# HSE Masters Exam
## Sample Variant 2. Solutions

**1.** *Convert the decimal number* $30_{10}$ *to binary.*

*Solution.* It suffices to convert $32_{10}$ to binary and then subtract 2. Since $32_{10} = 2^5$, its binary representation is 100000. When we subtract 1, we get 11111. When we subtract another 1, we get 11110.

**Answer:** 11110.

**2.** *Find the remainder when* $(14+18)\cdot(23-2)$ *is divided by* 11. *Write your answer as a nonnegative integer from* 0 *to* 10.

*Solution.* Essentially, we are asked to find the value of $(14+18)\cdot(23-2)$ *modulo* 11. Recall, that in *modulo arithmetic* we say that integers $a$ and $b$ are equivalent modulo $m$ iff $m$ divides $a-b$. We denote this equivalence by $a \equiv b \pmod m$. Furthermore, if $a \equiv a' \pmod m$ and $b$ is an arbitrary integer, then:

$$\begin{aligned} a+b &\equiv a'+b \pmod m \\ a\cdot b &\equiv a'\cdot b \pmod m \end{aligned}$$

Therefore, when computing a value of an expression modulo $m$, one may safely substitute any number and any subexpression with an equivalent one. Let us apply this idea to our problem. We have:

$$\begin{cases} 14 \equiv 3 \pmod{11}, \\ 18 \equiv -4 \pmod{11}, \\ 23 \equiv 1 \pmod{11}, \\ 2 \equiv 2 \pmod{11}, \end{cases}$$

hence

$$(14+18)\cdot(23-2) \equiv (3-4)\cdot(1-2) \equiv (-1)\cdot(-1) \equiv 1 \pmod{11}.$$

**Answer:** 1.

**3.** *The market price of a washing machine in March was* \$2000. *In April its price rose by* 30%. *Then, in May a shopkeeper allowed a discount of* 30% *on the machine. Find its selling price in May after the discount.*

*Solution.* The price in March was $x = 2000$ (in dollars). In April it rose by 30% and became equal to $y = 1.3x$. In a 30% was allowed, hence the price became equal to $z = 0.7y = 0.7\cdot 1.3x = 0.7\cdot 1.3\cdot 2000 = 1820$.

**Answer:** 1820.

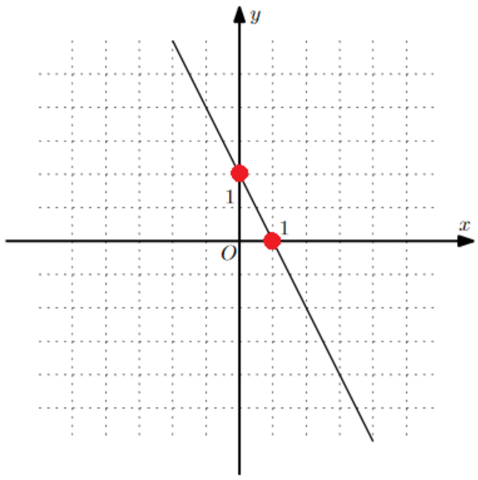
**4.** *Graph of which function (a, b, c or d) is given in the xy-plane below?*

*a)* $y = 2x - 2$;

*b)* $y = \frac{x}{2} + 2$;

*c)* $y = -\frac{x}{2} + 2$;

*d)* $y = -2x + 2$.




*Solution.* The line in the picture passes through points $(0; 2)$ and $(1; 0)$ (highlighted with red). This means that when we substitute $x = 0$ to the expression of the function, we should obtain $y = 2$, and when we substitute $x = 1$, we should obtain $y = 0$. The only function from the list which satisfies both properties is *d)* $y = -2x + 2$.

**Answer:** d.

**5.** *Given the parabola* $y = -x^2 + 4x - 4$,

*a) find the minimal x-coordinate of its intersection with the x-axis;*

*b) find the y-coordinate of its maximum.*

*Solution.*

a) the $x$-coordinates of the points where the parabola intersect the $x$-axis are exactly the solutions of

$$-x^2 + 4x - 4 = 0.$$

Observe, that $-x^2 + 4x - 4 = -(x^2 - 4x + 4) = -(x - 2)^2$. Hence the only solution of the above equation is $x = 2$.

