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Dissipativity analysis of explicit linearized finite-difference methods with regularization for gas dynamics equations

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General summary of the work

Statement of the problem. Numerical modeling of gas and fluid flows is one of the most important problems in mathematical modelling. There is a vast literature dedicated to numerical methods of solving systems of gas dynamics equations, see inter alia, Godunov S.K., Zabrodin A.V., Ivanov M.Ya., et al. (1976); Magomedov K.M., Kholodov A.S. (1988); Toro E.F. (2009); LeVeque R.J. (2004); Kulikovskii A.G., Pogorelov N.V., Semenov A.Yu. (2012); Goloviznin V.M. et al. (2013); Abgrall R., Shu C.-W. (eds., 2016).

One of the classes of such methods is connected to preliminary regularization of these systems of equations. One can attribute to those methods a so called quasi-gasdynamic (QGD) regularization, based inter alia on kinetic considerations, and simpler quasi-hydrodynamic (QHD) regularization. There are dissipative terms of special form with a small regularization parameter in these equations. This approach and based on it explicit two-level in time and three-point symmetric in space (in each of the spatial variables) finite-difference schemes and various applications are encountered in monographs of Chetverushkin B.N. (2004); Elizarova T.G. (2007); Sheretov Yu.V. (2009); Elizarova T.G., Shirokov I.A. (2017) and large number of papers. Meanwhile, explicit different schemes are of special interest for effective computer modeling on high performance computers. Nonetheless, the theory of such finite-difference schemes is developed comparably weakly.

Actuality of the theme. In the papers: Zlotnik A.A., Chetverushkin B.N. (2008); Zlotnik A.A.(2008) the Petrovsky parabolicity criteria were derived for QGD and QHD systems of equations respectively, as well as the problem of stability of small perturbations around a constant background was studied and the uniform in time estimates of the solutions of linearized (about a constant solution) variants of such systems were established. Inter alia part of the obtained estimates can be interpreted as L^2-dissipativity of the solutions of the Cauchy problem for the linearized systems. These estimates can be treated as the starting point of the present dissertation dedicated to deriving analogues of those estimates for approximating explicit difference schemes under the proper conditions on the mesh steps.

Besides, in the same papers simplified barotropic variants of QGD and QHD systems of equations were constructed, and analogous results were obtained for them. Later on in the paper: Zlotnik A.A. (2010) energetic equalities and estimates were obtained for such systems of equations, and in the paper: Zlotnik A.A. (2012) a simple derivation of QGD system of equations was proposed both in general and barotropic case. Note that the analysis of the barotropic case is a common approach in many questions of the theory of gas dynamics equations that supports gradual development of research technique. Along with that barotropic systems of equations also have their own significance and are relatively widely used in some applications, including one-dimensional (1D) and two-dimensional (2D) shallow water equations (corresponding to the isentropic case with adiabatic exponent $\gamma = 2$), equations of flow of two-phase two-component media, usually considered in the isothermic case (adiabatic exponent $\gamma = 1$) and others.

Difference methods for those and other barotropic problems based on QGD and QHD regularizations were constructed and used inter alia in the papers: Bulatov O.V., Elizarova T.G. (2011); Elizarova T.G., Ivanov A.V. (2016); Balashov V., Zlotnik A., Savenkov E. (2017); Zlotink A., Gavrilin V. (2016); Elizarova T.G., Zlotnik A.A., Istomina M.A. (2018).

In the papers: Elizarova T.G., Shilnikov E.V. (2009); Elizarova T.G., Shirokov I.A. (2017); Kraposhin M., Smirnova E., Elizarova T. et al. (2018) a verification of the finite-difference schemes based on the QGD approach was accomplished on a number of well-known 1D test problems, representing distinctive features of non-stationary non-viscous gas flows.

New spatial discretizations based on QGD and QHD approaches possessing additional property of energetic or entropic conservativeness (dissipativity) were constructed in one-dimensional case in the papers: Zlotnik A.A. (2012), Gavrilin V.A. (2015), firstly for the case of barotropic, and later on also full systems of gas dynamic equations.

