Introduction to stochastic differential equations – 7
Links between SDEs and PDEs

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Let us inspect the links between solution $X_t = X_t^{0,x}$ and parabolic PDEs.

Example (7.1)

Let $u(t,x) \in C_b^{1,2}([0,T] \times R^d)$ be a solution of the heat equation

$$\begin{aligned} u_t(t,x) + Lu(t,x) = 0, \quad 0 \leq t \leq T, \\ u(T,x) = g(x), \end{aligned}$$

with $g \in C_b^2(R^d)$. Then for any $0 \leq t \leq T$ the value $u(t,x)$ can be represented in the form

$$u(t,x) = Eg(X_T^{t,x}) \equiv E_x g(X_T^{t,x}).$$

Let us apply Ito's formula to $u(s, X_s^{t_0,x})$ for $0 \le t_0 \le s \le T$ (since $u(T,x) = g(x) \equiv Eg(X_T^{T,x})$):

$$du(s, X_s^{t_0,x}) = \sigma^* \nabla u(s, X_s^{t_0,x}) dW_s$$
$$+[u_s(s, X_s^{t_0,x}) + Lu(s, X_s^{t_0,x})]ds.$$

In the integral form with $t_0 + s = T$,

$$u(T, X_s^{t_0,x}) = u(t_0, x) + \int_{t_0}^{T} \sigma^* \nabla u(s, X_s^{t_0,x}) dW_s$$
$$+ \int_{t_0}^{T} [u_s(s,x) + Lu(s, X_s^{t_0,x})]ds.$$

Let us now take expectations from both sides of this equality:

$$Eu(T, X_T^{t_0,x}) = u(t_0, x),$$

because

$$E \int_{t_0}^T \sigma^* \nabla u(s, X_s^{t_0,x}) dW_s = 0,$$

$$\& \ \ [u_s(s,x) + Lu(s, X_s^{t_0,x})] = 0.$$

Remark

The condition $g \in C_b^2(R^d)$ follows automatically from $u(t,x) \in C_b^{1,2}([0,T] \times R^d)$. Both of them can be relaxed.

Example (7.2)

Let $u(t,x) \in C_b^{1,2}([0,T) \times R^d) \bigcap C_b([0,T] \times R^d)$ be a solution of the heat equation

$$u_t(t,x) + Lu(t,x) = 0, \quad 0 \le t \le T,$$
$$u(T,x) = g(x),$$

with $g \in C_b(R^d)$. Then for any $0 \le t \le T$ the value $u(t,x)$ can be represented in the form

$$u(t,x) = Eg(X_T^{t,x}).$$

The conditions of boundedness of g and u with its derivatives may be further considerably relaxed, too.

Proof of Example 7.2 $L = \frac{a_{ij}(t,x)\partial^2}{2\partial x^i \partial x^j} + \frac{b^i(t,x)\partial}{\partial x^i}$

NB: while $a = \sigma\sigma^*$, we may recover the *symmetric positive-definite* square root of the matrix $a(x)$ via the Cauchy – Dunford formula (see textbooks)

Note that the differential form of Ito's equation remains valid,

$$du(s, X_s^{t_0,x}) = \sigma^* \nabla u(s, X_s^{t_0,x}) dW_s$$
$$+[u_s(s, X_s^{t_0,x}) + Lu(s, X_s^{t_0,x})]ds.$$

