Introduction to stochastic differential equations – 8
Recurrence and ergodic properties

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Spring 2020

May 18, 2020

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online mini-course

Recurrence for homogeneous SDE

$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,\ t \geq 0,\ \tau := \inf(t \geq 0 : X_t \in \bar{D})$

Definition

Markov process $X_t, t \geq 0$ is called recurrent if it returns to any neighbourhood D of the origin with probability one starting from any initial data $X_0 = x$. If for any x its expected time to hit D is finite, $E_x\tau_D < \infty$, then the process is called positive-recurrent; if not, it is called null-recurrent. The process is called m-recurrent ($m \geq 1$) if for any domain $D \ni 0$ and for any x, $E_x\tau_D^m < \infty$, and exponentially-recurrent if (for any $D \ni 0$) there exists $\alpha > 0$ (which may depend on D but not on x) such that $E_x \exp(\alpha\tau_D) < \infty$ for any x.

The higher moment of τ_D, the better recurrence, which, in turn, usually leads to a faster rate of convergence to the invariant measure if it exists. So, our first question is: how can we verify recurrence, m-recurrence, etc., if we know the coefficients of the SDE?

Approaches to check recurrence

$\tau = \inf(t \geq 0 : X_t \in D)$: e.g., $|X_0| > R$, & $D = (z : |z| = R)$, or $(z : |z| \leq R)$

Evaluating moments of τ by using bounds for solutions of PDEs: for example, to evaluate $E\tau$ it suffices to solve, *or to estimate* the solution of the Poisson equation

$$Lu = -1, \quad u|_\Gamma = 0.$$

Indeed, this -1 in the rhs gives us the integral $-E_x \int_0^\tau 1 ds$ after we apply Ito's formula to $u(X_t)$. In fact, it also suffices to solve (or evaluate the solution of) the equation (inequality)

$$\boxed{Lu \leq -1,} \quad u|_\Gamma = 0.$$

To evaluate $E \exp(\alpha\tau)$ to solve (or evaluate a solution of) the equation (inequality)

$$\boxed{Lu + \alpha u \leq -1,} \quad u|_\Gamma = 0.$$

The **Lyapunov functions** method is based on these ideas.

SDE recurrence with probability one

Recall the example 4.11 (lecture 4). WP is null-recurrent in dimensions 1 and 2.

SDE 1-recurrence

Recall the examples 4.9 and 4.10. This is an option to evaluate the first moment for the hitting time to touch some domain. Consider an SDE in R^d with a real-valued function $0 < c_1 \le h(x) \le c_2 < \infty$,

$$dX_t = b(X_t)dt + dW_t, \; x_0 = x, \quad \text{with } b(x) = -xh(x).$$

For $h(x) \equiv \text{const} > 0$ solution of such an SDE is called Ornstein – Uhlenbeck process. Let $B_R = (x : |x| \le R)$, $\tau_R := \inf(t \ge 0 : X_t \in B_R)$.

Example (8.1)

If $2R^2 c_1 > d$ (that is, $R^2 > d/(2c_1)$), then

$$E_x \tau_R \le C(R) x^2, \quad \text{with } C(R) = (2R^2 c_1 - d)^{-1}. \qquad (1)$$

Also, $E_x \int_0^{\tau_R} X_s^2 ds \le x^2(1 + dC(R))$.

Let $T_N := \inf(t \ge 0 : |X_t| \ge N)$, $V(x) = x^2$, and apply Ito's formula to $V(X_t)$ for $t < \tau_R$:

$$dV(X_t) = 2X_t dX_t + \frac{1}{2}\Delta V(X_t)dt$$
$$= (-2X_t X_t h(X_t) + d)dt + 2X_t dW_t.$$

Then in the integral form,

$$V(X_{t\wedge\tau_R\wedge T_N}) = V(x) + \int_0^{t\wedge\tau_R\wedge T_N} (-2X_s^2 h(X_s) + d)ds + \int_0^{t\wedge\tau_R\wedge T_N} 2X_t dW_t.$$

So,

$$E_x V(X_{t\wedge\tau_R\wedge T_N}) = V(x) + E\int_0^{t\wedge\tau_R\wedge T_N} (-2X_s^2 h(X_s) + d)ds.$$

Example 8.1, proof, ctd. $dX_t = -X_t h(X_t)dt + dW_t$

$E_x V(X_{t\wedge\tau_R\wedge T_N}) = V(x) + E\int_0^{t\wedge\tau_R\wedge T_N}(-2X_s^2 h(X_s) + d)ds$

Since $-2X_s^2 h(X_s) + d \le -2R^2 c_1 + d < 0$ on $s \le \tau_R$ by the assumption (recall, $\tau_R := \inf(t \ge 0 : X_t \in B_R)$), we conclude

$$E_x V(X_{t\wedge\tau_R\wedge T_N}) \le V(x) - E_x\int_0^{t\wedge\tau_R\wedge T_N}(2R^2 c_1 - d)ds$$
$$= V(x) - (2R^2 c_1 - d)E(t\wedge\tau_R\wedge T_N).$$

In other words,

$$(2R^2 c_1 - d)E_x(t\wedge\tau_R\wedge T_N) \le V(x) - E_x V(X_{t\wedge\tau_R\wedge T_N}) \le V(x).$$

Letting $N\to\infty$ and then $t\to\infty$, by Fatou's lemma we get,

$$(2R^2 c_1 - d)E_x\tau_R \le V(x),$$

as required. The process is positive recurrent.