Level of progress in the theme. Stability issue is a central one in the theory of finite-difference schemes, and for a number of finite-difference methods of gas dynamics, mostly for model problems, is well presented in the literature, see inter alia Richtmyer R., Morton K. (1972); Godunov S.K., Ryabenkii V.S. (1977); Gustaffson B., Kreis H.-O., Oliger J. (1995); Ganzha V.G., Vorozhtsov E.V. (1996); LeVeque R.-J. (2004); Bakhvalov N.S., Zhidkov N.P., Kobelkov G.M. (2008); Coulombel J.-F. (2009); Martsinkevich G.L., Matus P.P., Chuiko M.M. (2010); Abgrall R., Shu C.-W. (eds., 2016). Inter alia, the spectral approach described in monographs: Godunov S.K., Ryabenkii V.S. (1977); Richtmyer R., Morton K. (1972) allows to obtain stability criteria for finite-difference schemes via reduction of the original problem to the corresponding algebraic one. Usually this approach is applied either to the Cauchy problem or to the problem with periodicity conditions over spatial variables.

As for the finite-difference methods based exactly on the QGD regularizations, the L^2-dissipativity of the explicit finite-difference schemes for 1D gas dynamic equations was studied by Sheretov Yu.V. (2004, 2009), and for the shallow water equations by Suhomozgii A.A., Sheretov Yu.V. (2013). However, only the case of zero background speed, that is, zero Mach number $M = 0$, was considered. It significantly simplifies the research but is completely insufficient for applications. Besides, in the first two papers only the case of Schmidt number equal to 1 was covered, and in the third one the general barotropic case was not considered.

In these papers the method of energetic inequalities had been used that allowed to obtain only sufficient conditions of L^2-dissipativity. The question about the corresponding necessary conditions and connected question about accuracy of obtained sufficient conditions remained open. Besides, it is difficult to generalize the proposed approach for 2D and 3D cases, which are much more important in practice.

Objectives and problems of the research. The main aim of this dissertation is the dissipativity analysis of linearized explicit finite-difference methods with QGD and QHDD regularizations for gas dynamics equations.

For achieving this aim the following problems are stated:

1. Developing a variant of spectral approach for dissipativity analysis of explicit finite-difference schemes both with convective and regularizing (viscous) terms.

2. Deriving criteria, necessary conditions, sufficient conditions of L^2-dissipativity for linearized explicit finite-difference schemes based on QGD and QHD regularizations of 1D barotropic system of gas dynamics equations, and proving the corresponding theorems for any background Mach number M.

3. Deriving criteria, necessary conditions, sufficient conditions of L^2-dissipativity of linearized explicit finite-difference schemes based on QGD regularization of 1D full system of gas dynamics equations for any M and general assumptions on the scheme parameters. Accomplishing verification of entropic-dissipative schemes and analysis of the applicability of the derived conditions in the non-linear statement. Besides, deriving L^2-dissipativity criteria for the schemes based on some other regularizations.

4. Deriving criteria, necessary conditions, sufficient conditions of L^2-dissipativity of linearized explicit finite-difference schemes based on QGD regularizations of 2D and 3D barotropic and full system of gas dynamics equations, for any M and general assumptions on the scheme parameters.

Scientific significance. In the present work, a new variant of spectral technique is developed for analysis of L^2-dissipativity of the Cauchy problem solutions for explicit two-level finite-difference schemes with both convective and regularizing terms.

For the first time, on that basis criteria and simpler necessary conditions of L^2-dissipativity of linearized explicit finite-difference schemes based on QGD regularizations of barotropic and full system of equations are obtained. Also, for the first time the corresponding sufficient conditions are obtained for any background Mach number M; for $M = 0$ they significantly improve already known ones in the 1D case.