Yet, now we cannot simply integrate it to T, because the derivatives are assumed only on the semi-open interval $[0, T)$.
Let $t_0 \geq 0$. Denote $T_n := T - \frac{1}{n}$. Then, for n such that $t_0 < T_n$ we have,

$$u(T_n, X_{T_n}^{t_0,x}) = u(t_0, x) + \int_{t_0}^{T_n} \sigma^* \nabla u(s, x + W_s) dW_s$$
$$+ \int_{t_0}^{T_n} [u_s(s, x) + Lu(s, x + W_s)]ds.$$

$$u(T_n, X_{T_n}^{t_0,x}) = u(t_0, x) + \int_{t_0}^{T_n} \sigma^* \nabla u(s, X_s^{t_0,x}) dW_s$$

$$+ \int_{t_0}^{T_n} [u_s(s,x) + Lu(t_0 + s, X_s^{t_0,x})] ds.$$

Let us take expectations here: since

$$[u_s(s,x) + Lu(s, x + W_s)] = 0$$

and because

$$E \int_{t_0}^{T_n} \sigma^* \nabla u(s, X_s^{t_0,x}) dW_s = 0,$$

we get

$$Eu(T_n, X_{T_n}^{t_0,x}) = u(t_0, x).$$

Equivalently,

$$u(t_0, x) = Eu(T_n, X_{T_n}^{t_0,x}).$$

Here we can pass to the limit as $T_n \uparrow T$ in the r.h.s.: since the function u is continuous and bounded up to T, and because X is continuous in time, we get by Lebesgue's bounded convergence theorem that again

$$u(t_0, x) = Eu(T, X_T^{t_0,x}) \equiv Eg(X_T^{t_0,x}),$$

as required. Recall that here $t_0 \geq 0$.

Now let us consider the equation with a non-zero r.h.s.

Example (7.3)

Let $u(t,x) \in C_b^{1,2}([0,T] \times R^d)$ be a solution of the heat equation

$$u_t(t,x) + Lu(t,x) = -f(t,x), \quad 0 \le t \le T,$$
$$u(T,x) = g(x),$$

with $g \in C_b^2(R^d)$, $f(t,x) \in C_b([0,T] \times R^d)$. Then for any $0 \le t \le T$ the value $u(t,x)$ can be represented in the form

$$u(t_0,x) = E\left[\int_{t_0}^T f(s, X_s^{t_0,x})ds + g(X_T^{t_0,x})\right].$$

Recall Ito's formula,

$$du(s, X_s^{t_0,x}) = \sigma^* \nabla u(s, X_s^{t_0,x})dW_s + [u_s(s, X_s^{t_0,x}) + Lu(s, X_s^{t_0,x})]ds.$$

Now it can be rewritten as follows,

$$du(s, X_s^{t_0,x}) = \sigma^* \nabla u(s, X_s^{t_0,x})dW_s - f(s, X_s^{t_0,x})ds,$$

or, in the integral form,

$$u(T, X_T^{t_0,x}) = u(t_0, x) + \int_{t_0}^T \sigma^* \nabla u(s, X_s^{t_0,x})dW_s - \int_{t_0}^T f(s, X_s^{t_0,x})ds.$$

Proof of Example 7.3, ctd. $L = \frac{1}{2}\frac{a_{ij}(t,x)\partial^2}{\partial x^i \partial x^j} + \frac{b^i(t,x)\partial}{\partial x^i}$

Taking expectations from both sides we get,

$$u(t_0, x) = Eu(T, X_T^{t_0,x}) + E\int_{t_0}^{T} f(s, X_s^{t_0,x})ds$$
$$= Eg(X_T^{t_0,x}) + E\int_{t_0}^{T} f(s, X_s^{t_0,x})ds,$$

as required.

Remark

Conditions of the Example may also be relaxed, as earlier, assuming derivatives only in the semi-open cylinder $([0, T) \times R^d)$ *along with continuity of u only in the closed cylinder* $([0, T] \times R^d)$. *Yet, it is not all that may be relaxed here.*

The issue is that for heat equations with a non-zero r.h.s. it is not often that solutions are classical, that is, from $C_b^{1,2}$.

How to verify that solution $u \in C_b^{1,2}$?