For the sequel, we note that also it follows

$$E_x \int_0^{t\wedge\tau_R\wedge T_N} (2X_s^2 c_1 - d)ds \le V(x).$$

By the standard argument of Fatou's lemma, as $N \to \infty$ & $t \to \infty$ it implies that

$$2c_1 E_x \int_0^{\tau_R} X_s^2 ds \le V(x) + dE_x\tau_R$$
$$\le V(x)(1 + dC(R)).$$

We established another useful for the sequel bound

$$\boxed{E_x \int_0^{\tau_R} X_s^2 ds \le x^2(1 + dC(R)).}$$

Example (8.2)

Let dimension $d = 1$, then $\forall R > 0$, $\exists a > 0, \alpha > 0$ such that

$$E_x \exp(\alpha \tau_R) \le \exp(a(|x| - R)_+).$$

Proof. Consider the case $x \ge R$. Let us apply Ito's formula to $\exp(aX_t + \alpha t)$ with some $a > 0$:

$$d\exp(aX_t + \alpha t) = \exp(aX_t + \alpha t)[adX_t + \frac{1}{2}a^2(dX_t)^2 + \alpha dt]$$

$$= \exp(aX_t + \alpha t)[(-aX_t h(X_t) + \alpha + \frac{a^2}{2})dt + adW_t]$$

We have on $t \le \tau_R$ due to the assumptions on h,

$$(-aX_t h(X_t) + \alpha + \frac{a^2}{2}) \le -aRc_1 + \alpha + \frac{a^2}{2}.$$

Example 8.2, proof ctd. $\tau_R := \inf(t \geq 0 : X_t \in B_R)$

$(-aX_t h(X_t) + \alpha + a^2/2) \leq -aRc_1 + \alpha + \frac{a^2}{2}$; $T_N := \inf(t \geq 0 : |X_t| \geq N)$

Let us **choose** $a > 0$ small enough, so that $-aRc_1 + \frac{a^2}{2} < 0$, and $0 < \alpha < aRc_1 - \frac{a^2}{2}$. Then

$$(-aX_t h(X_t) + \alpha + \frac{a^2}{2}) \leq -aRc_1 + \alpha + \frac{a^2}{2} < 0.$$

We write, with stopping times $T_N = \inf(t \geq 0 : X_t \geq N)$,

$$\exp(aX_{t\wedge\tau_R\wedge T_N} + \alpha(t \wedge \tau_R \wedge T_N)) - \exp(ax)$$
$$= \int_0^{t\wedge\tau_R\wedge T_N} \exp(aX_s + \alpha s)[(-aX_s h(X_s) + \alpha + \frac{a^2}{2})ds + adW_s],$$

and taking expectations,

$$E_x \exp(aX_{t\wedge\tau_R\wedge T_N} + \alpha(t \wedge \tau_R \wedge T_N)) - \exp(ax)$$
$$= E_x \int_0^{t\wedge\tau_R\wedge T_N} \exp(aX_s + \alpha s)(-aX_s h(X_s) + \alpha + \frac{a^2}{2})ds,$$

Ctd

$\tau_R := \inf(t \geq 0 : X_t \in B_R)$; $T_N := \inf(t \geq 0 : |X_t| \geq N)$

which implies

$$E_x \exp(aX_{t\wedge\tau_R\wedge T_N} + \alpha(t \wedge \tau_R \wedge T_N)) - \exp(ax)$$
$$\leq -(aRc_1 - \frac{a^2}{2} - \alpha)E_x \int_0^{t\wedge\tau_R\wedge T_N} \exp(aX_s + \alpha s)ds,$$

or,

$$E_x \exp(aX_{t\wedge\tau_R\wedge T_N} + \alpha(t \wedge \tau_R \wedge T_N))$$
$$+(aRc_1 - \frac{a^2}{2} - \alpha)E_x \int_0^{t\wedge\tau_R\wedge T_N} \exp(aX_s + \alpha s)ds \leq \exp(ax).$$

since on $t \leq \tau_R$ we have $X_t \geq R$, we get

$$E_x \exp(aR + \alpha(t \wedge \tau_R \wedge T_N))$$
$$+(aRc_1 - \frac{a^2}{2} - \alpha)E_x \int_0^{t\wedge\tau_R\wedge T_N} \exp(aR + \alpha s)ds \leq \exp(ax).$$

Ctd

$\tau_R := \inf(t \geq 0 : X_t \in B_R)$; $T_N := \inf(t \geq 0 : |X_t| \geq N)$

$$E_x \exp(aR + \alpha(t \wedge \tau_R \wedge T_N)) \\ +(aRc_1 - \frac{a^2}{2} - \alpha)E_x \int_0^{t\wedge\tau_R\wedge T_N} \exp(aR + \alpha s)ds \leq \exp(ax)$$

can be rewritten by integration as

$$\exp(aR)E_x \exp(\alpha(t \wedge \tau_R \wedge T_N)) \\ + \exp(aR)\frac{(aRc_1 - (a^2/2) - \alpha)}{\alpha}E_x \exp(\alpha(t \wedge \tau_R \wedge T_N)) \\ \leq \exp(ax).$$

Letting $N \to \infty$, then $t \to \infty$, we get by Fatou's lemma,

$$\exp(aR)E_x \exp(\alpha\tau_R) \leq \exp(ax).$$

Example 8.2, ctd

$\tau_R := \inf(t \geq 0 : X_t \in B_R)$; $T_N := \inf(t \geq 0 : |X_t| \geq N)$

In the case $x \leq -R$, a similar calculus leads to a similar bound,

$$\exp(aR) E_x \exp(\alpha \tau_R) \leq \exp(a|x|).$$

In the case if the initial value $|x| \leq R$, we have $\tau_R = 0$, and the same inequality holds true trivially:

$$\exp(aR) E_x \exp(0) \leq \exp(a|x|).$$

Remark. A similar result may be proved in $d > 1$; however, some additional technicalities are needed to overcome the difficulties related to $|x|$.

