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MATRIX-VECTOR APPROACH TO CONSTRUCT GENERALIZED CENTRALITY INDICES IN NETWORKS

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We introduce a class of new centrality indices in networks which take into account a parameter of a node, direct and indirect (with fixed length of the path) connections of nodes, and a group influence of nodes to a node. Total influence is evaluated as a linear combination of its components, however, other rules can be used as well.

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1. Introduction

In the large stream of works on network analysis one of the branches deals with the centrality indices. Until recently classic centrality measures disregard very important issues in network analysis. First, parameters of nodes have not been taken into account, though in the fundamental work (Newman, 2003) it was stated in an explicit way. Indeed, assume that we borrow $1 mln from some large bank, but cannot return the debt in time. Naturally, the bank will not be happy, but if it is really large bank, it will not be crushed. If the bank is small, it might lead to bankruptcy. This example shows that such parameter of banks as total assets should be taken into account.

Second, suppose Mrs. A and B borrow from a small bank $500 thous each. If any of them do not return the debt in time, the bank will survive. However, if both of them will not return debt, the bank will announce bankruptcy. This shows how important might be the influence of a group of nodes to the node in a network.

Third, sequential borrowing of one bank from another bank might create the situation when the bank at the end of borrower's path can start the falling domino reaction process and finally destroy the first bank of the path. This last shortage of the classic indices is taken into account partly by such indices as eigenvector, PageRank and few others.

To overcome the abovementioned shortages new indices were proposed in (Aleskerov et al., 2014; 2016; 2017), so-called Short- and Long-Range Interaction Centralities (SRIC and LRIC). However, to evaluate these indices it is necessary to reconstruct the graph in a very specific way leading to rather artificial constructions hardly explained in an explicit form. To escape this problem we propose new classes of centrality indices in networks.

The structure of the text is as follows. In the next Section 2 we propose new centrality indices. In Sections 3 and 4 we discuss some simple examples to explain new indices construction. Then in Section 5 we consider another way to construct a total influence of nodes.

2. New Centrality Indices

Let $G^0 = (V, W^0)$ be a weighted directed graph, where V be a set of vertices, $|V| = n$, and W^0 be a set of edges with weights w^0_{ji}. For each vertex $i \in V$ the quota q_i is defined. The maximum number of vertices which can simultaneously influence a node is defined as k.

a) Copeland in-degree index CI^0 on the initial graph G^0

Copeland in-degree index $CI^0 = (CI^0_i)$ for each vertex i is defined as the sum of weights w^0_{ji} of the incoming edges from connected vertices j, i.e. $CI^0(i) = \sum_j w^0_{ji}$.

Consider an example which we extensively use below. Let us consider the network shown on Fig. 1.

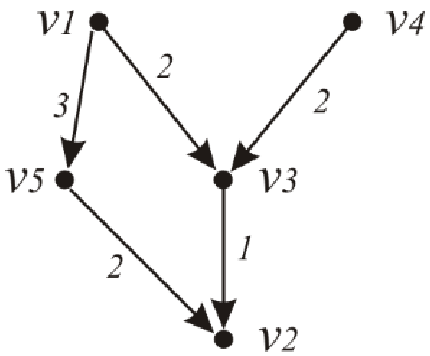

Fig. 1

The adjacency matrix is given in Table 1.

Table 1. Adjacency matrix W^0

	v1	v2	v3	v4	v5
v1	0	0	2	0	3
v2	0	0	0	0	0
v3	0	1	0	0	0
v4	0	0	2	0	0
v5	0	2	0	0	0

Then we obtain the following values for CI_i^0 (Table 2).

Table 2. CI^0-index

	v1	v2	v3	v4	v5
CI^0	0	3	4	0	3

b) Bundle index $BI^0 = (BI_i^0)$

To take into account the parameters of the vertices we first define for each vertex $i \in V$ the quota q_i. We define as well the notion of group influence of vertices to one vertex. For this we introduce the maximum number k of vertices which can simultaneously influence a node, i.e., the cardinality of the set influencing the node should not exceed k. We call these sets critical groups or critical sets.

Then $BI^0 = (BI_i^0)$ index is constructed as follows. For each set of vertices

$$S \subseteq V \backslash \{i\},\ |S| \leq k,\ \forall j \in S, w_{ji}^0 \neq 0,$$

the sum of weights $\sum_{j \in S} w_{ji}^0$ of incoming edges from node $j \in S$ to node i is compared to the quota q_i, and the value $BI_i^0(S)$ is calculated as

$$BI_i^0(S) = \begin{cases} 1, if \sum_{j \in S} w_{ji}^0 \geq q_i\ , \\ 0, else. \end{cases}$$

So, the $BI_i^0(S)$ is equal to 1, if the sum is not less than the quota, and it is equal to 0, otherwise. The $BI^0(i)$ index for the vertex i is defined as the sum of the $BI_i^0(S)$ on all considered subsets S, i.e.

$$BI^0(i) = \sum_S BI_i^0(S).$$

Thus, for each node $i \in V$, the value of the Bundle index $BI^0(i)$ is equal to the number of the subsets of incoming edges from not more than k vertices, with the sum of weights not less than the quota q_i.