In PDE theory often solutions are only with Sobolev derivatives!

$$u_t(t,x) + Lu(t,x) = -f(t,x), \quad 0 \le t \le T,$$
$$u(T,x) = g(x).$$

In general there is no option to differentiate explicit formulae for solutions as for the classical heat equation. However, there is another way, to use L_2 (or L_p) directional derivatives of SDEs. In principle, this approach is available *if the coefficients have sufficiently many derivatives with respect to x.* We do not show the details here. Without additional derivatives of coefficients, probabilists are not aware how to show existence of derivatives of expressions like $E\left[\int_{t_0}^T f(s, X_s^{t_0,x})ds + g(X_T^{t_0,x})\right]$ by purely probabilistic tools (i.e., without PDE techniques).

Similarly a PDE "with a potential" can be considered.

Example (7.4)

Let $u(t,x) \in C_b^{1,2}([0,T] \times R^d)$ be a solution of the heat equation with a potential

$$u_t(t,x) + Lu(t,x) - cu(t,x) = -f(t,x), \quad 0 \le t \le T,$$
$$u(T,x) = g(x),$$

with $g \in C_b^2(R^d)$, $f(t,x) \in C_b([0,T] \times R^d)$. Then for any $0 \le t \le T$ the value $u(t,x)$ can be represented in the form

$$u(t,x) = E\int_t^T e^{-cs} f(s, X_s^{t,x})ds + Ee^{-c(T-t)}g(X_T^{t,x}).$$

Let D be a bounded domain (by definition open one and connected; condition to be connected can be dropped, it is just for simplicity) in R^d. Consider the elliptic equation

$$Lu(x) = 0, \; x \in D, \quad \& \quad u|_\Gamma = \phi(x),$$

where $\Gamma = \partial D$ is the boundary of D. Denote $D^c := R^d \setminus D$. Let

$$\tau := \inf(t \geq 0 : X_t^{0,x} \in D^c).$$

Example (7.5)

Let $u(x) \in C_b^2(\bar{D})$ be a solution of the elliptic equation above with $\phi \in C(\bar{D})$, $a(x)$ uniformly nondegenerate. Then

$$u(x) = E\phi(X_\tau^{0,x}), \quad x \in D.$$

Let us apply Ito's formula to $u(X_t)$:

$$du(X_t) = \sigma^* \nabla u(X_t) dW_t + Lu(X_t)dt.$$

In the integral form we have (assuming $u \in C_b^2(R^d)$),

$$u(X_t) - u(x) = \int_0^t \sigma^* \nabla u(X_s) dW_s + \int_0^t Lu(X_s)ds.$$

However, it is not what we need because, in fact, we know nothing about u outside $\bar{D}$, or, at most, outside some its neighbourhood. So, we have to use stopping time τ. It follows from the nondegeneracy of $a(\cdot)$ that $\tau < \infty$ a.s. and, more than that, $\sup_{x \in D} E_\tau < \infty$. (Recall that domain D is bounded.)

Lemma

Let b and σ be bounded, $\sigma\sigma^*$ uniformly nondegenerate. Then

$$\sup_{x \in D} E_x \tau < \infty.$$

Proof consists of three easy steps. As we know, for a Markov process it suffices to show that there exists $T > 0$ such that

$$\inf_{x \in D} P_x(\exists t \in [0, T] \text{ such that } X_t \notin D) > 0.$$

I. Firstly, let us reduce the problem to the case with $b \equiv 0$. This can be done via Girsanov's measure transformation theorem. We will run the whole proof for $D = B_R$.

Proof of Lemma; $\tilde W_t = W_t + \int_0^t \tilde b(X_s)ds, t \le T$

$\tilde b = \sigma^{-1} b$

We have, with some (any) $T > 0$ and $P^\rho(A) = E\rho_T 1(A)$,

$$\rho_T = \exp(-\int_0^T \tilde b(X_s)dW_s - \frac{1}{2}\int_0^T \tilde b^2(X_s)ds),$$

due to the Cauchy – Buniakovskii – Schwarz inequality

$$P_x(\sup_{t\le T}|X_t| > R) = E_x^\rho \rho^{-1} 1(\sup_{t\le T}|X_t| > R)$$
$$\ge (E_x^\rho \rho_T)^{-1}(E_x^\rho 1(\sup_{t\le T}|X_t| > R))^2.$$