Example (8.3)

Let dimension $d = 1$, then $\underline{\forall \alpha > 0, a > 0}$, $\exists R > 0$ such that

$$E_x \exp(\alpha \tau_R) \leq \exp(a(|x| - R)_+).$$

Proof, hint. The only change is to notice that for any fixed $\alpha > 0$ the inequality

$$-aRc_1 + \alpha + \frac{a^2}{2} < 0$$

holds with any $a > 0$ if R is chosen large enough.

NB: However, we cannot let $a \downarrow 0$ for $R > 0$ fixed!

Example 8.4

$\tau_R := \inf(t \geq 0 : X_t \in B_R)$; $T_N := \inf(t \geq 0 : |X_t| \geq N)$

Consider now an SDE in R^d

$$dX_t = h(X_t)dt + dW_t,\ t \geq 0, \quad X_0 = x$$

with a condition that h is bounded Borel measurable, and

$$1(|x| \neq 0)\frac{xh(x)}{|x|} \leq -1(|x| \neq 0).$$

For $d = 1$ it means $1(x \neq 0)h(x)\text{sign}(x) \leq -1(x \neq 0)$.

Example (8.4)

Let dimension $d = 1$, then $\forall \underline{R > 0}$, $\exists \alpha > 0, a > 0$ such that

$$E_x \exp(\alpha \tau_R) \leq \exp(a(|x| - R)_+).$$

Example 8.4, Proof

$\tau_R := \inf(t \geq 0 : X_t \in B_R)$; $T_N := \inf(t \geq 0 : |X_t| \geq N)$

Apply Ito's formula firstly to $\exp(aX_t + \alpha t)$, in the case $x \geq R$:

$$d\exp(aX_t + \alpha t) = \exp(aX_t + \alpha t)[(\alpha + ah(X_t) + \frac{a^2}{2})dt + adW_t].$$

On the set $t < \tau_R = \inf(s \geq 0 : |X_s| \leq R)$ we have,

$$(\alpha + ah(X_t) + \frac{a^2}{2}) \leq -a + \frac{a^2}{2} + \alpha.$$

For a small enough, $-a + \frac{a^2}{2} < 0$; choose such $a > 0$, and take $0 < \alpha < a - \frac{a^2}{2}$; denote $\kappa := a - \frac{a^2}{2} - \alpha\,(> 0)$. Then

$$\exp(aX_{t\wedge\tau_R\wedge T_N} + \alpha(t \wedge \tau_R \wedge T_N)) - \exp(ax)$$
$$= \int_0^{t\wedge\tau_R\wedge T_N} \exp(aX_s + \alpha s)[(\alpha + ah(X_s) + \frac{a^2}{2})ds + adW_s]$$

Example 8.4

$\tau_R := \inf(t \geq 0 : X_t \in B_R)$; $T_N := \inf(t \geq 0 : |X_t| \geq N)$

implies

$$E_x \exp(aX_{t\wedge\tau_R\wedge T_N} + \alpha(t \wedge \tau_R \wedge T_N)) - \exp(ax)$$
$$= E_x \int_0^{t\wedge\tau_R\wedge T_N} \exp(aX_s + \alpha s)(\alpha + ah(X_s) + \frac{a^2}{2})ds$$
$$\leq -\kappa E_x \int_0^{t\wedge\tau_R\wedge T_N} \exp(aX_s + \alpha s)ds.$$

Therefore,

$$E_x \exp(aX_{t\wedge\tau_R\wedge T_N} + \alpha(t \wedge \tau_R \wedge T_N))$$
$$+\kappa E_x \int_0^{t\wedge\tau_R\wedge T_N} \exp(aX_s + \alpha s)ds \leq \exp(ax).$$

Example 8.4

$\tau_R := \inf(t \geq 0 : X_t \in B_R)$; $T_N := \inf(t \geq 0 : |X_t| \geq N)$

Letting $N \to \infty$ and then $t \to \infty$ we obtain by Fatou's lemma

$$E_x \exp(aX_{\tau_R} + \alpha(\tau_R)) + \kappa E_x \int_0^{\tau_R} \exp(aX_s + \alpha s) ds \leq \exp(ax).$$

It follows that $\tau_R < \infty$ a.s., hence, $X_{\tau_R} = R$, and we get

$$E_x \exp(aR + \alpha(\tau_R)) + \kappa E_x \int_0^{\tau_R} \exp(aX_s + \alpha s) ds \leq \exp(ax),$$

which implies

$$\exp(aR) \left(E_x \exp(\alpha(\tau_R)) + \kappa E_x \int_0^{\tau_R} \exp(\alpha s) ds \right) \leq \exp(ax),$$

Example 8.4

$\tau_R := \inf(t \geq 0 : X_t \in B_R)$; $T_N := \inf(t \geq 0 : |X_t| \geq N)$

In particular, it follows,

$$E_x \exp(\alpha(\tau_R)) \leq \exp(a(x - R)).$$

In the case $x \leq -R$ similarly obtain the bound

$$E_x \exp(\alpha(\tau_R)) \leq \exp(a(|x| - R)).$$

For all values of x

$$E_x \exp(\alpha(\tau_R)) \leq \exp(a(|x| - R)_+),$$

as required.

A similar bound can be established for $d > 1$, too.