Let us in our example go through all sets influencing vertices in the network. If q = 1, then for the vertex v_2 the following critical groups influence this vertex

– $\{v_3\}$ with the sum of weights being equal to 1,

– $\{v_5\}$ with the sum of weights 2, and

– $\{v_3, v_5\}$ with the sum of weights 3.

Then $BI^0(v_2)$ = 3. Similarly, $BI^0(v_3)$ = 3, and $BI^0(v_5)$ = 1. Since no critical group dominates v_1 and v_4, then $BI^0(v_1)$= $BI^0(v_4)$ = 0. The values of $BI^0(i)$ are given in Table 3.

Table 3. BI^0-index

	v1	v2	v3	v4	v5
$BI^0(i)$	0	3	3	0	1

c) Pivotal index $PI^0 = (PI_i^0)$

The Pivotal index $PI^0 = (PI_i^0)$ differs from the BI^0-index in the following way. Instead of the number of sets of edges, the number of pivotal nodes is calculated. The node $j_p \in S$ is called pivotal for the node $i \in V$ in the set $S \subseteq V\backslash\{i\}$ with the quota q_i, if

$$\sum_{j\in S} w_{ji}^0 \geq q_i, \text{ but } \sum_{j\in S\backslash\{j_p\}} w_{ji}^0 < q_i,$$

i.e. the sum of the weights of the incoming edges to node $i \in V$ from nodes of the set $S \subseteq V\backslash\{i\}$, is greater or equal to the quota q_i, but upon excluding the node j_p from the set S, the sum of the weights becomes less than the quota.

The PI^0 index is constructed as follows. For each of the subsets of the nodes $S \subseteq V\backslash\{i\}$, $|S| \leq k, \forall j \in S, w_{ji}^0 \neq 0$, the number of the pivotal nodes $PI_i^0(S)$ is calculated. The $PI^0(i)$ index is defined as the sum $PI_i^0(S)$ on all considered subsets S of cardinality not more than k of the vertices of the incoming edges to the node i, i.e.

$$PI^0(i)=\sum_S PI_i^0(S).$$

Consider our example. For the node v_2 the critical sets influencing the node are $\{v_3\}$, $\{v_5\}$, and $\{v_3, v_5\}$. In both groups, $\{v_3\}$ and $\{v_5\}$, both nodes are pivotal. But it is not the case in the set $\{v_3, v_5\}$. Hence, $PI^0(v_2)$= 2. Then we obtain

Table 4. PI^0-index

	v1	v2	v3	v4	v5
$PI^0(i)$	0	2	2	0	1

d) Now we can construct the total influence on the node i, or the centrality measure of level 0,

$$TI^0=\alpha_{01}CI^0 + \alpha_{02}BI^0 + \alpha_{03}PI^0.$$

We call it TI^0 emphasizing that there are 0 intermediate nodes between analyzed vertices. Then for our example with $\alpha_{01} = \alpha_{02} = \alpha_{03} = \frac{1}{3}$ we obtain for the case q = 1.

Table 5. Total influence TI^0, q = 1

	v1	v2	v3	v4	v5
CI^0	0	3	4	0	3
BI^0	0	3	3	0	1
PI^0	0	2	2	0	1
TI^0	0	2.67	3	0	1.67

Then we evaluate the indices in the normalized form, namely

$$\widetilde{CI}^0(i) = \frac{CI^0(i)}{\sum_j CI^0(j)},$$

$$\widetilde{BI}^0(i) = \frac{BI^0(i)}{\sum_j BI^0(j)},$$

$$\widetilde{PI}^0(i) = \frac{PI^0(i)}{\sum_j PI^0(j)}.$$

Then we evaluate the total influence $\widetilde{TI}^0(i) = \alpha_{01}\widetilde{CI}^0(i) + \alpha_{02}\widetilde{BI}^0(i) + \alpha_{03}\widetilde{PI}^0(i)$. The results with the same values of $\alpha_{01} = \alpha_{02} = \alpha_{03} = \frac{1}{3}$ are presented in Table 6.

Table 6. Normalized indices

	v1	v2	v3	v4	v5
$\widetilde{CI}^0$	0	0.3	0.4	0	0.3
$\widetilde{BI}^0$	0	0.43	0.43	0	0.14
$\widetilde{PI}^0$	0	0.4	0.4	0	0.2
$\widetilde{TI}^0$	0	0.373	0.406	0	0.211

e) Indirect influence

Now we take into account indirect influences. First, we construct the adjacency matrix in the following way. Consider all paths from node j to node i of the length d = 2 (one intermediate node, two edges in the path) and evaluate maxima of the weights w_{ji}^0

$$P_{jk_1i} = max\left(w_{j,k_1}^0, w_{k_1,i}^0\right),$$

$$P_{jk_2i} = max\left(w_{j,k_2}^0, w_{k_2,i}^0\right),$$

$$\ldots$$

$$P_{jk_ti} = max\left(w_{j,k_t}^0, w_{k_t,i}^0\right).$$

Then we put

$$w^1(ji) = min\left(P_{jk_1i}, P_{jk_2i}, \ldots, P_{jk_ti}\right),$$

and construct the adjacency matrix W^1 for d = 2.

For our example we obtain $P_{1,3,2}$ = 2, as maximum of $w_{1,3}^1$ = 2, and $w_{3,2}^1$ = 1, $P_{1,5,2}$ = 3, as maximum of $w_{1,5}^1$ = 3 and $w_{5,2}^1$ = 2, so indirect influence via one intermediate node, the minimum $w_{1,2}^1$ = min ($P_{1,3,2}$,$P_{1,5,2}$) = 2.