The listed results are obtained not only in 1D case, but for the first time in the 2D and 3D cases, as well as under general assumptions on the parameters of the schemes. Along with this, the best choice of regularization parameter is analyzed, and it is shown that under adequate choice of that the conditions on the time step are uniform in M that it is an important new property of QGD-regularization.

Additionally, criteria, necessary conditions, sufficient conditions of L^2-dissipativty of linearized explicit finite-difference schemes based on QHD regularizations in the 1D barotropic case, as well as some other regularizations for the 1D barotropic and full equations for any M are obtained.

Computational study is accomplished concerning practical applicability of the obtained conditions in the nonlinear 1D statement of barotropic and full equations for various different schemes, both of standard type and with energetic- or entropic-dissipative spatial discretizations, and considerable advantages of the latter ones are in a number of tests, including better correspondence to the obtained conditions.

Theoretical and practical significance is as follows.

1. A variant of spectral approach for L^2-dissipativity analysis of explicit two-level finite-difference schemes with both convective and viscous terms taking into account the commutator of their matrices.

2. Criteria and simpler both necessary conditions and sufficient conditions of L^2-dissipativity of linearized explicit finite-difference schemes based on QGD-regularizations for $1D$ full system of gas dynamics equations are derived for any background Mach number M and general assumptions on the parameters of the schemes.

3. Also, criteria, necessary conditions and sufficient conditions of L^2-dissipativity for explicit schemes based on QHD regularization in 1D barotropic case and some other regularizations in 1D barotropic and general cases are derived for any M.

4. Criteria and simpler both necessary conditions and sufficient conditions of L^2-dissipativity of linearized explicit finite-difference schemes based on QGD regularizations for the 2D and 3D barotropic and full system are derived for any M.

5. The derived necessary conditions and sufficient conditions of L^2-dissipativity in the form of the Courant type conditions may be used for adequate choice of parameters in QGD and QHD regularizations of gas dynamics equations in practical computations. This will potentially allow to significantly reduce time consumption both for the right adjustment of these parameters and for running the computations themselves, inter alia on multiprocessor computational systems.

Research methodology and methods. In the dissertation some methods of mathematical and functional analysis (theory of Hilbert spaces) are used. Matrix analysis including a number of theorems about eigenvalues of symmetric matrices and analysis of inequalities with Hermitian matrices is applied. Methods of discretization of systems of partial differential equations are applied. A variant of spectral approach is developed for analysis of L^2-dissipativity. A classification of de facto stability of 1D numerical solutions in non-linear statement is proposed, based on analysis of their variation in space.

For analysis, verification and visualization of various work results the computer algebra tools and program package Wolfram Mathematica were used. code implementation of the difference schemes is accomplished on the basis of methods of object-oriented programming using the language C# that allowed to easily include various difference schemes in the analysis.

Basic results presented to be defended.

1. Criteria of L^2-dissipativity of linearized explicit finite-difference schemes based on QGD and QHD regularizations of 1D barotropic system of gas dynamics equations are given for the case of zero Mach number $M = 0$, as well as criteria and both necessary and sufficient conditions of L^2-dissipativity are obtained in the case of any M.

2. Criteria and both necessary conditions and sufficient conditions of L^2-dissipativity of linearized explicit finite-difference scheme based on the QGD regularization of 1D full system of gas dynamics equations are obtained in the case of any M.

A code implementation of various finite-difference schemes is accomplished in non-linear statement, their verification is done, and the applicability of the found conditions of L^2-dissipativity in non-linear statement is analyzed.

3. Criteria of L^2-dissipativity for linearized finite-difference schemes based on some other regularizations of 1D barotropic and full system of gas dynamics equations are also obtained for any M.

4. Criteria and both necessary and sufficient conditions L^2-dissipativity of linearized explicit finite-difference schemes based on QGD regularization of the 2D and 3D barotropic and full system of gas dynamics equations are obtained for any M.

Reliability. Reliability of the theoretical results of the work is verified by rigorous mathematical proofs of the corresponding theorems. Reliability of the numerical verification is verified by comparing the results with the well-known analytic and other numerical solutions.