Here (as $+\frac{1}{2}\int_0^T \tilde b^2(X_s)ds \le -\frac{1}{2}\int_0^T \tilde b^2(X_s)ds + \|\tilde b\|^2 T$),

$$\sup_{x\in D} E_x^\rho \rho_T = \sup_{x\in D} E_x^\rho \exp(-\int_0^T \tilde b(X_s)d\tilde W_s + \frac{1}{2}\int_0^T \tilde b^2(X_s)ds) < \infty$$

So, to prove Lemma it remains to show that for *some* $T > 0$

$$\inf_{x\in D} E_x^\rho 1(\sup_{t\le T}|X_t| > R) > 0.$$

Proof of Lemma, ctd. Let $\sigma_t = (\sum_j \sigma_{1j}^2(X_t))^{1/2}$

Wanted: $\exists T > 0$ such that $\inf_{x \in B_R} E_x^\rho 1(\sup_{t \le T} |X_t| > R) > 0$

Note that under P^ρ the process X_t satisfies the equation **without a drift** removed by Girsanov, with a new WP $\tilde{W}$:

$$X_t = x + \int_0^t \sigma(X_s) d\tilde{W}_s, \ t \ge 0.$$

II. Now, consider the equation on one component of X_t, say, on X_t^1,

$$dX_t^1 = \sigma_{1j}(X_t) d\tilde{W}_t^j = (\sum_j \sigma_{1j}^2(X_t))^{1/2} d\bar{W}_t,$$

$$\text{where} \qquad \bar{W}_t := \int_0^t \frac{\sum_i \sigma_{1i}(X_t) d\tilde{W}_t^i}{(\sum_j \sigma_{1j}^2(X_t))^{1/2}}$$

The equation on X_t^1 can be rewritten as

$$dX_t^1 = \sigma_t d\bar{W}_t.$$

Proof of Lemma, ctd

Lévy characterisation of WP via the compensator

It is known[2] that a continuous martingale M_t is a WP iff its compensator $\langle M \rangle_t = t$. In our case $\bar{W}_t$ is a continuous martingale, and

$$\langle \bar{W} \rangle_t = \sum_i \int_0^t \frac{\sigma_{1i}^2(X_s)ds}{(\sum_j \sigma_{1j}^2(X_s))} = t,$$

so, $\bar{W}_t$ is a WP, as required. Moreover, the diffusion coefficient of X_t^1 is nondegenerate: with $\lambda^* = (1, 0, \ldots 0)$,

$$\sigma_t^2 = \sum_j \sigma_{1j}^2(X_t) = \lambda^* \sigma\sigma^*(X_t)\lambda \geq: c_1 > 0,$$

due to the assumption of the uniform nondegeneracy of $\sigma\sigma^*$. We will now show that $E \sup_{t \leq T} |X_t^1|^2 \to \infty, t \to \infty$.

[2] A separate topic, suitable for the homework or a seminar talk

Proof of Lemma, ctd

Wanted: $\exists T > 0$ such that $\inf_{x\in B_R} E_x^\rho 1(\sup_{t\le T}|X_t^1| > R) > 0$

III. The last step: we show that for T large enough

$$\inf_{x\in D} E_x^\rho 1(\sup_{t\le T}|X_t^1| > R) > 0.$$

We have,

$$E\sup_{t\le T}|X_t^1|^2 \ge E|X_T^1|^2 = x^2 + \int_0^T E\sigma_s^2 ds \ge c_1 T.$$

