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CONTRIBUTIONS TO THE THEORY OF VARIETIES OF SEMIGROUPS

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1 General Description of Thesis

Relevance of topics of thesis

A *variety* is a class of algebras that is closed under the formation of homomorphic images, subalgebras, and arbitrary direct products. By the celebrated theorem of Birkhoff [10], varieties are precisely equationally defined classes of algebras. The theory of varieties of semigroups—which include periodic varieties of groups—have been systematically and intensely investigated since the 1960s. A number of surveys have been published on various aspects of varieties of semigroups, such as equational properties and the finite basis problem [104, 114, 115], structural properties of semigroups in varieties [102], lattices of varieties [2, 17, 103, 110], and algorithmic problems [36]. Varieties of semigroups endowed with additional operations have also received much attention; see, for example, the monographs or surveys on varieties of groups [21, 74], varieties of involution semigroups [12], varieties of inverse semigroups [78], and varieties of completely regular semigroups [79].

There are three finiteness properties the investigation of which is highly popular in the theory of varieties: a variety is

- *finitely based* if its equational theory is finitely axiomatizable,
- *finitely generated* if it is generated by a finite algebra, and
- *small* if it contains finitely many subvarieties.

As observed by Sapir [93], these properties are independent in the sense that a variety that satisfies any two properties need not satisfy the third. A variety that satisfies all three properties is *Cross*.

The thesis consists of three parts (Parts I, II, and III) devoted to varieties of semigroups, varieties of involution semigroups, and varieties of monoids. To distinguish these three types of varieties, let $\mathsf{V}_{\mathsf{sem}}\mathfrak{C}$, $\mathsf{V}_{\mathsf{inv}}\mathfrak{C}$, and $\mathsf{V}_{\mathsf{mon}}\mathfrak{C}$, denote the variety of semigroups, variety of involution semigroups, and variety of monoids generated by

$\mathfrak{C}$, respectively. For any variety $\mathfrak{V}$, let $\mathfrak{L}(\mathfrak{V})$ denote the lattice of subvarieties of $\mathfrak{V}$. Results of the thesis can be roughly classified into four groups:

1) Rees–Suschkewitsch varieties;

2) aperiodic monoids with central idempotents;

3) sufficient conditions for equational properties;

4) results on involution semigroups.

Details of each group are given in the following.

1) Rees–Suschkewitsch varieties. The class of completely 0-simple semigroups was one of the first classes of semigroups to be studied, in the pioneering work of Rees [86] and Suschkewitsch [107], and remains one of the most significant classes of semigroups. For each $n \geq 1$, let $\mathbf{RS}_n$ denote the variety generated by all completely 0-simple semigroups over groups of exponent n. Following Kublanovskiĭ [40], semigroups in $\mathbf{RS}_n$ are called *Rees–Suschkewitsch semigroups* and subvarieties of $\mathbf{RS}_n$ are called *Rees–Suschkewitsch varieties*. One important result, due to Hall et al. [24] in the 1990s, is that the variety $\mathbf{RS}_n$ is finitely based. But not all Rees–Suschkewitsch varieties are finitely based, as non-finitely based examples exist in abundance [7, 34, 38, 61, 69, 70]. Since the publication of Hall et al. [24], Rees–Suschkewitsch varieties have received much attention; see, for instance, Kublanovskiĭ [40], Lee [42–45], Reilly [87–92], Volkov [117], and their collaborations [41, 60, 61, 63, 64].

One huge obstacle—and a unique characteristic—in the study of Rees–Suschkewitsch varieties is that not all of them are generated by completely simple or completely 0-simple semigroups, whence the convenient Rees Theorem is not applicable. Rees–Suschkewitsch varieties that are generated by completely simple or completely 0-simple semigroups are said to be *exact*. Even though exact Rees–

Suschkewitsch varieties have been fully characterized [89], a complete structural description of the lattice $\mathfrak{L}(\mathbf{RS}_n)$ of subvarieties of $\mathbf{RS}_n$ seems hopeless. In fact, this task is highly infeasible even in the basic case $n = 1$: only 13 subvarieties of $\mathbf{RS}_1$ are exact [89], but it follows from Vernikov and Volkov [111] that the variety $\mathbf{RS}_1$ is *finitely universal* in the sense that lattice $\mathfrak{L}(\mathbf{RS}_1)$ embeds every finite lattice. The following open problems are thus of fundamental importance, given especially that no subvariety of $\mathbf{RS}_1$ is known to be non-finitely based.

Problem 1. Investigate the lattice $\mathfrak{L}(\mathbf{RS}_1)$ and describe its varieties that are finitely based, finitely generated, or small.

Problem 2 (Jackson [27, Question 4.8])**.** Does the variety $\mathbf{RS}_1$ contain continuum many subvarieties?

The subvarieties of $\mathbf{RS}_1$ exclude all nontrivial groups and so are said to be *aperiodic*.

2) Aperiodic monoids with central idempotents. Over the years, varieties of monoids have received much less attention than varieties of semigroups. For several decades, the most understood varieties of monoids are probably the variety $\mathbb{C}\text{OM}$ of commutative monoids [25] and the variety $\mathbb{I}\text{DEM}$ of idempotent monoids [118]. The situation changed after the mid-1990s when the study of Rees quotients of free monoids became popular: for any set $\mathscr{W}$ of words over a countably infinite alphabet $\mathscr{A}$, let $\mathcal{R}_{\mathcal{Q}}\mathscr{W}$ denote the Rees quotient of the free monoid $\mathscr{A}^*$ over the ideal of all words that are not factors of any word in $\mathscr{W}$, and let $\mathbb{R}_{\mathcal{Q}}\mathscr{W} = \mathsf{V}_{\mathsf{mon}}\{\mathcal{R}_{\mathcal{Q}}\mathscr{W}\}$.