SDE polynomial recurrence

solution of an SDE may be positive recurrent under much weaker conditions

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \ t \geq 0.$$

Assume for simplicity $\sigma \equiv I$, and suppose

$$\boxed{\sup_{|x| \geq R} \langle x, b(x) \rangle \leq -r < 0.}$$

Example (8.5)

Let dimension $r > d/2$, $\tau_R = \inf(t \geq 0 : |X_t| \leq R)$. Then $\exists C > 0$ $(C = (2r - d))$, such that

$$E_x \tau_R \leq C(1 + |x|^2). \qquad (2)$$

(This suffice for the existence of the invariant measure.)

Apply Ito's formula to X_t^2 on $t < \tau$:

$$dX_t^2 = 2X_t dX_t + (dX_t)^2 = (2X_t b(X_t) + d)dt + 2X_t dW_t \\ \le (-2r + d)dt + 2X_t dW_t.$$

So, using the usual stopping procedure and $t \to \infty$, we get

$$E_x X_\tau^2 - x^2 \le -(2r - d)E_x\tau,$$

which implies by Fatou's lemma,

$$E_x\tau \le (2r - d)^{-1}x^2,$$

as required.

Note that for $r < d/2$ there is no invariant measure, so in some sense $r = d/2$ is a critical value.

Example 8.6: higher moments of $\tau = \tau_R$

$X_t = x + \int_0^t b(X_s)ds + W_t,\ t \geq 0.$

Assume again $\sigma \equiv I$, and suppose

$$\limsup_{|x|\to\infty} \langle x, b(x)\rangle = -\infty.$$

Example (8.6)

Let b be bounded, $\tau = \inf(t \geq 0 : |X_t| \leq R)$. Then $\underline{\forall k > 1}$, $\exists m > 0, R > 0, C > 0$, such that

$$E_x \tau_R^k \leq C(1 + |x|^m).$$

Without proof, because it is a bit more technically involved. The proof is based on Lyapunov functions of another type, $V(t, x) = (1 + t)^k x^m$.

Example 8.7 let $\sup_{|x|\ge R}\langle x, b(x)\rangle \le -r < 0$; $\tau = \tau_R$

$X_t = x + \int_0^t b(X_s)ds + W_t,\ t \ge 0.$; $E_x \int_0^\tau X_s^2 ds < \infty$? assume $d = 1$

Although evaluation of higher moments of τ is a bit involved, the estimation of $E_x \int_0^\tau X_s^2 ds < \infty$ is easier via $V(x) = x^4$.

Example (8.7)

If $d = 1$ & $r > 3/2$, then $E_x \int_0^\tau X_s^2 ds \le (4r-6)^{-1}|x|^4 < \infty$.

By Ito's formula,

$$dX_t^4 = 4X_t^3 dX_t + 6X_t^2(dX_t)^2 = (4X_t^3 b(X_t) + 6X_t^2)dt + 4X_t^3 dW_t$$
$$\le (-4X_t^2 r + 6X_t^2)dt + 4X_t^3 dW_t.$$

Using stopping times $t \wedge \tau \wedge T_N$, integrating, taking expectations, and using Fatou's lemma, we obtain

$$E_x X_\tau^4 + (4r-6)E_x \int_0^\tau X_s^2 ds \le |x|^4. \qquad \text{QED}$$

How to evaluate the moment $E_x\tau^2$ and higher ones via the lower ones? This is another idea based on so called Darling's or Dynkin's chain of equations (PDEs or ODEs). We just state the hint, without any theorem or example. Let $u_1(x) := E_x\tau$: as we know, $Lu_1(x) = -1, x \ge R, u_1(R) = 0$. Further,

$$E_x\tau^2 = 2E_x \int_0^\tau \int_t^\tau 1 ds dt = 2E_x \int_0^\infty 1(t<\tau) \int_t^\tau 1 ds dt$$

$$= 2\int_0^\infty E_x 1(t<\tau) \int_t^\tau 1 ds dt = 2\int_0^\infty E_x 1(t<\tau) E_{X_t} \int_0^\tau 1 ds dt$$

$$= 2\int_0^\infty E_x 1(t<\tau) u_1(X_t) dt = 2E_x \int_0^\tau u_1(X_t) dt.$$

Hence, $u_2(x) := E_x\tau^2$ is a solution of the equation

$$Lu_2(x) = -2u_1(x), \quad x \ge R, \ \& \ u_2(R) = 0.$$

Darling's or Dynkin's chain of equations

The role of "the second boundary conditions" for both equations is the claim that both functions u_1, u_2 grow no faster than some polynomials at $+\infty$.

The idea is now that if we can compute some upper bound for $u_1(x)$, it seems also reasonable that an upper bound for $u_2(x)$ may be found, too.

Further, by (possibly finite) induction it may be proved that

$$Lu_n(x) = -nu_{n-1}(x), \; x \geq R, \; \& \; u_n(R) = 0, \; 2 \leq n \leq N \leq \infty.$$

This may be the key for an evaluation of several moments of τ in the absence of any exponential moment. *Note, however, that if some exponential moment exists, it is usually much easier to estimate it by solving or evaluating the solution of just one single equation with a potential "of a wrong sign", $Lu + cu = -1$ with the appropriate boundary condition $u(R) = 0$.*

(left deliberately empty)

Invariant measure from positive recurrence

(Harris – Khasminsky principle)

If there exists $R > 0$ such that $\sup_{|x| \le R+1} E_x \tau_R < \infty$, then (under the condition of a locally uniform non-degeneracy of $a(t,x) = \sigma\sigma^*(t,x)$) there exists one invariant measure μ. Actually, the task may be simplified further if we assume $\sigma \equiv I$.

Coupling lemma
(Lemma on two r.v.)

