 \ldots i_r} \left( \begin{array}{c} s_1 \ldots s_m \\ l_1 \ldots l_m \end{array} \right) \partial^{1} \partial^{\hbar}_{s_1} F \ldots \partial^{m} \partial^{\hbar}_{s_m} F \]

\[ \text{for } 1 \leq s_i; 1 \leq l_i \leq r - 1. \]
Operators $\partial^\hbar_\lambda$ and corresponding variables $t^\hbar_\lambda$

A Young diagram $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ with rows of lengths $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ is described by $r_i = \text{card}\{j|\lambda_j = i\}$. Denote by $\rho(\lambda) = \lambda_1 \lambda_2 \ldots \lambda_\ell$ and $\sigma(\lambda) = \prod_{n \geq 1} r_n!$. We put also $\partial^\hbar_\lambda = \partial^\hbar_\lambda_1 \partial^\hbar_\lambda_2, \ldots, \partial^\hbar_\lambda_\ell$.

Let us consider the basis

$$m_\lambda(x_1, x_2, \ldots, x_n) = \frac{1}{(n - \ell(\lambda))! \sigma(\lambda)} \sum_{s \in S_n} x_1^{\lambda_s(1)} x_2^{\lambda_s(2)} \ldots x_n^{\lambda_s(n)}$$

in the space of symmetrical polynomials from $x_1, x_2, \ldots$.

These polynomials are linear combinations from symmetrical polynomials $t_k = \frac{1}{k} \sum_i x_i^k$. This gives new polynomials $m_\lambda(t)$ from $t = (t_1, t_2, \ldots)$. The first few functions $m_\lambda$ are:

$$m_{(1)}(t) = t_1,$$

$$m_{(2)}(t) = 2t_2, \quad m_{(1^2)}(t) = \frac{1}{2} t_1^2 - t_2,$$

$$m_{(3)}(t) = 3t_3, \quad m_{(12)}(t) = 2t_2 t_1 - 3t_3, \quad m_{(1^3)}(t) = \frac{1}{6} t_1^3 - t_2 t_1 + t_3. \quad (13)$$
Let us put
\[
    t^h_\lambda := \frac{\sigma(\lambda)}{\rho(\lambda)} \hbar^{l(\lambda)} m_\lambda(t/\hbar).
\]  

The first few \( t^h_\lambda \) for \( (\lambda) = (1^r_1 2^r_2, \ldots) \) are:

\[
    \begin{align*}
    t^h_{(1)} &= t_1, \\
    t^h_{(2)} &= t_2, \quad t^h_{(1^2)} = t_1^2 - 2\hbar t_2, \\
    t^h_{(3)} &= t_3, \quad t^h_{(2^1)} = t_2 t_1 - \frac{3}{2} \hbar t_3, \quad t^h_{(1^3)} = t_1^3 - 6\hbar t_2 t_1 + 6\hbar^2 t_3.
    \end{align*}
\]  

**Theorem**

*Any formal series \( F(t) = F(t_1, t_2, \ldots) \) has a representation in form of formal series*

\[
    F(t) = \sum_{\lambda} \partial^h_\lambda F(t') \bigg|_{t' = 0} \frac{t^h_\lambda}{\sigma(\lambda)},
\]  

\[ (16) \]
Construction of formal solutions

Consider now any family of infinitely differentiable functions

\[ f^{\hat{h}}_{[k]}(x) = f^{\hat{h}}_k(x) \quad (k = 1, 2, \ldots). \]

For other Young diagrams \( \lambda \) we put

\[ f^{\hat{h}}_{\lambda}(x) = \sum_{m \geq 1} \sum_{s_1 + l_1 + \ldots + s_m + l_m = |\lambda|}^{1 \leq s_i; \ 1 \leq l_i \leq \ell(\lambda) - 1} P^{\hat{h}}_{\lambda} \left( \begin{array}{c} s_1 \ldots s_m \\ l_1 \ldots l_m \end{array} \right) \partial^{l_1} f_{s_1}(x) \ldots \partial^{l_m} f_{s_m}(x), \]

(17)

where

\[ P^{\hat{h}}_{\lambda} \left( \begin{array}{c} s_1 \ldots s_m \\ l_1 \ldots l_m \end{array} \right) = P^{\hat{h}}_{\lambda_1 \ldots \lambda_r} \left( \begin{array}{c} s_1 \ldots s_m \\ l_1 \ldots l_m \end{array} \right) \]

for \( \lambda = [\lambda_1, \ldots, \lambda_\ell] \).
Theorem

For any $\hbar$ and any family of smooth or formal functions

$$f = \{f_0(x), f_1(x), f_2(x), \ldots\}$$

there exists a unique solution $F(x; t)$ of the $\hbar$-KP hierarchy such that

$$F(x; 0) = f_0(x) \quad \text{and} \quad \partial^\hbar_k F(x; t_1, t_2, \ldots) \bigg|_{t=0} = f_k(x).$$

This solution has the form

$$F(x; t) = f_0(x) + \sum_{|\lambda| \geq 1} \frac{f_\lambda^\hbar(x)}{\sigma(\lambda)} t^\hbar_\lambda. \quad (18)$$