Rees quotients of free monoids first appeared in the work of Perkins [76] back in the 1960s; one of the first published examples of non-finitely based finite semigroups is the monoid

$$\mathcal{P}_{25} = \mathcal{R}_{\mathcal{Q}}\{xyxy,\, xyzyx,\, xzyxy,\, x^2z\}$$

of order 25. But it was not until the turn of the millennium when, in the pioneering work of Jackson [27, 30], these monoids were considered at the varietal level. The monoid $\mathcal{R}_{\mathcal{Q}}\{xyx\}$ of order seven provides the most notable results: the variety $\mathsf{V}_{\mathsf{sem}}\{\mathcal{R}_{\mathcal{Q}}\{xyx\}\}$ of semigroups contains continuum many subvarieties [27] while the variety $\mathsf{V}_{\mathsf{mon}}\{\mathcal{R}_{\mathcal{Q}}\{xyx\}\}$ of monoids contains only five subvarieties [30]. Jackson [30] also proved that the varieties

$$\mathbb{J}_1 = \mathbb{R}_{\mathcal{Q}}\{xhxyty\} \quad \text{and} \quad \mathbb{J}_2 = \mathbb{R}_{\mathcal{Q}}\{xhytxy,\, xyhxty\}$$

are minimal non-finitely based varieties or *limit varieties*; these are in fact the first published examples of limit varieties of monoids. The lattices $\mathfrak{L}(\mathbb{J}_1)$ and $\mathfrak{L}(\mathbb{J}_2)$ are shown in Figure 1, where $\mathbf{0}$ is the variety of trivial monoids.

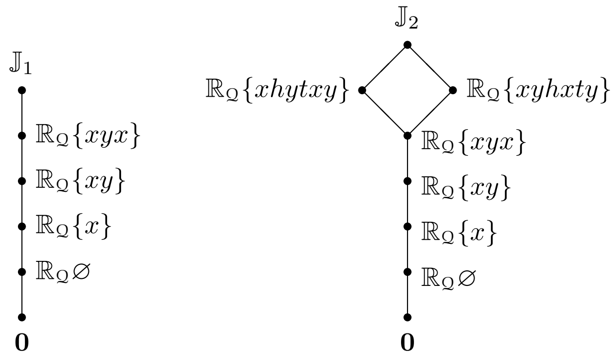

Figure 1: The lattices $\mathfrak{L}(\mathbb{J}_1)$ and $\mathfrak{L}(\mathbb{J}_2)$.

Since Rees quotients of free monoids belong to the class $\mathbb{A}^{\mathsf{zen}}$ of aperiodic monoids with central idempotents, Jackson [30, Question 1] asked the following question.

Problem 3. Are $\mathbb{J}_1$ and $\mathbb{J}_2$ the only finitely generated limit subvarieties of $\mathbb{A}^{\mathsf{zen}}$?

It is evident from Figure 1 that $\mathbb{J}_1$ and $\mathbb{J}_2$ are also minimal non-Cross or *almost Cross*. The following problem is thus relevant.

Problem 4. Are $\mathbb{J}_1$ and $\mathbb{J}_2$ the only almost Cross subvarieties of $\mathbb{A}^{\mathsf{zen}}$?

By referring to the words that appear in Figure 1, it seems that when a set $\mathscr{W}$ of words becomes more complicated, then so does the variety $\mathbb{R}_{\mathsf{Q}}\mathscr{W}$. This trend holds in many instances but not absolutely. For example, the word $xyxy$ is certainly less complicated than the word $xhytxy$, however, the variety $\mathbb{R}_{\mathsf{Q}}\{xyxy\}$ is non-finitely based [96] and contains continuum many subvarieties [121], while the variety $\mathbb{R}_{\mathsf{Q}}\{xhytxy\}$ is Cross.

3) Sufficient conditions for equational properties. Since an algebra satisfies the same identities as the variety it generates, it is unambiguous to say that an algebra is *finitely based* if it generates a finitely based variety. Finite members from classes of algebras such as groups [75], associative rings [39,68], Lie rings [8], and lattices [71] are finitely based. But as observed earlier, not all finite semigroups are finitely based.

The *finite basis problem*—the investigation of which algebras are finitely based—is one of the most prominent research problems in universal algebra. Besides being very natural by itself, this problem also has several interesting and unexpected connections with other topics of theoretical and practical importance, for example, feasible algorithms for membership in certain classes of formal languages [3] and classical number-theoretic conjectures such as the twin prime conjecture, Goldbach's conjecture, and the odd perfect number conjecture [77]. The algorithmic version of the finite basis problem, commonly known as *Tarski's finite basis problem*, was raised by Tarski [108] as a decision problem: given a finite algebra, decide if it is finitely based. Tarski's finite basis problem is undecidable for general algebras [72] but remains open for finite semigroups. A special case of this problem for semigroups was suggested by Sapir [95].

Problem 5 (Sapir [95, Problem 3.10.10]; see also Shevrin and Volkov [104, Question 7.1] and Volkov [115, Problem 4.1])**.** Is there an algorithm that when given a finite set $\mathscr{W}$ of words, decides whether or not the monoid $\mathcal{R}_{\mathcal{Q}}\mathscr{W}$ is finitely based?

A negative solution to Problem 5 clearly implies a negative solution to Tarski's finite basis problem for finite semigroups.

Over the years, there has been much focus on the study of the non-finite basis property of finite semigroups. For more information, refer to the surveys by Volkov [114, 115]. A common approach is to establish sufficient conditions under which a semigroup is non-finitely based. Perkins [76] presented such a condition and used it to show that the monoid $\mathcal{B}_2^1$ obtained from the Brandt semigroup

$$\mathcal{B}_2 = \left\langle \mathsf{a}, \mathsf{b} \,\middle|\, \mathsf{a}^2 = \mathsf{b}^2 = 0,\ \mathsf{aba} = \mathsf{a},\ \mathsf{bab} = \mathsf{b} \right\rangle$$

and the monoid $\mathcal{P}_{25}$ are both non-finitely based. Since then, more sufficient conditions were established, some of which can be found in Auinger et al. [4], Chen et al. [11], Hu et al. [26], Jackson [28–30], Jackson and Sapir [31], Li and Luo [66], Li et al. [67], Mikhailova and Sapir [73], Sapir [96, 97, 99–101], and Zhang and Luo [120]. These results exhibited many examples of finite semigroups and Rees quotients of free monoids that are non-finitely based.