Lemma (“Of two random variables”)

Let X^1 and X^2 be two random variables on their (without loss of generality different, which will be made independent after we take their direct product) probability spaces $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ and $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$ and with densities p^1 and p^2 with respect to some reference measure Λ, correspondingly. Then, if

$$q := \int \left(p^1(x) \wedge p^2(x)\right) \Lambda(dx) > 0,$$

then there exists one more probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two random variables on it $\tilde{X}^1, \tilde{X}^2$ such that

$$\mathcal{L}(\tilde{X}^j) = \mathcal{L}(X^j), \ j = 1, 2, \quad \& \quad \mathbb{P}(\tilde{X}^1 = \tilde{X}^2) = q.$$

This is a well-known technical tool in the coupling method.

Proof of Lemma

Assume $q < 1$, otherwise the Lemma is trivial

Let r.v. η_1, η_2, ξ, have the following densities:

$$p_{\eta^1}(t) = (1-q)^{-1}\left(p^1(t) - p^1(t)\wedge p^2(t)\right),$$
$$p_{\eta^2}(t) = (1-q)^{-1}\left(p^2(t) - p^1(t)\wedge p^2(t)\right),$$
$$p_{\xi}(t) = q^{-1}\left(p^1(t)\wedge p^2(t)\right).$$

Let ζ be a random variable independent of η^1, η^2 and ξ taking values in $\{0,1\}$ such that

$$P(\zeta=0)=q, \quad P(\zeta=1)=1-q.$$

Assume that $q \neq 0$ and $q \neq 1$ and let

$$\tilde X^1 := \eta^1 1(\zeta=1) + \xi 1(\zeta=0),$$
$$\tilde X^2 := \eta^2 1(\zeta=1) + \xi 1(\zeta=0).$$

Then $\tilde X^1 \overset{d}{=} X^1$, $\tilde X^2 \overset{d}{=} X^2$, and $P(\tilde X^1 = \tilde X^2) = q$. QED.

Ergodic theorem under Markov – Dobrushin's condition

[see A.Yu.V.,] Let $P_z(1, dz')$ be the transition kernel of the MC Z_n

Assume that

$$\kappa := \inf_{z_1, z_2} \int \left(\frac{P_{z_1}(1, dz')}{P_{z_2}(1, dz')} \wedge 1 \right) P_{z_2}(1, dz') > 0. \qquad (3)$$

Theorem (Ergodic)

Let the assumption (3) hold true. Then the process (Z_n) is ergodic, i.e., there exists a limiting probability measure ν, which is stationary and such that (3) holds true. Moreover, the uniform bound is satisfied for every n,

$$\sup_z \sup_{A \in \mathcal{S}} |P_z(n, A) - \mu(A)| \le (1 - \kappa)^n. \qquad (4)$$

(left deliberately empty)

Harnack inequality $(\Gamma_R = \partial B_R)$

Khasminsky's algorithm for invariant measure & Harnack inequality

Consider two sequences of stopping times with $E_x\gamma_n \stackrel{\forall n}{<} \infty$,

$$\tau_1 = \inf(t \ge 0 : |X_t| \le R),\ \gamma_1 = \inf(t \ge \tau_1 : |X_t| \ge R+1),$$
$$\tau_{n+1} = \inf(t \ge \gamma_n : |X_t| \le R),$$
$$\gamma_{n+1} = \inf(t \ge \tau_{n+1} : |X_t| \ge R+1), \ldots$$

Theorem (Elliptic Harnack inequality (Krylov & Safonov))

Under the boundedness of b, σ and nondegeneracy of σ, $\forall R > 0\ \exists C > 0$ (of course, $C \ge 1$) such that

$$\sup_{x,x' \in B_R} \sup_{A \subset \Gamma_{R+1}} \frac{P_x(X_{\gamma_1} \in A)}{P_{x'}(X_{\gamma_1} \in A)} \le C.$$

See [Krylov, Elliptic and parabolic PDEs], [Gilbarg, Trudinger, Elliptic PDEs]

Consider now the imbedded Markov chain

$$Z_n := X_{\gamma_n}, \quad n \ge 1.$$

From the Harnack inequality it follows that Z_n satisfies Markov – Dobrushin's condition. Hence, this chain has a unique invariant measure ν on Γ_{R+1}. For any $A \in \mathcal{B}^d$ let

$$\mu(A) := c \int_{\Gamma_{R+1}} \nu(dx) E_x \int_0^{\gamma_1} 1(X_t \in A)dt. \tag{5}$$

Theorem (Khasminsky)

If $\sup_{|x|=R+1} E_x\gamma_1 < \infty$, then the measure μ is invariant for X_t, and $c^{-1} = \int_{\Gamma_{R+1}} \nu(dx) E_x\gamma_1$.

For the proof see [Khasminsky, book]

Denote by T_t the semigroup acting on the measures as $\mu T_t(A) = \int \mu(dx) E_x 1(X_t \in A)$. Invariance of the measure μ means $\mu T_t = \mu$, $\forall\, t \geq 0$. As $\mathcal{L}(X_{T_1}) = \mathcal{L}(X_0) = \nu$, we have,

$$\begin{aligned}
\mu T_r &= \int_{\Gamma_{R+1}} \nu(dx) E_x \int_0^{\gamma_1} 1(X_{t+r} \in A) dt \\
&= \int_{\Gamma_{R+1}} \nu(dx) E_x \int_r^{\gamma_1 + r} 1(X_t \in A) dt \\
&= \int_{\Gamma_{R+1}} \nu(dx) E_x \int_0^{\gamma_1} 1(X_t \in A) dt \\
&\quad - \int_{\Gamma_{R+1}} \nu(dx) E_x \int_0^{r} 1(X_t \in A) dt \\
&\quad + \int_{\Gamma_{R+1}} \nu(dx) E_x \int_{\gamma_1}^{\gamma_1 + r} 1(X_t \in A) dt.
\end{aligned}$$