**Answer:** 2.

b) Since the parabola opens downwards (we have $a = -1 < 0$), its maximum point is its vertex. And since (as we have found several lines above) our parabola has a unique root $x = 2$, this root is exactly the x-coordinate of the vertex. Therefore, the y-coordinate of the vertex is 0.

**Answer:** 0.

**6.** *Find the value of* $x$ *if* $2 \cdot 2^{x+1} + 4 \cdot 2^x = 2^{2x+1}$.

*Solution.* Using $2 \cdot 2^{x+1} = 2^{x+2}$ and $4 \cdot 2^x = 2^{x+2}$, we rewrite the left part of the equation as $2^{x+2} + 2^{x+2}$, which is, in turn, equal to $2 \cdot 2^{x+2} = 2^{x+3}$. Hence we have an equation:

$$2^{x+3} = 2^{2x+1} \iff x + 3 = 2x + 1 \iff x = 2.$$

**Answer:** 2.

**7.** *Find the largest root of the quadratic equation* $x^2 - x - 20 = 0$.

*Solution.* By Vieta's formulae, the solutions of the quadratic equation $ax^2 + bx + c = 0$ are exactly those numbers $x_1$ and $x_2$ which satisfy $x_1 + x_2 = -b/a$ and $x_1 \cdot x_2 = c/a$. In this case we have:

$$\begin{cases} x_1 + x_2 = 1 \\ x_1 \cdot x_2 = -20. \end{cases}$$

It is not hard to see that this implies $x_1 = 5$, $x_2 = -4$, and $\max\{5, -4\} = 5$.

**Answer:** 5.

**8.** *Solve the inequality* $\log_2(x\sqrt{x} + 5\sqrt{x}) + log_{1/2}\sqrt{x} > 2$. *Choose the right answer:*

*a)* $[-1, +\infty)$*;*

*b)* $(0, +\infty)$*;*

*c)* $[1, +\infty)$*;*

*d)* $(-\infty; -1]$*;*

*e)* $(-\infty; 0)$*;*

*e)* $(0; 1]$.

*Solution.* First of all, we have to make sure that the final answer satisfies the constraints $x\sqrt{x} + 5\sqrt{x} > 0$, $\sqrt{x} > 0$ and $x \geqslant 0$, so that our expressions with logarithms and square roots make sense. Having that in mind, let us use the well-known properties of logarithms:

$$\log_a = -\log_{1/a} b$$

$$\log_a b - \log_a c = \log_a \frac{b}{c}$$

We rewrite the left part of the inequality as:

$$\log_2(x\sqrt{x} + 5\sqrt{x}) + \log_{1/2}\sqrt{x} = \log_2(x\sqrt{x} + 5\sqrt{x}) - \log_2\sqrt{x} =$$

$$= \log_2 \frac{x\sqrt{x} + 5\sqrt{x}}{\sqrt{x}} = \log_2(x + 5).$$

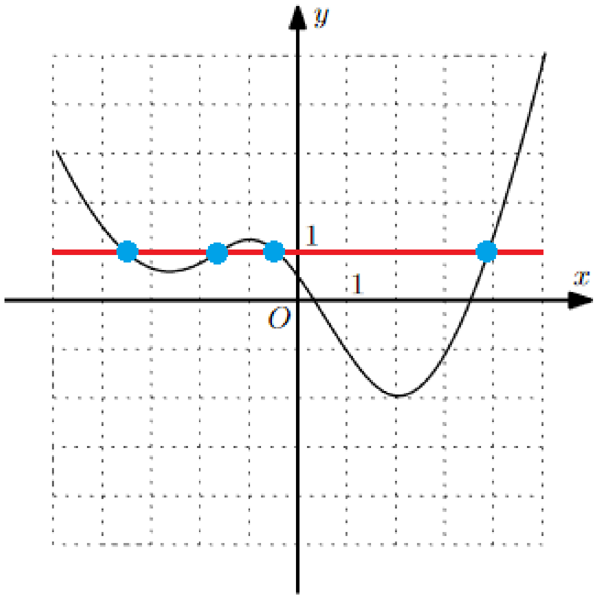
Therefore, it suffices to solve

$$\log_2(x + 5) > 2 \quad \Leftrightarrow \quad x + 5 > 4 \quad \Leftrightarrow \quad x > -1.$$

Recall, that the answer must satisfy $x\sqrt{x} + 5\sqrt{x} > 0$, $\sqrt{x} > 0$ and $x \geqslant 0$, hence the correct answer to the problem is $x > 0$.