Regarding techniques on establishing the finite basis property of finite semigroups, despite years of work, overall progress has remained "a collection of isolated theorems rather than a unified theory" [115, page 190]. One approach is to examine varieties that are *hereditarily finitely based* in the sense that all their subvarieties are finitely based. Indeed, if a system Σ of identities defines a hereditarily finitely based variety, then any semigroup that satisfies Σ is finitely based; such a system Σ is said to be *hereditarily finitely based.* An identity is *hereditarily finitely based* if it forms a hereditarily finitely based identity system on its own. Since the variety of commutative semigroups [76] and the variety of idempotent semigroups [9, 19, 20] are hereditarily finitely based, the identities $xy \approx yx$ and $x^2 \approx x$ are hereditarily finitely based.

Pollák initiated the investigation of hereditarily finitely based identities during the 1970s and, for over a decade, contributed immensely to their classification [80–84]. For more information on Pollák's accomplishments in hereditarily finitely based identities and varieties of semigroups in general, refer to the detailed survey of his work [116].

One type of identities whose hereditarily finitely based members have not been completely classified is the *alternating* identities, that is, identities of the form $\mathbf{u} \approx \mathbf{v}$ where $\mathbf{u}$ is a word that contains any variable at most once and $\mathbf{v}$ is a word that contains one variable twice and any other variable at most once.

Problem 6 (Pollák and Volkov [85])**.** Which alternating identities are hereditarily finitely based?

For more information on Pollák's accomplishments in hereditarily finitely based identities and varieties of semigroups in general, refer to the detailed survey of his work [116].

4) Involution semigroups. A unary operation $x \mapsto x^*$ on a semigroup S is an *involution* if S satisfies the identities

$$(x^*)^* \approx x, \quad (xy)^* \approx y^* x^*. \tag{inv}$$

An *involution semigroup* is a unary semigroup $\langle S, {}^* \rangle$ whose unary operation is an involution. An *inverse semigroup* is an involution semigroup $\langle S, {}^{-1} \rangle$ that satisfies the additional identities $xx^{-1}x \approx x$ and $xx^{-1}yy^{-1} \approx yy^{-1}xx^{-1}$. Examples of inverse semigroups include groups with inversion, while matrix semigroups with transposition are involution semigroups that are not inverse semigroups.

One main motivation for investigating involution semigroups—as in the case of semigroups—is to study natural generalization of groups; the presence of an involution is intended to better emulate the intricate symmetries of groups. Therefore it is counterintuitive that an involution semigroup $\langle S, {}^* \rangle$ and its semigroup *reduct* S can satisfy different equational properties such as the finite basis property. Infinite counterexamples demonstrating this phenomenon have long been available, as observed by Volkov [115, Section 2], while finite counterexamples have only recently been discovered: there exist non-finitely based involution semigroups whose semigroup reducts are finitely based [32, 57]. As observed by Auinger et al. [5, Section 1], there are no known finite examples of the opposite type, so the following open problem is of fundamental interest.

Problem 7. Is there a finitely based finite involution semigroup with non-finitely based semigroup reduct?

It is also possible for a semigroup to be the reduct of two involution semigroups that satisfy different equational properties. A prime example is the Brandt monoid $\mathcal{B}_2^1$ which admits two involutions: the operation $x \mapsto x^*$ that interchanges the elements ab and ba while fixing all other elements, and the operation $x \mapsto x^{-1}$ that interchanges the elements a and b while fixing all other elements. The involution semigroup $\langle \mathcal{B}_2^1, {}^* \rangle$ is *inherently non-finitely based* [6] in the sense that every locally finite variety containing it is non-finitely based. As for the involution semigroup $\langle \mathcal{B}_2^1, {}^{-1} \rangle$, which is an inverse semigroup, it is not inherently non-finitely based [94]. It is thus natural to question if other more extreme examples exist.

Problem 8. Are there two involution semigroups, both sharing the same semigroup reduct, such that one is finitely based and one is not?

Now the Brandt inverse monoid $\langle \mathcal{B}_2^1, {}^{-1} \rangle$ is not only non-finitely based [37] but has no irredundant identity bases [33]. In contrast, no finite involution semigroup is known to have an infinite irredundant identity basis, whence the following problem is relevant.

Problem 9. Is there a finite involution semigroup with an infinite irredundant identity basis?

The lattice $\mathfrak{L}_{\mathsf{sem}}$ of varieties of semigroups and the lattice $\mathfrak{L}_{\mathsf{inv}}$ of varieties of involution semigroups have been investigated since the 1960s and 1970s, respectively. But much less is known about the structure of the lattice $\mathfrak{L}_{\mathsf{inv}}$ compared with that of the lattice $\mathfrak{L}_{\mathsf{sem}}$. Although only a few small regions of $\mathfrak{L}_{\mathsf{inv}}$ have been considered, the results obtained so far are sufficient to demonstrate that the lattices $\mathfrak{L}_{\mathsf{inv}}$ and $\mathfrak{L}_{\mathsf{sem}}$ do not bear much resemblance to each other. For instance, the variety $\mathbf{IDEM}^*$ of all idempotent involution semigroups is not almost Cross [1] and the lattice $\mathfrak{L}(\mathbf{IDEM}^*)$ of subvarieties of $\mathbf{IDEM}^*$ is non-modular and of infinite width [13, 14]. In contrast, the variety $\mathbf{IDEM}$ of all idempotent semigroups is almost Cross and the lattice $\mathfrak{L}(\mathbf{IDEM})$ of subvarieties of $\mathbf{IDEM}$ is distributive and has width three [9,19,20]. Other differences between the lattices $\mathfrak{L}_{\mathsf{inv}}$ and $\mathfrak{L}_{\mathsf{sem}}$ can also be seen by comparing results related to the atoms of $\mathfrak{L}_{\mathsf{inv}}$ [15,18] and of $\mathfrak{L}_{\mathsf{sem}}$ [17,35]. Refer to Crvenković and Dolinka [12] for more information on the lattice $\mathfrak{L}_{\mathsf{inv}}$, and to Aĭzenshtat and Boguta [2], Evans [17], Shevrin et al. [103], and Vernikov [110] for more information on the lattice $\mathfrak{L}_{\mathsf{sem}}$.

Objectives of thesis

The thesis is concerned with varieties of semigroups, varieties of involution semigroups, and varieties of monoids. The main objectives are to investigate the following topics:

- Equational properties such as having or not having the following types of identity bases: finite, irredundant, and infinite irredundant.