Invariant measure, Proof

(Harris – Khasminsky principle)

Due to the invariance of ν on Γ_{R+1} for the MC Z_n we get,

$$\int_{\Gamma_{R+1}} \nu(dx) E_x \int_{\gamma_1}^{\gamma_1+r} 1(X_t \in A)dt$$

$$= \int_{\Gamma_{R+1}} \nu(dx) E_x E_x\Big(\int_{\gamma_1}^{\gamma_1+r} 1(X_t \in A)dt | \mathcal{F}_{\gamma_1}\Big)$$

$$= \int_{\Gamma_{R+1}} \nu(dx) E_x E_x\Big(\int_{\gamma_1}^{\gamma_1+r} 1(X_t \in A)dt | X_{\gamma_1}\Big)$$

$$= \int_{\Gamma_{R+1}} \nu(dx) E_x E_{X_{\gamma_1}} \int_0^r 1(X_t \in A)dt$$

$$= \int_{\Gamma_{R+1}} \nu(dx) E_x \int_0^r 1(X_t \in A)dt.$$

Hence, the calculus on the previous page shows that

$$\mu T_r = \mu.$$

Invariant measure: condition $\int x^2 \mu(dx) < \infty$

Recall the additional bound from Example 8.1

$$\boxed{E_x \int_0^{\tau_R} X_s^2 ds \leq x^2(1 + dC(R)).} \quad (6)$$

Corollary

Under the same assumptions leading to (6),

$$\int x^2 \mu(dx) < \infty. \quad (7)$$

Invariant measure: condition $\int x^2\mu(dx) < \infty$

Proof: recall that $\sup_{|x|\le R} E_x\gamma_1 < \infty$.

Proof. We have (firstly, two reminders),

$$E_x \int_0^{\tau_R} X_s^2 ds \le x^2(1 + dC(R)); \qquad (6)$$

$$\mu(A) = c \int_{\Gamma_{R+1}} \nu(dx) E_x \int_0^{\gamma_1} 1(X_t \in A) dt; \qquad (5)$$

so, $$\int f(x)\mu(dx) := c \int_{\Gamma_{R+1}} \nu(dx) E_x \int_0^{\gamma_1} f(X_t) dt;$$

hence, $$\int x^2\mu(dx) := c \int_{\Gamma_{R+1}} \nu(dx) E_x \int_0^{\gamma_1} X_t^2 dt$$

$$= c \int_{\Gamma_{R+1}} \nu(dx) E_x \int_0^{\tau_R} X_t^2 dt$$

$$+c \int_{\Gamma_{R+1}} \nu(dx) E_x E_{X_{\tau_R}} \int_0^{\gamma_1} \underbrace{X_t^2}_{\le (R+1)^2} dt < \infty.$$

Indeed, e.g., from (2) (and a similar bound follows from (1)),

$$\int_{\Gamma_{R+1}} \nu(dx) E_x \int_0^{\tau_R} X_t^2 dt \le (R+1)^2(1+dC(R));$$

$$\int_{\Gamma_{R+1}} \nu(dx) E_x E_{X_{\tau_R}} \int_0^{\gamma_1} X_t^2 dt \le \sup_{|x|=R} E_x \int_0^{\gamma_1} X_t^2 dt$$

$$\le (R+1^2) E_x \gamma_1 < \infty,$$

the last inequality due to lemma from the lecture 3 (about a finite expectation for the exit time from a bounded domain for a Markov process) and another lemma from the lecture 7 (about a finite expectation for the exit time from a bounded domain for a Markov diffusion).

Invariance equation $L^* f(x) = 0$ writes (recall, $a(x) = \sigma^2(x)$),

$$(\frac{1}{2}a(x)f'(x))' - (b(x)f(x))' = 0.$$

In other words,

$$(\frac{1}{2}a(x)f'(x)) - (b(x)f(x)) = C_1 \qquad \text{(const)},$$

or,

$$f'(x) - \frac{2b(x)}{a(x)}f(x) = \frac{C_1}{a(x)}.$$

Let $f(x) = g(x)e(x) := g(x)\exp(\int_0^x \frac{2b(y)}{a(y)}dy)$, then

$$g'(x)e(x) + \frac{2b(x)}{a(x)}g(x)e(x) - \frac{2b(x)}{a(x)}g(x)e(x) = \frac{C_1}{a(x)}.$$

Hence,

$$g'(x) = \frac{C_1}{a(x)} e(x)^{-1} = \frac{C_1}{a(x)} \exp(-\int_0^x \frac{2b(y)}{a(y)}dy).$$

So,

$$g(x) = C_2 + \int_0^x \frac{C_1}{a(y)} \exp(-\int_0^y \frac{2b(z)}{a(z)}dz)dy,$$

and

$$f(x) = C_2 \exp(\int_0^x \frac{2b(y)}{a(y)}dy)$$
$$+ \exp(\int_0^x \frac{2b(y)}{a(y)}dy) \int_0^x \frac{C_1}{a(y)} \exp(-\int_0^y \frac{2b(z)}{a(z)}dz)dy$$
$$= C_2 \exp(\int_0^x \frac{2b(y)}{a(y)}dy) + \int_0^x \frac{C_1}{a(y)} \exp(\int_y^x \frac{2b(z)}{a(z)}dz)dy.$$

Let $C_1 = 0$**.** Then

$$\boxed{f(x) = C_2 \exp\left(\int_0^x \frac{2b(y)}{a(y)} dy\right).}$$

Assuming that

$$\int_{-\infty}^{\infty} \exp\left(\int_0^x \frac{2b(y)}{a(y)} dy\right) dx < \infty,$$

we find the normalizing constant

$$C_2^{-1} = \int_{-\infty}^{\infty} \exp\left(\int_0^x \frac{2b(y)}{a(y)} dy\right) dx.$$

Under conditions like $\lim_{|x| \to \infty}(xb(x)) = -\infty$, boundedness and nondegeneracy of σ this is a stationary density.