**Answer:** b.

**9.** *The complete graph of the function* $y = f(x)$ *is shown in the xy-plane below. Find* $x$ *such that the value* $f(x)$ *is the smallest possible. What is the number of all preimages of* $y = 1$*? Express your answers as integers separated by a comma.*




*Solution.* It is easy to see from the picture that the smallest possible value is $-2$, which is achieved at $x = 2$. To answer the second question, recall, that a preimage of $y = 1$ is such value $x_0$ that $f(x_0) = 1$. There are exactly four points (highlighted with blue in the picture) where the graph of $f(x)$ crosses the line $y = 1$.

**Answer:** 2, 4.

**10.** *A geometric progression has 625 as the first term. The product of its first 3 terms is equal to the product of its first 6 terms. Find the common ratio of the progression. Give your answer as a decimal fraction rounded to one decimal place.*

*Solution.* Let $b_i$ denote the $i$th term of the progression (with $b_1 = 625$ being the first term) and let $q$ be its common ratio. We have that

$$b_1 b_2 b_3 = b_1 b_2 b_3 b_4 b_5 b_6,$$

hence

$$b_4 b_5 b_6 = 1.$$

Further, by the definition of a geometric progression, $b_i = b_1 \cdot q^{i-1}$, $i \geqslant 1$. Therefore we obtain

$$b_1^3 \cdot q^{12} = 1,$$

$$q^{12} = \frac{1}{b_1^3} = \frac{1}{625^3} = \frac{1}{5^{12}}.$$

Thus $q = \frac{1}{5} = 0.2$.

**Answer:** 0.2.

**11.** *Given $f(x) = x^3 - 5x + 8$, find $f'(-1)$.*

*Solution.* Let us use the standard rules to find the derivative $f'(x)$ as a function of $x$:

- $\big(g(x) + h(x)\big)' = g'(x) + h'(x)$ for any two functions $g(x)$ and $h(x)$;

- $\big(c \cdot g(x)\big)' = c \cdot g'(x)$ for any function $g(x)$ and any constant $c$;
- $(x^n)' = n \cdot x^{n-1}$ for $n \geqslant 1$;
- $c' = 0$ for any constant $c$.

Applying these rules, we obtain

$$f'(x) = (x^3 - 5x + 8)' = 3x^2 - 5.$$

When evaluated at $x = -1$, this function gives us $f'(-1) = -2$.

**Answer:** $-2$.

**12.** *Calculate the binary number that equals* $101001_2 - 11110_2 + 10110_2$. *Write your answer as a binary number.*

*Solution.* The calculation is easy to perform bitwise:

$$\begin{array}{r} 101001 \\ -\ \ 11110 \\ \underline{+\ \ 10110} \\ 100001 \end{array}$$

**Answer:** 100001.

**13.** *There is a swimming pool and three hoses: red, green, and blue. It takes 3 hours to fill the pool with the red hose. It also takes 45 minutes to fill it with the red and blue hoses at same time and 90 minutes to fill it with red and green hoses at same time. How long does it take to fill the pool with all three hoses at same time? Write your answer in minutes.*

*Solution.* Let the pool have volume $V$ and let $r$, $b$ and $g$ be the volumes of water pumped in one minute by the red, blue and green hoses respectively. Since 3 hours equals 180 minutes, we have:

$$\frac{V}{r} = 180, \quad \frac{V}{r+b} = 45, \quad \frac{V}{r+g} = 90.$$

Therefore, $\dfrac{r+g}{r} = 2$ and $\dfrac{r+b}{r} = 4$, hence $g = r$ and $b = 3r$. The task is to compute