- Varietal lattice-related properties such as being finitely generated and small.
- Other stronger properties of varieties such as being hereditarily finitely based, inherently non-finitely generated, and Cross.
- Varieties that are minimal with respect to certain properties such as limit varieties and almost Cross varieties.
- Structure of lattices of subvarieties.
- Relationship between an involution semigroup and its semigroup reduct with respect to various equational properties.
- Differences between the lattices $\mathfrak{L}_{\mathsf{inv}}$ and $\mathfrak{L}_{\mathsf{sem}}$.

A more specific goal of the thesis is to provide complete solutions to Problems 1–4 and 6–9, and to some weakened version of Problem 5.

Research methods

The majority of results in the thesis is concerned with equational properties. Therefore syntactic methods—direct manipulation of identities—are often employed throughout. The standard method to show that an algebra is finitely based is to compute an explicit identity basis, but this method is also used to establish (1) the existence of infinite irredundant identity bases and (2) the absence of irredundant identity bases. On the other hand, to show that an algebra is non-finitely based, it suffices to show that "long" identities are not deducible from "shorter" ones without the need to have an explicit identity basis. To show that a variety satisfies a stronger property such as being hereditarily finitely based or Cross, it is often necessary to first partition its lattice of subvarieties into subintervals before further investigations.

The extreme majority of proofs in the thesis is self-contained. However, several results from the literature, some of which are not syntactical, have also been used on a few occasions, for example,

- results by Lee and Volkov [63] on idempotent-separable semigroups and the minimal finitely universal variety of Vernikov and Volkov [111] are crucial to the study of aperiodic Rees–Suschkewitsch varieties;
- the characterization by Straubing [106] of finite nilpotent monoids is essential to the description of Cross subvarieties and inherently non-finitely generated subvarieties of $\mathbb{A}^{\mathsf{zen}}$;
- the sufficient condition of Volkov [113] for the finite basis property is required to establish the hereditary finite basis property of an overcommutative variety of monoids, that is, a variety that contains $\mathbb{C}\mathrm{OM}$;
- the finite basis property of the direct product of a semigroup with a nilpotent semigroup, due to Volkov [112], is crucial to the construction of semigroups S that are *conformable* in the sense that S is non-finitely based while S^1 is finitely based.

Scientific novelty

Major new results of the thesis are listed below.

Part I: Varieties of semigroups.

- Every aperiodic Rees–Suschkewitsch variety is finitely based [122].
- A characterization of aperiodic Rees–Suschkewitsch varieties that are either Cross or finitely generated [123].
- Complete description of hereditarily finitely based identities that are alternating [55].
- A sufficient condition under which a semigroup is non-finitely based [124].

- A sufficient condition under which a non-finitely based finite algebra of any type has no irredundant identity bases [129].

Part II: Varieties of involution semigroups.

- First examples of finitely based finite involution semigroups with non-finitely based semigroup reducts [56].

- First examples of finite involution semigroups with infinite irredundant identity bases [127].

- First examples of a trio of finite involution semigroups, sharing the same semigroup reduct, such that one has a finite identity basis, one has an infinite irredundant identity basis, and one has no irredundant identity bases [129].

- A twisted involution semigroup is non-finitely based whenever its semigroup reduct is non-finitely based [128].

Part III: Varieties of monoids.

- Existence and uniqueness of two maximal hereditarily finitely based overcommutative varieties of monoids [125].

- A characterization of varieties of monoids from several large classes that are hereditarily finitely based [125].

- A characterization of varieties of monoids that are either Cross or inherently non-finitely generated within the class $\mathbb{A}^{\mathsf{zen}}$ [48, 51, 54].

- First examples finite conformable semigroups and recipe of their construction [53].

Scientific significance of results

- Techniques used and results established in the study of aperiodic Rees–Suschkewitsch varieties (Chapter 3) have facilitated investigations in several directions, for example, Rees–Suschkewitsch varieties with nontrivial groups [40, 64], varieties generated by monoids obtained from Rees–Suschkewitsch semigroups [46, 47, 49, 50], varieties generated by small semigroups [16, 52, 59, 65], and finite semigroups generating pseudovarieties that are join irreducible in the lattice of pseudovarieties of finite semigroups [62].

- The description of aperiodic Rees–Suschkewitsch varieties that are finitely based, finitely generated, or small (Chapter 3) has solved Problem 1.

- The finite basis property established for all aperiodic Rees–Suschkewitsch varieties (Chapter 3) has solved Problem 2 in the negative.

- The complete description of all alternating hereditarily finitely based identities (Chapter 4) has solved Problem 6.

- The sufficient condition for the non-finite basis property of semigroups (Chapter 5) and the equational results on varieties of monoids (Chapters 11 and 14) have motivated further work on the finite basis problem [97–99].

- The new examples of involution semigroups with extreme and contrasting properties (Chapters 7–9) pushed the boundaries on how vast the difference can be between (1) a finite involution semigroup and its semigroup reduct, (2) several finite involution semigroups sharing the same semigroup reduct, and (3) the lattices $\mathfrak{L}_{\mathsf{inv}}$ and $\mathfrak{L}_{\mathsf{sem}}$. As a result, Problems 7–9 are positively solved.

- The condition under which the non-finite basis property of an involution semigroup is inherited from its semigroup reduct (Chapter 10) can be used to convert many results on equational properties of semigroups to results applicable to involution semigroups.

- The investigation of varieties of involution semigroups (Chapters 7–10) is ongoing [57, 58].

- The hereditary finite basis property established for varieties of monoids (Chapter 11) has (1) positively solved Problem 3, (2) provided a method to determine the hereditary finite basis property of monoids of the form $\mathcal{R}_{\mathcal{Q}}\mathscr{W}$ (cf. Problem 5), and (3) led to further work on limit varieties of monoids [119].

- Techniques used and results established in the study of varieties of aperiodic monoids (Chapters 12 and 13) have (1) positively solved Problem 4 and (2) played a role in the description of chain varieties of monoids [23] and the construction of varieties of monoids with continuum many subvarieties [22, 121].