Invariant measure in $d = 1$: Examples

$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,\ t \ge 0, \quad d = 1, \quad \sigma$ is nondegenerate

I: let $b(x) = -\text{sign}(x)$, $\sigma \equiv 1$. Then $\int\limits_0^x \frac{2b(y)}{a(y)}dy = -2|x|$,

$$\int\limits_{-\infty}^{\infty} \exp(\int\limits_0^x \frac{2b(y)}{a(y)}dy)dx = 2\int\limits_0^{\infty} \exp(-2x)dx = 1.$$

II: $b(x) = -\frac{r\,\text{sign}(x)1(|x|>1)}{|x|}$, $\sigma \equiv 1$; $\int\limits_0^x \frac{2b(y)}{a(y)}dy = -2r(\ln|x|)_+$,

$$\int\limits_{-\infty}^{\infty} \exp(\int\limits_0^x \frac{2b(y)}{a(y)}dy)dx = 2 + 2\int\limits_1^{\infty} \exp(-2r\ln x)dx$$

$$= 2 + 2\int\limits_1^{\infty} x^{-2r}dx = 2 + \frac{2}{2r-1}.$$ ($\boxed{r > 1/2}$ compulsory).

Invariant measure in $d = 1$: Examples

Recall, $f(x) = C_2 \exp(\int_0^x \frac{2b(y)}{a(y)} dy) + \int_0^x \frac{C_1}{a(y)} \exp(\int_y^x \frac{2b(z)}{a(z)} dz) dy$.

Q: Why not use the term with C_1? There are examples [see S. Shaposhnikov] of non-uniqueness of solution of stationarity equations where "other solutions" are usually not everywhere positive, or do not determine a probability measure. Let us have a look at one of our examples above.

I: $b(x) = -\text{sign}(x)$, $\sigma \equiv 1$. Then $\int\limits_0^x \frac{2b(y)}{a(y)} dy = -2|x|$, and

$$\int\limits_0^\infty dx \int_0^x \frac{1}{a(y)} \exp(\int_0^x \frac{2b(z)}{a(z)} dz) \exp(-\int_0^y \frac{2b(z)}{a(z)} dz) dy$$

$$= \ldots = \int\limits_0^\infty dx\ e^{-2x} \frac{1}{2}(e^{2x} - 1) = \infty.$$

Hence, this term cannot serve as a density or as its part.

Invariant measure in $d > 1$: explicit formula

Special case of SDE: $\sigma = I$, $b = \nabla U$; $a = I/2$; $dX_t = dW_t + \nabla U(X_t)dt$

The stationarity equation reads,

$$\mathrm{div}(\frac{1}{2} I \nabla f) - \mathrm{div}(bf) = 0.$$

Let $\boxed{f(x) = c \exp(2U(x))}$, $\nabla U(x) = b(x)$. Then

$$\mathrm{div}(\frac{1}{2}\nabla f) - \mathrm{div}(bf) = \mathrm{div}(\frac{1}{2}\nabla f) - \mathrm{div}(f \nabla U)$$

We have,

$$c^{-1}(\frac{1}{2}\nabla f - f\nabla U) = \exp(2U(x))[\nabla U - \nabla U] = 0.$$

So, $\mathrm{div}(\frac{1}{2}\nabla f - f\nabla U) = 0$. This shows that the density $f(x) = c \exp(2U(x))$ is, indeed, stationary.

A little example: let $U(x) \equiv 0$. Then $f(x) \equiv 1$ is a stationary density, but the measure which corresponds to it is infinite.

What may $U(\cdot)$ be at ∞ s.t f were a probability density? $\sigma = I$, $b = \nabla U$; $a = I/2$; $dX_t = dW_t + \nabla U(X_t)dt$

If $U \geq 0$, $U(0) = 0$, $\lim_{|x|\to\infty}$, then $f(x) = c\exp(2U(x))$ is a stationary density. What is needed for this density to be a probability one? Consider the "radial symmetric" case where $U(x) = U(|x|)$. Then

$$c^{-1}\int f(x)dx = \int_{R^d} \exp(2U(x))dx = \int_{R_+} \exp(2U(t))S(d)t^{d-1}dt.$$

($S(d) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the "area" of 1-sphere in R^d.) Assume that

$$U(t) \overset{<}{\sim} -r\ln t,\ t \to \infty,\ r > 0.$$

For $\boxed{r > d/2}$ (which corresponds to $r > 1/2$ for $d = 1$),

$$c^{-1}\int f(x)dx = \int_0^\infty \exp(2U(r))S(d)t^{d-1}dt$$

$$" \asymp "\ 2\int_1^\infty t^{-2r}t^{d-1}dt\ " \asymp "\ 2\int_1^\infty t^{d-1-2r}dt < \infty.$$

(left empty)

What next: convergence to invariant measure

case $d = 1$; we concentrate on an SDE $dX_t = b(t, X_t)dt + dW_t$

So, let us consider a positive, or even m-recurrent, or even exponentially-recurrent solution of an SDE. What next? For a technical simplification and because of some lack of time we will deal only with dimension $d = 1$ here, although, similar results under appropriate assumptions are valid in $d > 1$, too. For a positive-recurrent process we know that there exists an invariant probability measure, say, μ. Let us show how to check that

$$\|\mu_t^x - \mu\|_{TV} \to 0,$$

and how to evaluate the rate of this convergence. For this aim we consider a couple of independent processes X_t and Y_t each of which solves the same SDE (above in the subtitle), but with different initial data: let $X_0 = x$, and let Y_0 have a stationary distribution μ. They both solve the same SDE but with different independent Wiener processes, say, W_t for X and $\tilde{W}_t$ for Y; weak solutions suffice.