Table 7. Adjacency matrix W^1 for indirect influence with d = 2

	v1	v2	v3	v4	v5
v1	0	2	0	0	0
v2	0	0	0	0	0
v3	0	0	0	0	0
v4	0	2	0	0	0
v5	0	0	0	0	0

For this matrix and q = 1 there are three critical sets influencing v_2 – $\{v_1\}$, $\{v_4\}$ and $\{v_1, v_4\}$. Then the corresponding indices which we denote as $CI^1(i)$, $BI^1(i)$, and $PI^1(i)$ are presented in Table 8.

Table 8. The indices for indirect influence on the adjacency matrix W^1, d = 2

	v1	v2	v3	v4	v5
CI^1	0	4	0	0	0
BI^1	0	3	0	0	0
PI^1	0	2	0	0	0

As the total direct influence can be called the Total Influence of level 0, since there are 0 intermediate nodes between the considered nodes, the total indirect influences can be called Total Influence of level 1, 2, etc., for different numbers of intermediate nodes, i.e.,

$$TI^1(i) = \alpha_{11}CI^1(i) + \alpha_{12}BI^1(i) + \alpha_{13}PI^1(i),$$

where $\alpha_{11} + \alpha_{12} + \alpha_{13} = 1$.

Finally, we can construct total influence as the linear combination of total direct and indirect influences of levels 0, …, l as

$$TI(i) = \beta_0 TI^0(i) + \beta_1 TI^1(i) + \ldots + \beta_l TI^l(i),$$

where $\beta_0 + \beta_1 + \ldots + \beta_l = 1$.

For our example with q = 1 we obtain the following values for TI^1 and TI with $\alpha_{11} = \alpha_{12} = \alpha_{13} = \frac{1}{3}$, $\beta_1 = \beta_2 = \frac{1}{2}$ presented in Table 9.

Table 9. The indices for d = 1, 2, and q = 1

	v1	v2	v3	v4	v5
CI^0	0	3	4	0	3
CI^1	0	4	0	0	0
BI^0	0	3	3	0	1
BI^1	0	3	0	0	0
PI^0	0	2	2	0	1
PI^1	0	2	0	0	0
TI^0	0	2.67	3	0	1.67
TI^1	0	3	0	0	0

Then $TI(v_1) = TI(v_4) = 0$, $TI(v_2) = 2.84$, $TI(v_3) = 1.5$, $TI(v_5) = 0.84$.

In other words, the most influential central node is v_2, next is v_3, and the third one is v_5.

Consider now another distribution of weights. We assume that

$$\alpha_{11} = \alpha_{13} = \frac{1}{4}, \ \alpha_{12} = \frac{1}{2};$$
$$\alpha_{21} = \alpha_{23} = \frac{1}{4}, \ \alpha_{22} = \frac{1}{2};$$
$$\beta_1 = \beta_2 = \frac{1}{2}.$$

In this case we give more weight to the Bundle index, i.e. to the number of critical groups, influencing the nodes. The results are given in Table 10.

Table 10. The TI indices with alternative weights, d = 1, 2; q = 1

	v1	v2	v3	v4	v5
TI^0	0	2.75	2	0	1.5
TI^1	0	3	0	0	0

Then $TI(v_1) = TI(v_4) = 0$, $TI(v_2) = 2.88$, $TI(v_3) = 1$, $TI(v_5) = 0.75$. Again, the most important node is v_2.

f) The example with the quota $q_i = 2$

If in our example we choose the quota q_i to be equal to 2, one can check that we obtain the following results.

Table 11. The indices on the initial matrix W^0, q = 2

	v1	v2	v3	v4	v5
CI^0	0	3	4	0	3
BI^0	0	2	3	0	1
PI^0	0	2	2	0	1

Remark. In fact, we can define several levels of Copeland in-degree Index and construct first total in-degree index for all levels 1, …, l, i.e.

$$CI(i) = \gamma_0 CI^0(i) + \gamma_1 CI^1(i) + \ldots + \gamma_l CI^l(i).$$

Similarly, we can construct total Bundle Index and total Pivotal Index for all levels as

$$BI(i) = \delta_0 BI^0(i) + \delta_1 BI^1(i) + \ldots + \delta_l BI^l(i);$$
$$PI(i) = \varepsilon_0 PI^0(i) + \varepsilon_1 PI^1(i) + \ldots + \varepsilon_l PI^l(i).$$

Then we can evaluate the total influence as

$$TI'(i) = \omega_1 CI(i) + \omega_2 BI(i) + \omega_3 PI(i),$$

where $\omega_1 + \omega_2 + \omega_3 = 1$.

Let us evaluate centrality indices for this case. We put $\gamma_0 = \gamma_1 = \frac{1}{2}$, $\delta_0 = \delta_1 = \frac{1}{2}$, $\varepsilon_0 = \varepsilon_1 = \frac{1}{2}$. Then we obtain the following values for the indices CI, BI, and PI.