Approbation of results

Results of the thesis were presented at the following conferences and meetings:

- The 3rd Novi Sad Algebraic Conference (Novi Sad, Serbia, 2009);

- Workshop on Groups and Semigroups (Porto, Portugal, 2015);

- Rhodesfest 81: a conference in honor of John Rhodes's 81st birthday (Ramat Gan, Israel, 2018);

- AMS Sectional Meeting #999, Special Session on Semigroup Theory (Nashville, TN, USA, 2004);

- Algebraic Systems Seminar #1275, Ural Federal University (Ekaterinburg, 2017);
- Algebraic Seminar, Lanzhou University (Lanzhou, China, 2019) — a series of 4 reports;
- Algebraic Seminar, Shaanxi University of Science and Technology (Xi'an, China, 2019).

2 Summary of Thesis

Part I: Semigroups

Results of this part can be classified into the four groups.

1) Aperiodic Rees–Suschkewitsch varieties. The largest aperiodic Rees–Suschkewitsch variety $\mathbf{RS}_1$ coincides with the variety $\mathbf{A_2}$ generated by the 0-simple semigroup

$$\mathcal{A}_2 = \big\langle \mathsf{a}, \mathsf{e} \,\big|\, \mathsf{e}^2 = \mathsf{eae} = \mathsf{e},\ \mathsf{aea} = \mathsf{a},\ \mathsf{a}^2 = 0 \big\rangle.$$

Therefore the investigation of aperiodic Rees–Suschkewitsch varieties coincides with the study of the lattice $\mathfrak{L}(\mathbf{A_2})$ of subvarieties of $\mathbf{A_2}$. Some important subvarieties of $\mathbf{A_2}$ include $\mathbf{B_2} = \mathsf{V}_{\mathsf{sem}}\{\mathcal{B}_2\}$, $\mathbf{A_0} = \mathsf{V}_{\mathsf{sem}}\{\mathcal{A}_0\}$, and $\mathbf{B_0} = \mathsf{V}_{\mathsf{sem}}\{\mathcal{B}_0\}$, where the semigroups

$$\mathcal{A}_0 = \big\langle \mathsf{e}, \mathsf{f} \,\big|\, \mathsf{e}^2 = \mathsf{e},\ \mathsf{f}^2 = \mathsf{f},\ \mathsf{ef} = 0 \big\rangle$$

$$\text{and} \quad \mathcal{B}_0 = \big\langle \mathsf{a}, \mathsf{e}, \mathsf{f} \,\big|\, \mathsf{e}^2 = \mathsf{e},\ \mathsf{f}^2 = \mathsf{f},\ \mathsf{ef} = \mathsf{fe} = 0,\ \mathsf{ea} = \mathsf{af} = \mathsf{a} \big\rangle$$

are isomorphic to subsemigroups of $\mathcal{A}_2$ and $\mathcal{B}_2$, respectively.

For each $\mathbf{V} \in \{\mathbf{A_2}, \mathbf{B_2}, \mathbf{A_0}, \mathbf{B_0}\}$, there exists a unique subvariety $\overline{\mathbf{V}}$ of $\mathbf{A_2}$ that is largest with respect to excluding $\mathbf{V}$. It follows that $\overline{\mathbf{A_2}}$ is the unique maximal subvariety of $\mathbf{A_2}$, and that the lattice $\mathfrak{L}(\overline{\mathbf{A_2}})$ is disjoint union of the intervals

$$\mathfrak{I}_1 = [\mathbf{A_0} \vee \mathbf{B_2}, \overline{\mathbf{A_2}}], \quad \mathfrak{I}_2 = [\mathbf{A_0}, \overline{\mathbf{B_2}}], \quad \mathfrak{I}_3 = [\mathbf{B_2}, \overline{\mathbf{A_0}}],$$

$$\mathfrak{I}_4 = [\mathbf{B_0}, \overline{\mathbf{A_0}} \cap \overline{\mathbf{B_2}}], \quad \text{and} \quad \mathfrak{I}_5 = \mathfrak{L}(\overline{\mathbf{B_0}}).$$

The intervals $\mathfrak{I}_1$, $\mathfrak{I}_2$, $\mathfrak{I}_3$, and $\mathfrak{I}_4$ are each isomorphic to a direct product of two countably infinite chains. But a description of the interval $\mathfrak{I}_5$ is not feasible since it embeds every finite lattice [111].

Proposition 10 (Section 3.3)**.** *The lattice $\mathfrak{L}(\mathbf{A_2})$ is the disjoint union of the countably infinite distributive interval*

$$[\mathbf{B_0}, \mathbf{A_2}] = \mathfrak{I}_1 \cup \mathfrak{I}_2 \cup \mathfrak{I}_3 \cup \mathfrak{I}_4 \cup \{\mathbf{A_2}\}$$

and the lattice $\mathfrak{L}(\overline{\mathbf{B_0}}) = \mathfrak{I}_5$.

Several results can be deduced from the aforementioned partial description of the lattice $\mathfrak{L}(\overline{\mathbf{A_2}})$.

Theorem 11 (Theorem 1.5)**.** *The variety $\mathbf{A_2}$ is hereditarily finitely based.*

It follows that the lattice $\mathfrak{L}(\mathbf{A_2})$ is countably infinite and so Problem 2 is solved in the negative.

An identity of the form $x_1 x_2 \cdots x_n \approx x_{1\pi} x_{2\pi} \cdots x_{n\pi}$, where π is some nontrivial permutation on $\{1, 2, \ldots, n\}$, is called a *permutation identity*. An identity of the form $x_1 x_2 \cdots x_n \approx \mathbf{w}$ that is not a permutation identity is said to be *diverse*. Diverse identities and the variety $\mathbf{H}_{\mathsf{com}}$ defined by the identity system $\{xyx \approx y^2, \, xy \approx yx\}$ are crucial in the characterization of Cross subvarieties of $\mathbf{A_2}$.

Theorem 12 (Theorem 1.6)**.** *The following conditions on any subvariety $\mathbf{V}$ of $\mathbf{A_2}$ are equivalent:*

(a) $\mathbf{V}$ *is Cross*;

(b) $\mathbf{V}$ *is small*;

(c) $\mathbf{V}$ *is finitely generated and* $\mathbf{B_0} \nsubseteq \mathbf{V}$;

(d) $\mathbf{V}$ *satisfies some diverse identity*;

(e) $\mathbf{H}_{\mathsf{com}} \nsubseteq \mathbf{V}$.