Positive recurrence for the pair: intersection

Let $X_0 = x$ and $\mathcal{L}(Y_0) = \mu$ (the stationary measure)

The pair (X_t, Y_t) is a Markov and strong Markov process on the product probability space. Let us consider the following stopping time:

$$T := \inf(t \geq 0 : X_t = Y_t \text{ and } |X_t| = |Y_t| \leq R).$$

The value R will be chosen in a while.

Lemma (8.7)

Let appropriate recurrence assumptions hold true and, in particular, $\int x^2 \mu(dx) < \infty$. Then

$$P(T < \infty) = 1,$$

and, moreover,

$$E_{x,\mu} T \leq C(1 + |x|^2).$$

Intersection: idea of proof

NB: under stronger assumptions higher moments of T can be estimated

Idea is very simple (although the realisation is technically a bit involved). We already know that $E_x\tau_R^X < \infty$, and also $E_\mu\tau_R^Y < \infty$, where τ_R^X stands for the hitting time of X to touch $\bar{B}_R$, while τ_R^Y stands for the similar hitting time of Y. Let us denote $\tau_R^{X\wedge Y} := \tau_R^X \wedge \tau_R^Y$. Clearly, $\tau_R^{X\wedge Y} < \infty$ a.s., and, more than that, $E_{x,\mu}\tau_R^{X\wedge Y} \le C(1+x^2)$. So, we wait until the first process hits $\bar{B}_R$. *It can be shown* that the second process at this moment is on average no further away from $\bar{B}_R$ than it was at $t = 0$. Hence, we now wait until the second process also hits $\bar{B}_R$; this time may be estimated by virtue of Bienaymé – Chebyshev – Markov inequality. In the meanwhile, there is a positive probability that the first process remains in B_{R+1}. If this occurs (at the moment when the second process hits $\bar{B}_R$), we wait another *unit of time*. During this unit of time there is a positive probability that the trajectories of X and Y intersect, which is T.

Intersection: idea of proof, ctd.

Alternatively, if the first process decides to leave B_{R+1}, we stop both of them at this moment of hitting B^c_{R+1} for the first process. After that, we repeat the first step again.

If the event of intersection does not happen, we stop the processes either at the end of this unit interval of time, or at the moment of one them them hitting B^c_{R+1}. Overall this time does not exceed 1, this is a stopping time, and at this moment both processes do not exceed $R+1$ by modulus. Hence, we repeat this step again: wait until one of the hits $\bar B_R$ (if both are outside in the beginning), then wait until the second also hits $\bar B_R$, remembering that with a positive probability the first one remains, at least, in $\bar B_{R+1}$. If this is achieved, we wait whether or not their trajectories intersect during another unit of time interval. A careful estimation allows to conclude that $E_{x,y} T \le C(1+x^2+y^2)$, and due to the condition $\int y^2 dy < \infty$, we obtain the desired bound.

(left empty)

Upper bound for the total variation

Coupling inequality

The pair (X, Y) is Markov *and strong Markov*. Hence, define the new process

$$\tilde{X}_t = X_t 1(t \le T) + Y_t 1(t > T).$$

Due to the strong Markov property it is also (strong) Markov, has the same generator as X, and has the same distribution in the space of trajectories. So,

$$
\begin{aligned}
|P_x(X_t \in A) - P_\mu(Y_t \in A)| &= |P_x(\tilde{X}_t \in A) - P_\mu(Y_t \in A)| \\
= |E_x 1(\tilde{X}_t \in A) - E_\mu 1(Y_t \in A)| &= |E_{x,\mu}(1(\tilde{X}_t \in A) - 1(Y_t \in A))| \\
= E_\mu 1(Y_t \in A)| &= |E_{x,\mu}(1(\tilde{X}_t \in A) - 1(Y_t \in A))1(t \le T)| \\
\le P_{x,\mu}(T \ge t) &\le (t^{-1} E_{x,\mu} T \wedge 1) \le (t^{-1} C(1 + x^2)) \wedge 1.
\end{aligned}
$$

Note that the rhs here does not depend on A.

Total variation bounds

Corollary (8.8)

Under the same conditions,

$$\|\mu_t^x - \mu\|_{TV} \leq \frac{2C(1+|x|^2)}{t} \wedge 2.$$

Corollary (8.9)

Suppose $E_{x,\mu} T^k \leq C(x)$, $k > 0$. *Then*

$$\|\mu_t^x - \mu\|_{TV} \leq 2P_{x,\mu}(T > t) \leq \frac{2E_{x,\mu}T^k}{t^k} \wedge 2 \leq \frac{2C(x)}{t^k} \wedge 2.$$

If $E_{x,\mu} \exp(\alpha T) \leq C(x)$, *then*

$$\|\mu_t^x - \mu\|_{TV} \leq 2P_{x,\mu}(T > t) \leq (2C(x)\exp(-\alpha t)) \wedge 2.$$

Multidimensional SDE case?

In the case $d > 1$ it is not possible to use intersections. This case will be considered separately in the next lecture.