Table 12. The indices CI, BI, and PI, q = 1

	v1	v2	v3	v4	v5
CI^0	0	3	4	0	3
CI^1	0	4	0	0	0
CI	0	3.5	2	0	1.5
BI^0	0	3	3	0	1
BI^1	0	3	0	0	0
BI	0	3	1.5	0	0.5
PI^0	0	2	2	0	1
PI^1	0	2	0	0	0
PI	0	2	1	0	0.5

Now put $\omega_1 = \omega_2 = \omega_3 = \frac{1}{2}$. Then we obtain

$$TI'(v_1) = TI'(v_4) = 0;$$
$$TI'(v_2) = \frac{1}{3} \cdot 3.5 + \frac{1}{3} \cdot 3 + \frac{1}{3} \cdot 2 = 2.79;$$
$$TI'(v_3) = \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 0.5 + \frac{1}{3} \cdot 1 = 1.17;$$
$$TI'(v_5) = \frac{1}{3} \cdot 1.5 + \frac{1}{3} \cdot 0.5 + \frac{1}{3} \cdot 0.5 = 0.83.$$

Again, the most influential node is v_2, then v_3, and v_5.

If we change the weights as $\omega_1 = 0.1$, $\omega_2 = 0.1$, and $\omega_3 = 0.8$, i.e. we consider most important component being pivotal nodes, then we obtain the following values:

$$TI'(v_1) = TI'(v_4) = 0;$$
$$TI'(v_2) = 0.1 \cdot 3.5 + 0.1 \cdot 3 + 0.8 \cdot 2 = 2.25;$$
$$TI'(v_3) = 0.1 \cdot 2 + 0.1 \cdot 0.5 + 0.8 \cdot 1 = 1.05;$$
$$TI'(v_5) = 0.1 \cdot 1.5 + 0.1 \cdot 0.5 + 0.8 \cdot 0.5 = 0.6.$$

The most influential node is the same, v_2, then v_3, and v_5.

<u>Remark.</u> We would like to emphasize that any function can be used on each step of aggregation, not necessary min or max or summation, it might be other more complicated functions.

3. Another example of network with 5 vertices

Consider the network of 5 vertices (Fig. 2) with the adjacency matrix of the network given in Table 13.

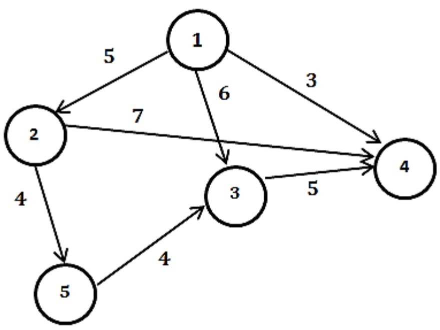

Fig. 2

Table 13. Adjacency matrix W^0 for the network with 5 vertices

	v1	v2	v3	v4	v5
v1	0	5	6	3	0
v2	0	0	0	7	4
v3	0	0	0	5	0
v4	0	0	0	0	0
v5	0	0	4	0	0

The indices CI^0, BI^0, and PI^0 for the network with matrix W^0 for the quota *q* = 3 have the following values presented in Table 14.

Table 14. The indices for the quota *q* = 3

	v1	v2	v3	v4	v5
CI^0	0	5	10	15	4
BI^0	0	1	3	7	1
PI^0	0	1	2	3	1

The normalized values of the indices $\widetilde{CI}^0$, $\widetilde{BI}^0$, and $\widetilde{PI}^0$ are given in Table 15.

Table 15. The normalized values of the indices, *q* = 3

	v1	v2	v3	v4	v5
$\widetilde{CI}^0$	0	0.15	0.29	0.44	0.12
$\widetilde{BI}^0$	0	0.08	0.25	0.58	0.08
$\widetilde{PI}^0$	0	0.14	0.29	0.43	0.14

Note that for the vertex v_1 all indices are equal to zero, since there are no incoming arcs to this vertex.

Now consider indirect influence with the length of the path *d* = 2.

Table 16. Adjacency matrix W^1 for the length of the path *d* = 2

	v1	v2	v3	v4	v5
v1	0	0	0	6	5
v2	0	0	4	0	0
v3	0	0	0	0	0
v4	0	0	0	0	0
v5	0	0	0	5	0

For the quota *q* = 3 the indices are as follows.

Table 17. The indices for the matrix W^1, *d* = 2, *q* = 3

	v1	v2	v3	v4	v5
CI^1	0	0	4	11	5
BI^1	0	0	1	3	1
PI^1	0	0	1	2	1

The normalized values of the indices are given in Table 18.

Table 18. The normalized indices for the matrix W^1, d = 2, q = 3

	v1	v2	v3	v4	v5
$\widetilde{CI}^1$	0	0	0.20	0.55	0.25
$\widetilde{BI}^1$	0	0	0.20	0.60	0.20
$\widetilde{PI}^1$	0	0	0.25	0.50	0.25

For the indirect influence with the length of the path d = 3 the adjacency matrix is presented in Table 19.

Table 19. Adjacency matrix W^2 for the length of the path d = 3

	v1	v2	v3	v4	v5
v1	0	0	5	0	0
v2	0	0	0	5	0
v3	0	0	0	0	0
v4	0	0	0	0	0
v5	0	0	0	0	0

And the indices for the quota q = 3 are presented in Table 20.