It follows that the variety $\mathbf{H}_{\mathsf{com}}$ is the unique almost Cross subvariety of $\mathbf{A_2}$, and the non-Cross subvarieties of $\mathbf{A_2}$ constitute the interval $[\mathbf{H}_{\mathsf{com}}, \mathbf{A_2}]$. Further, Theorems 11 and 12 provide a solution to Problem 1.

Corollary 13 (Propositions 3.48 and 3.49)**.**

(i) *The Cross subvarieties of $\mathbf{A_2}$ constitute an incomplete sublattice of $\mathfrak{L}(\mathbf{A_2})$.*

(ii) *The finitely generated subvarieties of $\mathbf{A_2}$ constitute an incomplete sublattice of $\mathfrak{L}(\mathbf{A_2})$.*

2) Hereditarily finitely based identities. The following result provides a complete description of alternating hereditarily finitely based identities and so also a solution to Problem 6.

Theorem 14 (Theorem 4.1)**.** *An alternating identity is hereditarily finitely based if and only if it is not of the form*

$$x_1x_2\cdots x_\ell\, y^2 z_1z_2\cdots z_r \approx x_1x_2\cdots x_\ell\, yz_1z_2\cdots z_r,$$

where $\ell, r \geq 0$ and $(\ell, r) \neq (0,0)$.

3) Semigroups without finite identity bases.

Theorem 15 (Theorem 1.23)**.** *A semigroup is non-finitely based if for some fixed $n \geq 2$, it satisfies the identities*

$$x^{2n} \approx x^n, \quad x^n\left(\prod_{i=1}^{m} y_i^n\right)x^n \approx x^n\left(\prod_{i=m}^{1} y_i^n\right)x^n, \quad m = 2,3,4,\ldots$$

but violates the identity $(x^ny^nx^n)^{n+1} \approx x^ny^nx^n$.

This result can be stated more concisely if only finite semigroups are involved.

Corollary 16 (Corollary 5.8). *A finite semigroup is non-finitely based if it satisfies the pseudoidentities*

$$x^\omega\left(\prod_{i=1}^{m} y_i^\omega\right)x^\omega \approx x^\omega\left(\prod_{i=m}^{1} y_i^\omega\right)x^\omega, \quad m = 2, 3, 4, \ldots$$

but violates the pseudoidentity $(x^\omega y^\omega x^\omega)^{\omega+1} \approx x^\omega y^\omega x^\omega$.

An example to which Theorem 15 applies is the direct product

$$\mathcal{L}_{\ell,n} = \mathcal{L}_\ell \times \mathcal{Z}_n$$

of the $\mathscr{J}$-trivial semigroup

$$\mathcal{L}_\ell = \left\langle \mathsf{e}, \mathsf{f} \,\middle|\, \mathsf{e}^2 = \mathsf{e},\ \mathsf{f}^2 = \mathsf{f},\ \underbrace{\mathsf{efefe}\cdots}_{\text{length } \ell \geq 2} = \mathsf{0} \right\rangle$$

of order 2ℓ with the cyclic group $\mathcal{Z}_n$ of order $n \geq 1$.

Corollary 17 (Corollary 5.3). *The semigroup* $\mathcal{L}_{\ell,n}$ *is finitely based if and only if* $\ell = 2$.

4) Semigroups without irredundant identity bases.

Theorem 18 (Theorem 6.3). *Let A be any non-finitely based finite algebra. Suppose that*

(a) *an identity basis $\Lambda = \Lambda_0 \cup \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ for A exists, where Λ_0 is some finite set of identities of A and $\lambda_1, \lambda_2, \lambda_3, \ldots$ are identities of A such that $\Lambda_0 \cup \{\lambda_{k+1}\} \vdash \lambda_k$ for all $k \geq 1$;*

(b) *any identity basis for A contains identities $\theta_1, \theta_2, \theta_3, \ldots$ such that $\Lambda_0 \cup \{\theta_k\} \vdash \lambda_k$ for all $k \geq 1$.*

Then A has no irredundant identity bases.

For each fixed $n \geq 1$, the identities

$$x^{n+2} \approx x^2, \quad x^{n+1}yx^{n+1} \approx xyx, \quad xhykxty \approx yhxkytx,$$

$$x\left(\prod_{i=1}^{m}(y_i\, h_i\, y_i)\right)x \approx x\left(\prod_{i=m}^{1}(y_i\, h_i\, y_i)\right)x, \quad m = 2, 3, 4, \ldots$$

constitute an identity basis for the semigroup $\mathcal{L}_{3,n}$; Theorem 18 is applicable to this identity basis to obtain the following result.

Theorem 19 (Theorem 6.20)**.** *The semigroup $\mathcal{L}_{3,n}$ has no irredundant identity bases.*

The identity basis for $\mathcal{L}_{3,n}$ can also be used to deduce an upper bound for the complexity of its varietal membership problem.

Theorem 20 (Theorem 6.21)**.** *The variety membership problem for the semigroup $\mathcal{L}_{3,n}$ belongs to the class* $\mathsf{co\text{-}NP}$ *of problems with complements that are nondeterministic polynomial time.*

Part II: Involution Semigroups

Results of this part can be classified into the two groups.

1) Counterintuitive examples. For each $\ell \geq 2$, the semigroup $\mathcal{L}_\ell$ admits a unique involution * which is completely determined by its action on the generators $\{\mathsf{e}, \mathsf{f}\}$:

$$(\mathsf{e}^\star, \mathsf{f}^\star) = \begin{cases} (\mathsf{f}, \mathsf{e}) & \text{if } \ell \text{ is even,} \\ (\mathsf{e}, \mathsf{f}) & \text{if } \ell \text{ is odd.} \end{cases}$$

For instance, in the semigroup $\mathcal{L}_3 = \{\mathsf{0}, \mathsf{e}, \mathsf{f}, \mathsf{ef}, \mathsf{fe}, \mathsf{fef}\}$, the involution * interchanges ef and fe but fixes all other elements.