Probation of the work. The results of the work were reported on the following conferences:

— Annual interuniversity scientific-technical conference of students, PhD students and young specialists in the name of E.V. Armensky (Moscow, MIEM NRU HSE, 2017, 2018, 2019)

— XVII and XVIII All-Russian conferences-schools “Modern problems of mathematical modelling” (Durso, Russia, 2017 and 2019)

— 23rd International Conference Mathematical Modelling and Analysis (Druskininkai, Lithuania, 2017)

— 24th International Conference Mathematical Modelling and Analysis (Sigulda, Latvia, 2018)

— Fall School Hyperbolic Conservation Laws and Mathematical Fluid Dynamics (Julius-Maximilians-Universität, Würzburg, Germany, 2018)

— 25th International Conference Mathematical Modelling and Analysis (Tallinn, Estonia, 2019)

— International scientific conference “Modern problems of computational mathematics and mathematical physics” dedicated to 100-th anniversary of academician A.A. Samarsky (Moscow, 2019).

References to the published abstracts of reports are given in the dissertation text.

Author's personal contribution. Theoretical results in the dissertation on deriving conditions of L^2-dissipativity of finite-difference schemes, based on QGD and QHD regularizations of systems of gas dynamics equations, are obtained by the applicant in collaboration with A.A. Zlotnik. The corresponding results for other regularizations as well as the code implementation of the finite-difference schemes presented in the dissertation and running numerical experiments, constructing various plots and tables are done by the applicant personally.

Size and structure of the work. The dissertation consists of Introduction, three chapters, Conclusion and Appendix. The full size of the dissertation is 118 pages. The work includes 26 figures and 6 tables. Bibliography contains 67 titles.

Main inferences of the research. In Chapter 1 L^2-dissipativility of discretizations of the one-dimensional barotropic QGD and QHD systems of equations is analyzed. In Section 1.1 the simplest case of such discretizations is studied in the case of the zero background speed. In Theorem 1 a necessary spectral condition of von Neumann type is derived, and in Theorem 2 a spectral criterion of dissipativity for discretization of QGD system of equations is proved. In Theorem 3 both necessary condition of von Neumann type and the spectral criterion for discretization of the QHD system of equations are derived. Even in the simplest case the necessary von Neumann condition turns out to differ from the criterion and is not a sufficient one. Note that the discretization of the QGD system remains stable even for the zero Schmidt number $\alpha_s = 0$ (i.e., the zero artificial Navier-Stokes viscosity), but the discretization of the QHD system does not. Besides, practical probation of the obtained criteria is accomplished for two finite-difference schemes: a scheme of standard type and an "enthalpic" scheme, where the spatial discretization possesses the property of dissipativity with respect to energy. The experimental results for the second scheme correspond well to the obtained criterion. The results for the scheme of the standard type are inferior to them despite the fact that the schemes coincide in the linearized variant.

The conclusion that the necessary von Neumann condition is not sufficient leads to the necessity of developing a new variant of spectral approach. Such an approach for abstract explicit two-level finite-difference schemes both with convective and regularizing (viscous) terms is described in Section 1.2. Theorem 4 is proved where criterion (necessary and sufficient condition), and simpler both necessary condition and sufficient condition of L^2-dissipativity of the mentioned abstract schemes are derived. Some remarks are also added to the theorem, that allow to simplify verification of the derived conditions.