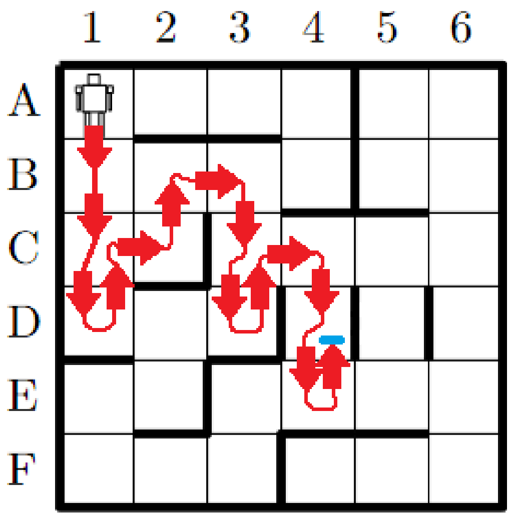
$$\frac{V}{r+b+g} = \frac{V}{5r} = \frac{1}{5} \cdot \frac{V}{r} = \frac{180}{5} = 36.$$

**Answer:** 36.

**14.** *A robot is placed in the maze below in A1 position. It is programmed to do the following:*

- *if the robot is able to move down, then it moves down;*
- *if the robot is not able to move down, then it makes one step up and one step right;*
- *if the robot is not able to move right or up, then it stops and the program terminates.*

*Where will the robot stop after executing the program? Give your answer as a pair of a letter and a number, e.g. B4.*




*Solution.* To find the answer it suffices to execute the program step by step as shown in the picture.

**Answer:** *D*4.

**15.** *There are two operations:*

***A****: multiply a given number by 2 and add 1;*

***B****: add 3 to a given number.*

*How to get* 60 *from* 1 *in* 7 *steps using operations* ***A*** *and* ***B****? Write your answer as a sequence of symbols* ***A*** *and* ***B****. (Example:* ***ABB*** *turns* 1 *to* 9*).*

*Solution.* We are to find a sequence of eight numbers, such that the first number is 1, the last is 60, and each number except the first is a result of applying operation **A** or **B** to the previous. One can solve this problem using a pure intuition. However, it is interesting to use a couple of mathematical observations to guide this intuition to the right place.

Since 60 is even and operation **A** always outputs an odd number, the last operation in the sequence must be **B**. Which means that we need to convert 1 to 57 in 6 steps. Since 57 is an odd number, one can (feeling optimistic) try to assume that the previous operation was **A**. With this assumption, it suffices to obtain 28 from 1 in 5 steps.

Our strategy of solving this problem will be as follows. When possible, we assume **A** to be the previous operation (as it brings us closer to 1 — the beginning of the sequence), otherwise we have to assume **B** as the previous operation. This strategy appears to be successful:

$$60 \xleftarrow{\mathbf{B}} 57 \xleftarrow{\mathbf{A}} 28 \xleftarrow{\mathbf{B}} 25 \xleftarrow{\mathbf{A}} 12 \xleftarrow{\mathbf{B}} 9 \xleftarrow{\mathbf{A}} 4 \xleftarrow{\mathbf{B}} 1$$

It now suffices to read the operation (backwards!) from the obtained sequence.

**Answer:** *BABABAB*.

**16.** *James has* \$1000 *and wants to invest it in a project. He knows that each dollar brings* \$2 *income per month. He may rent a number of billboards for* \$100 *each. Each billboard increases one dollar's income by* \$1 *per month. Find the James' maximal total month income. Write your answer in dollars.*

*Solution.* Let $x$ denote the number of billboards rented by James. The total cost of the rent is then $100x$, so James has $1000 - 100x$ dollars left. The income of one dollar is now $2 + x$, hence James' total month income equals

$$(1000 - 100x)(2 + x) = -100x^2 + 800x + 2000$$

dollars.

To answer the question it now suffices to find the maximum of this function of $x$, whose graph is a parabola. Since this parabola opens downwards, the maximum is achieved at its vertex. For a parabola of the form $y = ax^2 + bx + c$ the well-known formula tells the x-coordinate of its vertex:

$$x_v = -\frac{b}{2a}.$$

In our case we have $x_v = -\frac{800}{-200} = 4$. Hence the maximum function's value is $-100x_v^2 + 800x_v + 2000 = 3600$.