In contrast, for any $n \geq 3$, the cyclic group $\mathcal{Z}_n$ admits more than one involution: if $\Re \in \{1, 2, \ldots, n\}$ is a *square root of unity modulo n* in the sense that $\Re^2 \equiv 1 \pmod{n}$, then the unary operation $x \mapsto x^{\Re}$ on $\mathcal{Z}_n$ is an involution. For example, if $n \geq 3$, then the group inverse of $\mathcal{Z}_n$ coincides with the involution $x \mapsto x^{n-1}$.

Theorem 21 (Theorem 1.30)**.** *The direct product*

$$\langle \mathcal{L}_{3,n}, {}^{\star\Re} \rangle = \langle \mathcal{L}_3, {}^{\star} \rangle \times \langle \mathcal{Z}_n, {}^{\Re} \rangle$$

has a finite identity basis if $\Re = 1$; otherwise, it has an infinite irredundant identity basis.

Consequently, the involution semigroup $\langle \mathcal{L}_{3,n}, {}^{\star\Re} \rangle$ always has an irredundant identity basis. In contrast, by Theorem 19, the semigroup reduct of $\langle \mathcal{L}_{3,n}, {}^{\star\Re} \rangle$ has no irredundant identity bases.

Corollary 22. *The involution semigroup $\langle \mathcal{L}_{3,n}, {}^{\star 1} \rangle$ is finitely based while its semigroup reduct $\mathcal{L}_{3,n}$ is non-finitely based.*

Note that Problems 8 and 9 are affirmatively answered by Theorem 21, while Problem 7 is affirmatively answered by Corollary 22.

Theorem 21 is established by computing identity bases for the involution semigroup $\langle \mathcal{L}_{3,n}, {}^{\star\Re} \rangle$: the identities (inv) and

$$x^{n+2} \approx x^2, \quad x^{n+1}yx^{n+1} \approx xyx, \quad xhykxty \approx yhxkytx,$$

$$xy^*x^* \approx xyx^*, \quad x^*yx^* \approx xyx, \quad xhx^*kx \approx xhxkx$$

constitute an identity basis for the involution semigroup $\langle \mathcal{L}_{3,n}, {}^{\star 1} \rangle$, while an infinite irredundant identity basis for $\langle \mathcal{L}_{3,n}, {}^{\star\Re} \rangle$ with $\Re > 1$ can be formed by (inv) and some of the following identities:

$$x^{n+2} \approx x^2, \quad x^{n+1}yx^{n+1} \approx xyx, \quad xhykxty \approx yhxkytx,$$

$$x^*yx^* \approx x^{\Re}yx^{\Re}, \quad xhx^*kx \approx xhx^{\Re}kx,$$

$$x^*\,\mathbf{h}_1\,y^{\circledast_1}\,\mathbf{h}_2\,x^{\circledast_2}\,\mathbf{h}_3\,y^{\circledast_3} \approx x^{\Re}\,\mathbf{h}_1\,y^{\circledast_1}\,\mathbf{h}_2\,x^{\circledast_2}\,\mathbf{h}_3\,y^{\circledast_3},$$

$$x^{\circledast_1}\,\mathbf{h}_1\,y^{*}\,\mathbf{h}_2\,x^{\circledast_2}\,\mathbf{h}_3\,y^{\circledast_3} \approx x^{\circledast_1}\,\mathbf{h}_1\,y^{\Re}\,\mathbf{h}_2\,x^{\circledast_2}\,\mathbf{h}_3\,y^{\circledast_3},$$

$$x^{\circledast_1}\,\mathbf{h}_1\,y^{\circledast_2}\,\mathbf{h}_2\,x^{*}\,\mathbf{h}_3\,y^{\circledast_3} \approx x^{\circledast_1}\,\mathbf{h}_1\,y^{\circledast_2}\,\mathbf{h}_2\,x^{\Re}\,\mathbf{h}_3\,y^{\circledast_3},$$

$$x^{\circledast_1}\,\mathbf{h}_1\,y^{\circledast_2}\,\mathbf{h}_2\,x^{\circledast_3}\,\mathbf{h}_3\,y^{*} \approx x^{\circledast_1}\,\mathbf{h}_1\,y^{\circledast_2}\,\mathbf{h}_2\,x^{\circledast_3}\,\mathbf{h}_3\,y^{\Re},$$

$$x\Bigg(\prod_{i=1}^{m}(y_i\mathbf{h}_i y_i^*)\Bigg)x^* \approx x\Bigg(\prod_{i=m}^{1}(y_i\mathbf{h}_i y_i^*)\Bigg)x^*, \quad m = 2, 3, 4, \ldots,$$

where $\mathbf{h}_i \in \{\varnothing, h_i\}$ and $\circledast_i \in \{1, *\}$.

For any disjoint semigroups S_1 and S_2 that exclude the symbol 0, the *0-direct union* of S_1 and S_2 is the semigroup

$$S_1 \uplus S_2 = S_1 \cup S_2 \cup \{0\},$$

where the product of two elements from S_i is their usual product in S_i, and $ab = ba = 0$ for all $a \in S_1 \cup \{0\}$ and $b \in S_2 \cup \{0\}$. In general, it is possible to form the 0-direct union of any two semigroups S_1 and S_2 simply by renaming their elements so that $S_1 \cap S_2 = \varnothing$ and $0 \notin S_1 \cup S_2$.

Theorem 23 (Theorem 9.1)**.** *For each $n \geq 3$, there exist three involution semigroups, all sharing the semigroup reduct $\mathcal{L}_{3,n} \uplus \mathcal{L}_{3,n}$, such that one has a finite identity basis, one has an infinite irredundant identity basis, and one has no irredundant identity bases.*

The involution semigroups $\{\langle \mathcal{L}_\ell, {}^\star \rangle \mid \ell = 2, 3, 4, \ldots\}$ highlight the structural difference between the lattices $\mathfrak{L}_{\mathsf{inv}}$ and $\mathfrak{L}_{\mathsf{sem}}$.