On the other hand,

$$\begin{aligned} E\sup_{t\le T}|X_t^1|^2 &= E\sup_{t\le T}|X_t^1|^2 1(\sup_{t\le T}|X_t^1|^2 \ge R^2) \\ &+ E\sup_{t\le T}|X_t^1|^2 1(\sup_{t\le T}|X_t^1|^2 < R^2) \\ &\le E\sup_{t\le T}|X_t^1|^2 1(\sup_{t\le T}|X_t^1|^2 \ge R^2) + R^2 \\ &\le (E\sup_{t\le T}|X_t^1|^4)^{1/2}(P(\sup_{t\le T}|X_t^1|^2 \ge R^2))^{1/2} + R^2. \end{aligned}$$

Proof of Lemma, ctd $\inf_{x\in B_R} E_x^\rho 1(\sup_{t\le T}|X_t^1|>R)>0$

$Y_t=\int_0^t \sigma_s d\tilde W_s$; $dY_t^4=4Y_t^3\sigma_t d\tilde W_t+6Y_t^2 dt$; $EY_t^4=6\int_0^t EY_s^2 ds$; $EY_s^2\le c_2 s$

We estimate $E\sup_{t\le T}|X_t^1|^4$ via Doob's inequality for continuous martingales ($E\sup_{t\le T}|M_t|^p\le C(p)EM_T^p$, $p>1$):

$$E\sup_{t\le T}|X_t^1|^4\le 2^3x^4+24C(4)(\int_0^T EY_s^2 ds)\le 2^3(R^4+3C(4)c_2T^2),$$

since $EY_t^4=6\int_0^t EY_s^2 ds\le 3c_2t^2$. Thus, from

$$c_1T\le (E\sup_{t\le T}|X_t^1|^4)^{1/2}(P(\sup_{t\le T}|X_t^1|^2\ge R^2))^{1/2}+R^2,$$

we find, for $T>R^2/c_1$,

$$(P(\sup_{t\le T}|X_t^1|^2\ge R^2))^{1/2}\ge\frac{c_1T-R^2}{(E\sup_{t\le T}|X_t^1|^4)^{1/2}}$$

$$\ge\frac{c_1T-R^2}{(8(R^4+3C(4)c_2T^2))^{1/2}}=c>0,\qquad \text{as required.}$$

Proof of Example 7.5, ctd. $L = \frac{1}{2}\frac{a_{ij}(x)\partial^2}{\partial x^i \partial x^j} + \frac{b^i(x)\partial}{\partial x^i}$

$u(X_t) - u(x) = \int_0^t \sigma^* \nabla u(X_s) dW_s + \int_0^t Lu(X_s)ds$ – "last equation", & $Lu = 0$

Resume our Example 7.5! It is also true that the lhs in the last equation equals the rhs if we integrate from 0 to $t \wedge \tau$:

$$u(x + X_{t\wedge\tau}) - u(x) = \int_0^{t\wedge\tau} \sigma^* \nabla u(X_s) dW_s.$$

Let us take expectations:

$$Eu(X_{t\wedge\tau}) - u(x) = E\int_0^{t\wedge\tau} \sigma^* \nabla u(X_s) dW_s = 0.$$

Since $u \in C_b^2(\bar{D})$, we obtain as $t \to \infty$,

$$u(x) = E_x u(X_\tau),$$

as required, where E_x stands to recall that the expectation is computed given the initial data $X_0 = x$.

Remark

Similar representations can be established for unbounded domains, in particular, for the complement B_R^c of any ball B_R under the assumption that, due to certain conditions,

$$E_x\tau < \infty,$$

where

$$\tau := \inf(t \geq 0 : X_t \in B_R).$$

This will be explored in the lectures about recurrence and ergodic properties.

Example 7.6, Poisson equation

Now $b \,\&\, \sigma$ do not depend on time; $L = \frac{1}{2}\frac{a_{ij}(x)\partial^2}{\partial x^i \partial x^j} + \frac{b^i(x)\partial}{\partial x^i}$, $a(x) = \sigma\sigma^*(x)$

Let D be a bounded domain in R^d. Consider the Poisson equation

$$Lu(x) = -\psi(x),\ x \in D, \quad \& \quad u(x)|_\Gamma = \phi(x),$$

where $\Gamma = \partial D$ is the boundary of D. Recall that $D^c := R^d \setminus D$, $\tau := \inf(t \geq 0 : X_t \in D^c)$.