Table 20. The indices on the matrix W^2, d = 3, q = 3

	v1	v2	v3	v4	v5
CI^2	0	0	5	5	0
BI^2	0	0	1	1	0
PI^2	0	0	1	1	0

The normalized values of the indices are given in Table 21.

Table 21. The normalized indices for the matrix W^2, d = 3, q = 3

	v1	v2	v3	v4	v5
$\widetilde{CI}^2$	0	0	0.5	0.5	0
$\widetilde{BI}^2$	0	0	0.5	0.5	0
$\widetilde{PI}^2$	0	0	0.5	0.5	0

Note, that only vertices v_3 and v_4 have non-zero influence, which are obtained from the arcs from the vertices v_1 and v_2, respectively.

Now we evaluate the total influence indices for different sets of values of coefficients α.

For each set of α_{ij} coefficients, denote them as A1, A2, A3, and A4, defined as

A1: α_{ij}= $1/3$ for all i= 0,1,2; j = 1, 2, 3;

A2: α_{i1}= 0.8, i= 0,1,2; α_{ij}= 0.1 for i = 0, 1, 2; j= 2, 3;

A3: α_{i2}= 0.8, i= 0, 1,2; α_{ij}= 0.1 for i = 0, 1, 2; j= 1, 3;

A4: α_{i3}= 0.8, i= 0, 1,2; α_{ij}= 0.1 for i = 0, 1, 2; j= 1,2.

We present the values of $\widetilde{TI}^0$, $\widetilde{TI}^1$, and $\widetilde{TI}^2$ indices in Table 22.

Table 22. The indices $\widetilde{TI}^0$, $\widetilde{TI}^1$, and $\widetilde{TI}^2$ for different sets of α coefficients, q = 3

	$\widetilde{TI}^0$					$\widetilde{TI}^1$					$\widetilde{TI}^2$				
	v1	v2	v3	v4	v5	v1	v2	v3	v4	v5	v1	v2	v3	v4	v5
A1	0	0.12	0.28	0.48	0.11	0	0	0.22	0.55	0.23	0	0	0.5	0.5	0
A2	0	0.14	0.29	0.45	0.12	0	0	0.21	0.55	0.25	0	0	0.5	0.5	0
A3	0	0.1	0.26	0.55	0.09	0	0	0.21	0.59	0.21	0	0	0.5	0.5	0
A4	0	0.14	0.28	0.45	0.13	0	0	0.24	0.52	0.25	0	0	0.5	0.5	0

Next we evaluate only for the set A3, i.e. for the values α_{i2}= 0.8, i = 0, 1,2; α_{ij} = 0.1 for i = 0, 1,2; j= 1, 3 the total influence $\widetilde{TI}$ with different values of the β coefficients. These results are presented in Table 23.

Table 23. Index $\widetilde{TI}$ for different values of β coefficients, q = 3

		v1	v2	v3	v4	v5
B1	β_0= β_1= β_2=$^1/_3$	0	0.03	0.32	0.55	0.10
B2	β_0= 0.7, β_1= β_2= 0.15	0	0.07	0.29	0.55	0.10
B3	β_0= β_2= 0.15 β_1= 0.7	0	0.01	0.26	0.57	0.16
B4	β_0= β_1= 0.15 β_2= 0.7	0	0.01	0.42	0.52	0.05

Below in Table 24 we present classic indices for this network.

Table 24. Classic indices for the network

	v1	v2	v3	v4	v5
In-degree	0	5	10	15	4
Out-degree	14	16	15	15	8
Betweenness	0	2	1	0	1
Closeness	0.067	0.083	0.077	0.077	0.053
PageRank	0.097	0.126	0.248	0.393	0.136
Eigenvector	0.939	0.976	0.901	1	0.53

One can see that the values of the In-degree index coincide with CI^0, the most influential node is v_4 with respect to PageRank, Eigenvector and $\widetilde{TI}$.

4. Example of networks with 10 vertices

Now consider the network with 10 vertices (see Fig. 3).

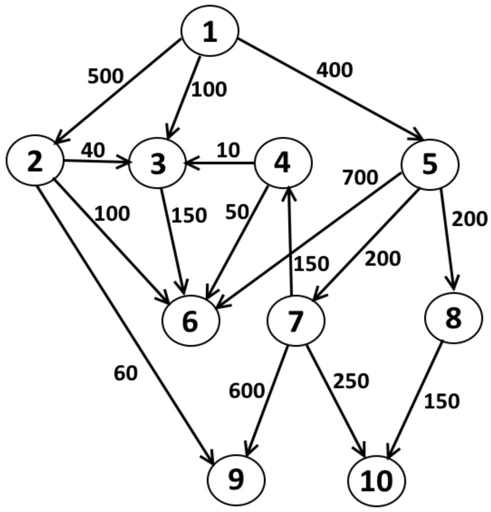

Fig. 3

The adjacency matrix of the network is presented in Table 25.

Table 25. Adjacency matrix W^0 for the network of 10 vertices

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
v1	0	500	100	0	400	0	0	0	0	0
v2	0	0	40	0	0	100	0	0	60	0
v3	0	0	0	0	0	150	0	0	0	0
v4	0	0	10	0	0	50	0	0	0	0
v5	0	0	0	0	0	700	200	200	0	0
v6	0	0	0	0	0	0	0	0	0	0
v7	0	0	0	150	0	0	0	0	600	250
v8	0	0	0	0	0	0	0	0	0	150
v9	0	0	0	0	0	0	0	0	0	0
v10	0	0	0	0	0	0	0	0	0	0

The quota will be constructed in the following way. For each vertex, the half of the sum of the weights of all incoming arcs will be evaluated and then 1 will be added. Then one can construct the indices CI^0, BI^0, and PI^0 (see Table 26) as well as these indices in the normalized form (see Table 27). The indices are calculated with the maximum size of coalition k = 3.