**Answer:** 3600.

**17.** *A polynomial $p(x) = (x^4 - 3x^3 - x + 1)^9$ is written in standard form. Find the sum of all its coefficients.*

*Solution.* Suppose that $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ (the "standard form"). Therefore,

$$p(1) = a_n + a_{n-1} + \ldots + a_1 + a_0.$$

Therefore, $p(1)$ is the answer to the problem, and $p(1) = -2^9 = -512$.

**Answer:** $-512$.

**18.** *Let us say that an integer $n$*

- *is of type **A** if it is divisible by 4 **or** is divisible by 6;*
- *is of type **B** if it is even **and** is divisible by 9.*

*Which of the following statements are true?*

1. *If a number is of type **A**, then it is also of type **B**;*
2. *If a number is of type **B**, then it is also of type **A**;*
3. *70 is not of type **A** and is not of type B;*
4. *If $a$ is of type **A** and $b$ is of type **B**, then $45a + 20b$ is always divisible by 90.*

*Write down the numbers of the true statements in ascending order.*

*Solution.*

1. Statement 1 is false: 4 is of type **A** (since it is divisible by 4) but not of type **B** (since it is not divisible by 9).
2. Statement 2 is true. Indeed, let $b$ be of type **B**. Then b is divisible by 9, hence by 3. Furthermore, $b$ is even, hence divisible by 2. Since 2 and 3 are coprime, $b$ is divisible by $2 \cdot 3 = 6$, and hence is of type **A**.

3. Statement 3 is true: 70 is no divisible by 9, hence it is not of type **B**, and it is not divisible by neither 4 nor 6, hence it is not of type **A**.

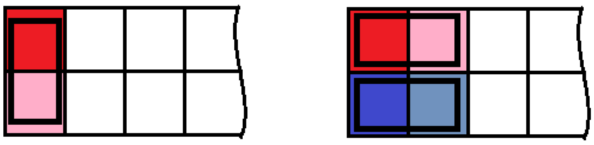
4. Statement 4 is also true. Indeed, $a$ is divisible either by 4 or 6, but it is in any case divisible by 2. Hence $a = 2a'$ and $45a = 90a'$ is divisible by 90. Further, $b$ is divisible by 9, hence $b = 9b'$ and $20b = 180b'$ is also divisible by 90. Therefore, the sum $45a + 20b$ is divisible by 90.

**Answer:** 234.

**19.** *How many ways are there to tile a $2 \times 7$ rectangle with 7 dominoes (identical $1 \times 2$ rectangles)?*

*Solution.* Let $w(n)$ be the number of ways to tile a $2 \times n$ rectangle with $n$ dominoes. We are to compute $w(7)$.

There are two options to start a tiling: either put one domino vertically, or two dominoes horizontally:



The remaining $2 \times (n-1)$ (in the first case) and $2 \times (n-2)$ rectangles may be tiled in $w(n-1)$ and $w(n-2)$ ways, respectively.

Therefore, we obtain a recursive formula $w(n) = w(n-1) + w(n-2)$ for $n \geqslant 3$. It is also clear, that $w(1) = 1$ and $w(2) = 2$. What remains is to fill the table using our recursive formula:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $w(n)$ | 1 | 2 | 3 | 5 | 8 | 13 | 21 |

**Answer:** 21.

**20.** *Find the minimal positive integer $n$ such that the decimal representation of $n \cdot 18$ consists of all twos.*

*Solution.* We have

$$18 \cdot n = 222\ldots22.$$

Let us divide both sides of the equation by 2. We thus obtain

$$9 \cdot n = 111\ldots11$$

A well-known divisibility-by-nine rule asserts that a number with decimal representation $\overline{d_1 d_2 \ldots d_k}$, where $d_1, d_2, \ldots, d_k$ are its decimal digits, has the same remainder, when divided by 9, as the number $d_1 + d_2 + \ldots + d_k$.

In the equation above the left part is divisible by 9, hence so is the right part. But this means that the some of 1's on the right side is also divisible by 9. This is true iff there are $9m$ 1's on the right side, and, since $n$ must be positive, $m > 0$. To obtain a minimal possible $n$, we put $m = 1$.

Hence

$$9n = 111111111 \quad \Longleftrightarrow \quad n = \frac{111111111}{9} = 12345679.$$

**Answer:** 12345679.