In Section 1.3 discretizations for the 1D barotropic QGD and QHD system of equations with arbitrary background speed are analyzed and criteria as well as simpler both necessary and sufficient conditions of dissipativity are derived: in Theorem 5 for the discretization of the QGD system of equations, and in Theorem 6 for the discretization of the QHD system of equations. Necessary conditions differ from the corresponding sufficient ones by at most twice. It is shown that under the adequate choice of the regularization parameter in the QGD system of equations the conditions on time step are uniform in M that is an important new property of the QGD regularization. Note that unlike the QGD regularization, the QHD regularization of the equations does not possess this property. Also, practical verification of the obtained conditions is accomplished and their applicability in nonlinear 1D statement is analyzed for various finite-difference schemes, both of the standard type and two schemes with energetically dissipative spatial discretizations. Classification of de facto stability of 1D numerical solutions in non-linear statement is done on the basis of analysis of their variation in space. For the scheme of standard type stable results are much worse than those corresponding to the necessary condition, and moreover, the correspondence strongly worsens as the Mach number grows. For the other two schemes the results are significantly better. These results point out the advantage of using in practice namely the energetically dissipative schemes, especially as the Mach number grows.

In Chapter 2 L^2-dissipativity of discretizations of the 1D full QGD system of equations is analyzed. In Section 2.1 a discretization of the 1D QGD system of equations is written. In Theorem 7 both necessary condition and sufficient condition of L^2-dissipativity of the discretization in the case of zero background speed are derived(these conditions differ by at most twice), and in Theorem 8 the corresponding conditions are derived in the case of arbitrary background speed. As is in the barotropic case, for the full system of equations under the correct choice of regularization parameter the conditions can also be written as uniform with respect to Mach number.

In Section 2.2 verification of entropically dissipative schemes and their comparison with the scheme of the standard type (all of them are discretizations of the full QGD equations) are accomplished on a Riemann problem. The Riemann problem, where the initial data are piecewise-constant discontinuous functions, is traditionally used to analyze the properties of numerical methods for solving gas dynamics equations. Five tests are considered which reflect specific situations in the arising flows. For a version of Sod's test, additionally, practical rates of convergence in mesh L^1-norm are analyzed, and they are shown to be close to 0.5. It turns out that for the both entropically dissipative schemes the results are mainly similar, and they run well on these tests. Inter alia in the Einfeldt test during computations by the entropically dissipative schemes the entropy wake (nonphysical local maximum of internal energy) is very small, whereas the computations under other QGD (and not only) schemes are far from such a result. In the other tests the entropically dissipative schemes demonstrate numerical solutions of better (on the same mesh) and/or that larger time steps (sometimes up to two orders of magnitude) can be used than for the QGD schemes of standard type.

The applicability of the obtained conditions of L^2-dissipativity in non-linear statement is also analyzed. Numerical experiments for a model Riemann problem are run, and, besides, the deviation of their relative variations from 1 at the final time moment is used to analyze the quality of the numerical solutions. It is inferred that to a certain extent the conditions can be used as well in the non-linear statement as the Mach number grows, at least, in some computations. As a rule the conditions are more adequate for the entropically dissipative QGD discretizations, whereas for the QGD schemes of standard type they overestimate the admissible time step (though their linearization is the same). The accomplished analysis of L^2-dissipativity still holds even when the so called Schmidt number and inverse Prandtl number are zero. Recall that the case of zero Schmidt number corresponds to the absence of artificial Navier-Stokes viscosity, and in the conclusion of Section 2.2 it is shown that for that case numerical solutions given by the entropically dissipative schemes are not destroyed (though the results are not the best) unlike the schemes of standard type when this as a rule happens.

In Section 2.3 the spectral approach is applied for some other regularizations of gas dynamics equations. Just recently in the M. Svärd (2018) paper some new model of the equations for viscous conductive gas flow has been proposed. In Theorem 9 a criterion of L^2-dissipativity of linearized explicit two-level discretization for the model is derived for the 1D barotropic case, and in Theorem 10 this is done in the case of the 1D full system of gas dynamics equations with artificial viscosity. It turns out that in both cases the criterion has the same form, and, moreover, it cannot be written as uniform in Mach number. In the remarks to the theorem it is stated how to apply the obtained results when the physical viscosity or the combination of physical and artificial viscosity is present. Their applicability to some other regularizations of systems of gas dynamics equations is also discussed.