Table 26. The indices CI^0, BI^0, and PI^0 on the initial matrix W^0, q = 50% + 1

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
CI^0	0	500	150	150	400	1000	200	200	660	400
BI^0	0	1	4	1	1	7	1	1	2	2
PI^0	0	1	4	1	1	7	1	1	2	2

Table 27. The normalized indices $\widetilde{CI}^0$, $\widetilde{BI}^0$, and $\widetilde{PI}^0$ on matrix W^0, *q* = 50% + 1

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
$\widetilde{CI}^0$	0	0.14	0.04	0.04	0.11	0.27	0.05	0.05	0.18	0.11
$\widetilde{BI}^0$	0	0.05	0.2	0.05	0.05	0.35	0.05	0.05	0.1	0.1
$\widetilde{PI}^0$	0	0.05	0.2	0.05	0.05	0.35	0.05	0.05	0.1	0.1

Now the adjacency matrix W^1 is constructed for the length of path *d* = 2.

Table 28. Adjacency matrix W^1, *d* = 2

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
v1	0	0	500	0	0	150	400	400	500	0
v2	0	0	0	0	0	150	0	0	0	0
v3	0	0	0	0	0	0	0	0	0	0
v4	0	0	0	0	0	150	0	0	0	0
v5	0	0	0	200	0	0	0	0	600	200
v6	0	0	0	0	0	0	0	0	0	0
v7	0	0	150	0	0	150	0	0	0	0
v8	0	0	0	0	0	0	0	0	0	0
v9	0	0	0	0	0	0	0	0	0	0
v10	0	0	0	0	0	0	0	0	0	0

Let the quota be formed in the same way, then the indices are as follows.

Table 29. The indices on the indirect influence matrix W^1, *d* = 2, *q* = 50% + 1

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
CI^1	0	0	650	200	0	600	400	400	1100	200
BI^1	0	0	2	1	0	4	1	1	2	1
PI^1	0	0	2	1	0	12	1	1	2	1

Table 30. The normalized indices on the indirect influence matrix W^1, *d* = 2, *q* = 50% + 1

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
$\widetilde{CI}^1$	0	0	0.18	0.06	0	0.17	0.11	0.11	0.31	0.06
$\widetilde{BI}^1$	0	0	0.17	0.08	0	0.33	0.08	0.08	0.17	0.08
$\widetilde{PI}^1$	0	0	0.1	0.05	0	0.6	0.05	0.05	0.1	0.05

And for the length of path *d* = 3, the adjacency matrix W^2 is presented in Table 31.

Table 31. Adjacency matrix W^2, *d* = 3

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
v1	0	0	0	400	0	500	0	0	600	400
v2	0	0	0	0	0	0	0	0	0	0
v3	0	0	0	0	0	0	0	0	0	0
v4	0	0	0	0	0	0	0	0	0	0
v5	0	0	200	0	0	200	0	0	0	0
v6	0	0	0	0	0	0	0	0	0	0
v7	0	0	0	0	0	150	0	0	0	0
v8	0	0	0	0	0	0	0	0	0	0
v9	0	0	0	0	0	0	0	0	0	0
v10	0	0	0	0	0	0	0	0	0	0

The indices for the indirect influence with the length of the path d = 3 are presented in Table 32.

Table 32. The indices on the indirect influence matrix W^2, d = 3, q = 50% + 1

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
CI^2	0	0	200	400	0	850	0	0	600	400
BI^2	0	0	1	1	0	4	0	0	1	1
PI^2	0	0	1	1	0	4	0	0	1	1

The normalized indices are presented on Table 33.

Table 33. The normalized indices on the indirect influence matrix W^2, d = 3, q = 50% + 1

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
$\widetilde{CI}^2$	0	0	0.08	0.16	0	0.35	0	0	0.24	0.16
$\widetilde{BI}^2$	0	0	0.13	0.13	0	0.5	0	0	0.13	0.13
$\widetilde{PI}^2$	0	0	0.13	0.13	0	0.5	0	0	0.13	0.13

Next to evaluate the total influence indices we consider two sets of coefficients, A,

$$\alpha_{01}= 0.2;\ \ \alpha_{02}= \alpha_{03} = 0.4;$$

$$\alpha_{11}= 0.2;\ \ \alpha_{12}= \alpha_{13} = 0.4;$$

$$\alpha_{21}= 0.2;\ \ \alpha_{22}= \alpha_{23} = 0.4,$$

and we can construct the $\widetilde{TI}$ index using β_0= 0.2; β_1= β_2 = 0.4.

The results are presented in Table 34.