Example (7.6)

Let $u(x) \in C_b^2(\bar{D})$ be a solution of the Poisson equation with $\phi \in C(\Gamma)$, $\psi \in C(\bar{D})$. Then $u(x)$ in D can be represented as

$$u(x) = E_x[\int_0^\tau \psi(X_s)ds + \phi(X_\tau)].$$

By Ito's formula, on the set $t < \tau$ we have,

$$\begin{aligned} du(X_t) &= \sigma^* \nabla u(X_t) dW_t + Lu(X_t)dt \\ &= \sigma^* \nabla u(X_t) dW_t - \psi(X_t)dt. \end{aligned}$$

So, in the integral form with a stopping time,

$$\begin{aligned} u(X_{t\wedge\tau}) - u(x) = \int_0^{t\wedge\tau} \sigma^* \nabla u(X_s) dW_s \\ - \int_0^{t\wedge\tau} \psi(X_s) ds. \end{aligned}$$

Taking expectations, we get

$$E_x u(X_{t\wedge\tau}) - u(x) = -E_x \int_0^{t\wedge\tau} \psi(X_s) ds.$$

Since[3] $\sup_x E_x\tau < \infty$ and letting $t \to \infty$, we have due to continuity of u, X and the integral wrt t and by virtue of Lebesgue's dominated convergence theorem,

$$E_x u(X_\tau) - u(x) = -E_x \int_0^\tau \psi(X_s)ds,$$

or, equivalently,

$$u(x) = E_x \psi(X_\tau) + E_x \int_0^\tau \psi(X_s)ds,$$

as required.

[3] A homework!

Let D be a bounded domain in R^d. Consider the Poisson equation with a (variable) potential $0 \leq c(x) \in C(\bar{D})$

$$Lu(x) - c(x)u(x) = -\psi(x),\ x \in D, \quad \& \quad u(x)|_\Gamma = \phi(x).$$

Denote $\kappa(t) := \int_0^t c(X_s)ds$. Recall that $D^c := R^d \setminus D$, $\tau := \inf(t \geq 0 : X_t \in D^c)$.

Example (7.7)

Let $u(x) \in C_b^2(\bar{D})$ be a solution of the Poisson equation with $\phi \in C(\Gamma)$, $\psi \in C(\bar{D})$. Then $u(x)$ in D can be represented as

$$u(x) = E_x[\int_0^\tau e^{-\kappa(s)}\psi(X_s)ds + e^{-\kappa(\tau)}\phi(X_\tau)].$$

By Ito's formula,

$$\begin{aligned} de^{-\kappa(t)}u(X_t) &= e^{-\kappa(t)}\sigma^*\nabla u(X_t)dW_t \\ &\quad + e^{-\kappa(t)}[Lu(X_t) - c(X_t)u(X_t)]dt \\ &= e^{-\kappa(t)}\sigma^*\nabla u(X_t)dW_t - e^{-\kappa(t)}\psi(X_t)dt. \end{aligned}$$

So, in the integral form with a stopping time,

$$\begin{aligned} e^{-\kappa(t\wedge\tau)}u(X_{t\wedge\tau}) - u(x) &= \int_0^{t\wedge\tau} e^{-\kappa(s)}\sigma^*\nabla u(X_s)dW_s \\ &\quad - \int_0^{t\wedge\tau} e^{-\kappa(s)}\psi(X_s)ds. \end{aligned}$$

Taking expectations, we get

$$E_x e^{-\kappa(t\wedge\tau)}u(X_{t\wedge\tau}) - u(x) = -E\int_0^{t\wedge\tau} e^{-\kappa(s)}\psi(X_s)ds.$$