In Chapter 3 L^2-dissipativity of explicit two-level three-point in each spatial direction discretization of linearized multidimensional (2D or 3D) quasi-gasdynamic (QGD) system of equations on a uniform rectangular mesh is analyzed. Generalization to the multidimensional case significantly complicates the problem as n matrices of convective terms and $\frac{n(n+1)}{2}$ matrices of viscous terms arise, $n = 2, 3$. In Section 3.1 discretization of the barotropic QGD system of equations is analyzed. In Theorem 11 a multidimensional spectral criterion of L^2-dissipaitivity is derived. In Theorem 12 a criterion (necessary and sufficient condition) as well as simpler necessary condition and sufficient condition of L^2-dissipativity of explicit two-level abstract schemes are proved. In Theorem 13 a necessary dissipativity condition for discretization of the barotropic QGD system is derived. It is analyzed how two ways of choosing the "average" step of the rectangular mesh in the regularization parameter affect it together with the choice of optimal value of the regularization parameter. The sufficient condition depends on the upper estimate of the maximal eigenvalue of matrix that is the symbol of viscous terms. In Theorem 14 such an estimate is derived, and that concludes the analysis.

Besides, 2D numerical experiments on practical analysis of the stability of the linearized schemes and matching the results with the necessary L^2-dissipativity condition are accomplished on the mesh with the big scatter of the cell size. Such scatters often arise during the automatic mesh generation in the domains of complex shape, and they may significantly negatively affect the choice of time step.

In Section 3.2 L^2-dissipativity of the explicit two-level discretization of linearized full 2D and 3D QGD-systems of equations is analyzed. Note that here all matrices of convective and regularizing terms have 4th and 5th order in 2D and 3D cases respectively. In Theorem 15 a criterion (necessary and sufficient) condition as well as simpler necessary condition and sufficient condition of L^2-dissipativity of explicit two-level abstract schemes are given. In Theorem 16 necessary condition of L^2-dissipativity is derived. The sufficient condition looks analogously to the barotropic case and also depends on the maximal eigenvalue of matrix that is the symbol of viscous terms. In Theorem 17 the upper estimate of this eigenvalue is derived. An important property is the ability to write the obtained conditions as uniform in M both for barotropic and full equations.

List of publications on the topic of the dissertation. The main results on the topic of dissertation are presented in [1–6]. All papers are indexed in Scopus and/or WoS, and journal [6] has rating Q1, journals [2–4] have rating Q2 in Scopus and/or WoS, journal [5] has rating Q3 in Scopus.

1. *Zlotnik A., Lomonosov T.* On conditions for weak conservativeness of regularized explicit finite-difference schemes for 1D barotropic gas dynamics equations // Differential and difference equations with applications. ICDDEA 2017. Springer Proceedings in Mathematics & Statistics. Vol. 230. Cham: Springer International Publishing AG, 2018. P. 635—647.

2. *Zlotnik, A.A., Lomonosov, T.A.* On conditions for L^2-dissipativity of linearized explicit QGD finite-difference schemes for one-dimensional gas dynamics equations // Dokl. Math. 2018. Vol. 98, no. 2, P. 458–463.

3. *Zlotnik, A.A., Lomonosov, T.A.* Conditions for L^2-dissipativity of linearized explicit difference schemes with regularization for 1D barotropic gas dynamics equations // Comput. Math. Math. Phys. 2019. Vol. 59., no. 3, P. 452–464.

4. *Zlotnik A., Lomonosov T.* Verification of entropic QGD-schemes for 1D gas dynamics equations // Math. Model. Anal. 2019. Vol. 24, no. 2. P. 179—194.

5. *Lomonosov T.* L^2-dissipativity criteria for linearized explicit finite difference schemes for regularization of one-dimensional gas dynamics equations // J. Math. Sci. 2020. Vol. 244, no. 4. P. 97—102.

6. *Zlotnik A., Lomonosov T.* L^2-dissipativity of the linearized explicit finite-difference scheme with a kinetic regularization for 2D and 3D gas dynamics system of equations // Appl. Math. Lett. 2020. Vol. 103.