Table 34. $\widetilde{TI}$ indices for set of coefficients A, for d = 1, 2, 3, and total influence, q = 50% + 1

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
$\widetilde{TI}^0$	0	0.07	0.17	0.05	0.06	0.33	0.05	0.05	0.12	0.1
$\widetilde{TI}^1$	0	0	0.14	0.06	0	0.41	0.08	0.08	0.17	0.06
$\widetilde{TI}^2$	0	0	0.12	0.13	0	0.47	0	0	0.15	0.13
$\widetilde{TI}$	0	0.01	0.14	0.09	0.01	0.42	0.04	0.04	0.15	0.1

Another set of coefficients, A', is given below

$$\alpha_{01}= 0.8;\ \ \alpha_{02}= \alpha_{03} = 0.1;$$

$$\alpha_{11}= 0.8;\ \ \alpha_{12}= \alpha_{13} = 0.1;$$

$$\alpha_{21}= 0.8;\ \ \alpha_{22}= \alpha_{23} = 0.1;$$

$$\beta_0= 0.8;\ \beta_1= \beta_2 = 0.1.$$

The corresponding results are given in Table 35.

Table 35. $\widetilde{TI}$ indices for set of coefficients A', for d = 1, 2, 3, and total influence, q = 50% + 1

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
$\widetilde{TI}^0$	0	0.12	0.07	0.04	0.1	0.29	0.05	0.05	0.16	0.11
$\widetilde{TI}^1$	0	0	0.17	0.06	0	0.23	0.1	0.1	0.27	0.06
$\widetilde{TI}^2$	0	0	0.09	0.16	0	0.38	0	0	0.22	0.16
$\widetilde{TI}$	0	0.1	0.08	0.06	0.08	0.29	0.05	0.05	0.18	0.11

In the first version, A, we prescribe more importance to the critical sets of nodes and pivotal nodes influencing a node, while in the second version, A', we put more importance to the standard in-degree values.

Now we can compare the obtained results with the classic centrality measures and the indices SRIC and LRIC. Note, the $\widetilde{TI}$ indices are presented for above considered quota q = 50% + 1 and for q = 25% as well.

Table 36. Classic and $\widetilde{TI}$ indices

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
In-degree	0	500	150	150	400	1000	200	200	660	400
Out-degree	1000	200	150	60	1100	0	1000	150	0	0
Closeness	0.00015	0.00051	0.00158	0.00071	0.00019	0.0111	0.00028	0.00158	0.0111	0.0111
Betweenness	0	1	0	3	5	0	6	0	0	0
Page Rank	0.06	0.08	0.09	0.07	0.08	0.25	0.07	0.07	0.11	0.13
Eigenvector	0.67	0.46	0.21	0.11	1.00	0.81	0.45	0.23	0.31	0.15
SRIC, q = 25%	0	0.152	0	0	0.121	0.356	0.019	0.019	0.212	0.121
LRIC maxmin, q = 25%	0	0.087	0	0	0.087	0.217	0.091	0.091	0.238	0.190
$\widetilde{TI}$ for A, q = 50%+1	0	0.01	0.14	0.09	0.01	0.42	0.04	0.04	0.15	0.1
$\widetilde{TI}$ for A', q = 50%+1	0	0.1	0.08	0.06	0.08	0.29	0.05	0.05	0.18	0.11
$\widetilde{TI}$ for A, q = 25%	0	0.01	0.13	0.08	0.01	0.44	0.04	0.04	0.15	0.09
$\widetilde{TI}$ for A', q = 25%	0	0.09	0.09	0.05	0.08	0.3	0.05	0.05	0.18	0.11

We see that the eigenvector centrality orders the first 5 most important nodes as

$$v_5 \succ v_6 \succ v_1 \succ v_2 \succ v_7,$$

for LRIC with q = 25% the order is

$$v_9 \succ v_6 \succ v_{10} \succ v_7 \sim v_8,$$

for $\widetilde{TI}$ (for the set A) with q = 25%

$$v_6 \succ v_9 \succ v_3 \succ v_{10} \succ v_4,$$

and for $\widetilde{TI}$ (for the set A) with q = 50% +1, the ordering is the same,

$$v_6 \succ v_9 \succ v_3 \succ v_{10} \succ v_4.$$

5. Another method to construct centrality indices

As we mentioned above, the approach presented in this work allows to construct many new indices depending on our understanding what features are important in the problem under study. To illustrate this we present the centrality index constructed as a linear combination of the eigenvector centrality and PI-index. We use the example of the network considered in the previous Section.

The values of the eigenvector of the nodes are

$$v_1=0.67,\ v_2=0.46,\ v_3=0.21,\ v_4=0.11,\ v_5=1.00,$$
$$v_6=0.81,\ v_7=0.45,\ v_8=0.23,\ v_9=0.31,\ v_{10}=0.15.$$

In Table 37 the matrix of pivotal nodes for each node is presented, the coalitions are constructed as before. i.e. on the basis on critical groups with total influence exceeding the quota which is equal to 50% +1 of the value of all incoming arcs.

Table 37. Matrix of pivotal nodes

WP	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
v1	0	1	4	0	1	0	0	0	0	0
v2	0	0	0	0	0	0	0	0	0	0
v3	0	0	0	0	0	0	0	0	0	0
v4	0	0	0	0	0	0	0	0	0	0
v5	0	0	0	0	0	7	1	1	0	0
v6	0	0	0	0	0	0	0	0	0	0
v7	0	0	0	1	0	0	0	0	2	2
v8	0	0	0	0	0	0	0	0	0	0
v9	0	0	0	0	0	0	0	0	0	0
v10	0	0	0	0	0	0	0	0	0	0

Below, the normalized values of PI index are given.