From the equation

$$E_x e^{-\kappa(t\wedge\tau)} u(X_{t\wedge\tau}) - u(x) = -E \int_0^{t\wedge\tau} e^{-\kappa(s)} \psi(X_s)ds,$$

by letting $t \to \infty$, we obtain due to continuity of all terms in t, because of $\sup_x E\tau < \infty$, and by virtue of the Lebesgue dominated convergence theorem,

$$E_x e^{-\kappa(\tau)} u(X_\tau) - u(x) = -E \int_0^{\tau} e^{-\kappa(s)} \psi(X_s)ds,$$

or, equivalently,

$$u(x) = E_x e^{-\kappa(\tau)} u(X_\tau) + E_x \int_0^{\tau} e^{-\kappa(s)} \psi(X_s)ds,$$

as required. Note that the condition $c \geq 0$ was essential.

Lévy characterisation of WP, particular case

Recall the lemma inside the example 7.5

Recall that in the lemma we dealt with the process X_t satisfying under the probability measure $\tilde{P} = P^{\rho}$ the equation

$$X_t = x + \int_0^t \sigma(X_s) d\tilde{W}_s, \ t \geq 0.$$

Then the equation on X_t^1 reads,

$$dX_t^1 = \sigma_{1j}(X_t) d\tilde{W}_t^j = \underbrace{(\sum_j \sigma_{1j}^2(X_t))^{1/2}}_{=:\sigma_t} d\bar{W}_t,$$

$$\text{where} \qquad \bar{W}_t := \int_0^t \frac{\sum_i \sigma_{1i}(X_t) d\tilde{W}_t^i}{(\sum_j \sigma_{1j}^2(X_t))^{1/2}}$$

We pretend that $\boxed{\bar{W}_t \text{ is a WP}}$ & the equation on X_t^1 reads,

$$dX_t^1 = \sigma_t d\bar{W}_t.$$

Lévy characterisation of WP, particular case, proof

Recall that $\bar{W}_t$ is a continuous martingale,

$$\bar{W}_t := \int_0^t \frac{\sum_i \sigma_{1i}(X_t) d\tilde{W}_t^i}{(\sum_j \sigma_{1j}^2(X_t))^{1/2}}; \text{ let } \psi_t := \exp(i\lambda(\bar{W}_t - \bar{W}_r)), \; t > r.$$

We want to show that $\bar{W}_t$ is, in fact, a Wiener process. Consider a conditional expectation (conditional characteristic function) for $r < t$,

$$\phi(\lambda) := E(\exp(i\lambda(\bar{W}_t - \bar{W}_r))|\mathcal{F}_r)$$
$$= E_{X_r} \exp(i\lambda \int_r^t \frac{\sum_i \sigma_{1i}(X_s) d\tilde{W}_s^i}{(\sum_j \sigma_{1j}^2(X_s))^{1/2}}),$$

the latter equality due to the Markov property of the process X. It suffices to show $\phi(\lambda) = \exp(-\lambda^2(t-r)/2)$.

Lévy characterisation of WP, particular case, proof, ctd

Denote

$$f_s^i := \frac{\sum_i \sigma_{1i}(X_s)}{(\sum_j \sigma_{1j}^2(X_s))^{1/2}}; \text{ note that } \sum_i (f_s^i)^2 = 1.$$

By Ito's formula we have,

$$d\psi_t = d\exp(i\lambda(\bar{W}_t - \bar{W}_r)) = d\exp(i\lambda\int_r^t \sum_i f_s^i d\tilde{W}_s^i)$$

$$= \exp(i\lambda(\bar{W}_t - \bar{W}_r))(i\lambda \sum_i f_t^i d\tilde{W}_t^i - \frac{\lambda^2}{2}\underbrace{\sum_i (f_t^i)^2}_{=1} dt);$$

hence,

$$E_{X_r}\psi_t = 1 - \frac{\lambda^2}{2}\int_r^t E_{X_r}\psi_s ds \implies \phi(\lambda) = E_{X_r}\psi_t = e^{-\lambda^2(t-r)/2}, \; QED$$