Table 38. Normalized values of PI index ($\widetilde{PI}^0$), q = 50% + 1

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10
$\widetilde{PI}^0$	0	0.05	0.2	0.05	0.05	0.35	0.05	0.05	0.1	0.1

We consider the following linear combination $TI(v_i)$ = 0.5$\cdot$ $EV(v_i)$ +0.5$\cdot$ $PI(v_i)$, where $EV(v_i)$ is the eigenvector centrality for v_i, $PI(v_i)$ is the pivotal index for v_i. Then

$TI(v_1)$ = 0.5·0.67 + 0.5·0 = 0.34;
$TI(v_2)$ = 0.5·0.46 + 0.5·0.05 = 0.26;
$TI(v_3)$ = 0.5·0.21 + 0.5·0.2 = 0.21;
$TI(v_4)$ = 0.5·0.11 + 0.5·0.05 = 0.08;
$TI(v_5)$ = 0.5· 1.00 + 0.5·0.05 = 0.53;
$TI(v_6)$ = 0.5·0.81 + 0.5·0.35 = 0.58;
$TI(v_7)$ = 0.5·0.45 + 0.5·0.05 = 0.25;
$TI(v_8)$ = 0.5·0.23 + 0.5·0.05 = 0.14;
$TI(v_9)$ = 0.5·0.31 + 0.5·0.1 = 0.21;
$TI(v_{10})$ = 0.5·0.15 + 0.5·0.1 = 0.13.

Now consider another linear combination of parameters $TI'(v_i)$ = 0.1$\cdot$ $EV(v_i)$ +0.9$\cdot$ $PI(v_i)$

$TI'(v_1)$ = 0.1·0.67 + 0.9·0 = 0.07;
$TI'(v_2)$ = 0.1·0.46 + 0.9·0.05 = 0.09;
$TI'(v_3)$ = 0.1·0.21 + 0.9·0.2 = 0.2;
$TI'(v_4)$ = 0.1·0.11 + 0.9·0.05 = 0.06;
$TI'(v_5)$ = 0.1· 1.00 + 0.9·0.05 = 0.15;
$TI'(v_6)$ = 0.1·0.81 + 0.9·0.35 = 0.4;
$TI'(v_7)$ = 0.1·0.45 + 0.9·0.05 = 0.09;
$TI'(v_8)$ = 0.1·0.23 + 0.9·0.05 = 0.07;

$$TI'(v_9) = 0.1\cdot 0.31 + 0.9\cdot 0.1 = 0.12;$$
$$TI'(v_{10}) = 0.1\cdot 0.15 + 0.9\cdot 0.1 = 0.11.$$

The orderings of the first five most important nodes for these two cases are

$$TI : v_6 \succ v_5 \succ v_1 \succ v_2 \succ v_7;$$
$$TI': v_6 \succ v_3 \succ v_5 \succ v_9 \succ v_{10}.$$

6. Conclusion

We have introduced a class of new centrality indices which take into account a parameter of a node, direct and indirect (with fixed length of the path) connections of nodes, and a group influence of nodes to a node. Total influence is evaluated as a linear combination of its components, however, other rules can be used as well.

There are two parts of the model with high computational complexity. First, it is the construction of the matrix in which the critical sets influencing nodes are defined. One of the ways to decrease complexity is to consider the sets of fixed cardinality, say, not more than 5. Second, the construction of the paths naturally increases the complexity of the model. Here we can limit ourselves with only some selected paths, e.g. those in which the weight on edges exceed the quota on, say, not less than 10%. There are many other ways to decrease complexity depending on the problem in hand.

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References

Newman M.E.J. (2003) The Structure and Function of Complex Networks // SIAM Review. Vol. 45 (2). P. 167–256.

Aleskerov F., Andrievskaya I., Permjakova E. (2014) Key borrowers detected by the intensities of their short-range interactions: National Research University Higher School of Economics Basic Research Program Working Paper. Series Financial Economics. WR BRP, 33/FE/2014 <http://www.hse.ru/data/2014/08/12/1314030953/33FE2014.pdf>

Aleskerov F., Meshcheryakova N., Shvydun S. (2016) Centrality Measures in Networks based on Nodes Attributes, Long-Range Interactions and Group Influence: Working Paper WP7/2016/04. Moscow: Higher School of Economics Publ. House, 2016 <https://www.hse.ru/en/org/hse/wp/wp7en https://arxiv.org/abs/1610.05892>

Aleskerov F., Meshcheryakova N., Shvydun S. (2017) Power in Network Structures // Models, Algorithms, and Technologies for Network Analysis. / V. Kalyagin, A. Nikolaev, P. Pardalos et al. (eds). NET, 2016. Springer Proceedings in Mathematics & Statistics. Vol. 197. Springer, Cham. First Online: 24 June 2017. DOI: 10.1007/978-3-319-56829-4_7

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Вводится новый класс индексов центральности в сетях, в которых учитываются параметры вершин, прямые и косвенные (с фиксированной длиной пути) связи между вершинами, а также влияние групп вершин на вершину. Полное влияние оценивается в виде линейной комбинации его компонент, но могут использоваться и другие правила.

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(*на английском языке*)

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