Theorem 24 (Theorem 1.33)**.** *The varieties*

$$\mathsf{V}_{\mathsf{sem}}\{\mathcal{L}_2\} \subset \mathsf{V}_{\mathsf{sem}}\{\mathcal{L}_3\} \subset \mathsf{V}_{\mathsf{sem}}\{\mathcal{L}_4\} \subset \cdots$$

form a single infinite chain in the lattice $\mathfrak{L}_{\mathsf{sem}}$, while the varieties

$$\mathsf{V}_{\mathsf{inv}}\{\langle \mathcal{L}_2, {}^\star \rangle\} \subset \mathsf{V}_{\mathsf{inv}}\{\langle \mathcal{L}_4, {}^\star \rangle\} \subset \mathsf{V}_{\mathsf{inv}}\{\langle \mathcal{L}_6, {}^\star \rangle\} \subset \cdots$$

$$\textit{and} \quad \mathsf{V}_{\mathsf{inv}}\{\langle \mathcal{L}_3, {}^\star \rangle\} \subset \mathsf{V}_{\mathsf{inv}}\{\langle \mathcal{L}_5, {}^\star \rangle\} \subset \mathsf{V}_{\mathsf{inv}}\{\langle \mathcal{L}_7, {}^\star \rangle\} \subset \cdots$$

form two incomparable infinite chains in the lattice $\mathfrak{L}_{\mathsf{inv}}$.

2) Equational theories of involution semigroups. An involution semigroup is *twisted* if the variety it generates contains the involution semilattice $\langle \mathcal{S}\ell_3, {}^* \rangle$, where

$$\mathcal{S}\ell_3 = \left\langle \mathsf{e}, \mathsf{f} \,\middle|\, \mathsf{e}^2 = \mathsf{e},\ \mathsf{f}^2 = \mathsf{f},\ \mathsf{ef} = \mathsf{fe} = \mathsf{0} \right\rangle$$

and * interchanges e and f.

Theorem 25 (Theorem 1.36)**.** *A twisted involution semigroup is non-finitely based if its semigroup reduct is non-finitely based.*

Since the variety $\mathsf{V}_{\mathsf{inv}}\{\langle \mathcal{S}\ell_3, {}^* \rangle\}$ is an atom in the lattice $\mathfrak{L}_{\mathsf{inv}}$ [18], twistedness is a weakest assumption in Theorem 25—the only way to weaken it is to omit it altogether. But twistedness cannot be omitted since by Theorems 19 and 21, the non-twisted involution semigroup $\langle \mathcal{L}_3, {}^\star \rangle$ is finitely based while its semigroup reduct $\mathcal{L}_3$ is not. Further, the converse of Theorem 25 does not hold because a finite counterexample exists [32]. Hence Theorem 25 has reached its full potential.

A number of results can be deduced from Theorem 25.

Theorem 26 (Theorem 1.37)**.** *Any involution semigroup with non-finitely based semigroup reduct is embeddable in some non-finitely based involution semigroup with at most three more elements.*

Theorem 27 (Auinger et al. [5]; Theorem 1.38)**.** *A twisted involution semigroup is inherently non-finitely based if its semigroup reduct is inherently non-finitely based.*

A finite algebra that is not inherently non-finitely based is said to be *weakly finitely based*. The class of weakly finitely based finite semigroups is a pseudovariety [114, Subsection 4.4]. In contrast, this result does not hold for involution semigroups.

Proposition 28 (Proposition 1.39)**.** *The class of weakly finitely based finite involution semigroups is not a pseudovariety.*

For example, the inverse monoid $\langle \mathcal{B}_2^1, {}^{-1} \rangle$ is weakly finitely based [94] and the involution semilattice $\langle \mathcal{S}\ell_3, {}^* \rangle$ is even finitely based [14], but their direct product $\langle \mathcal{B}_2^1, {}^{-1} \rangle \times \langle \mathcal{S}\ell_3, {}^* \rangle$ is inherently non-finitely based by Theorem 27.

Part III: Monoids

Results of this part can be classified into the three groups.

1) Hereditarily finitely based varieties.

Theorem 29 (Theorem 11.1)**.** *The variety $\mathbb{O}$ of monoids defined by the following identities is hereditarily finitely based*:

$$xhxyty \approx xhyxty, \quad xhytxy \approx xhytyx.$$

The variety $\mathbb{O}^{\triangleleft}$ that is dual to $\mathbb{O}$ is obviously also hereditarily finitely based.

Theorem 30 (Theorem 1.52)**.** *Let $\mathbb{V}$ be any variety of monoids such that $\mathbb{V} \subseteq \mathbb{A}^{\mathsf{zen}}$ or $\mathbb{V} \subseteq \mathsf{V}_{\mathsf{mon}}\{\mathcal{B}_2^1\}$ or $\mathbb{R}_{\mathbb{Q}}\{xyx\} \subseteq \mathbb{V}$ or $\mathbb{C}\mathbb{OM} \subseteq \mathbb{V}$. Then the following conditions on $\mathbb{V}$ are equivalent*:

(a) $\mathbb{V}$ *is hereditarily finitely based*;

(b) $\mathbb{J}_1, \mathbb{J}_2 \not\subseteq \mathbb{V}$;

(c) $\mathbb{V} \subseteq \mathbb{O}$ *or* $\mathbb{V} \subseteq \mathbb{O}^{\triangleleft}$.

An affirmative solution to Problem 3 and a few other results can be deduced from Theorem 30.

Theorem 31 (Theorems 1.53 and 1.54 and Corollary 11.18)**.**

(i) *The varieties $\mathbb{J}_1$ and $\mathbb{J}_2$ are the only limit subvarieties of $\mathbb{A}^{\mathsf{zen}}$.*

(ii) *The varieties $\mathbb{O}$ and $\mathbb{O}^{\triangleleft}$ are the only maximal hereditarily finitely based overcommutative varieties of monoids.*

(iii) *Overcommutative limit varieties of monoids do not exist.*

Recall that Sapir [95, Problem 3.10.10] asks for a solution to the finite basis problem for Rees quotients $\mathcal{R}_{\mathbb{Q}}\mathscr{W}$ with finite set $\mathscr{W}$ of words; see Problem 5. This problem is still open, but since $\mathcal{R}_{\mathbb{Q}}\mathscr{W} \in \mathbb{A}^{\mathsf{zen}}$, it is possible to determine by Theorem 30 if $\mathcal{R}_{\mathbb{Q}}\mathscr{W}$ is hereditarily finitely based as a monoid.

Theorem 32 (Theorem 1.55)**.** *For any finite set $\mathscr{W}$ of words, the Rees quotient $\mathcal{R}_{\mathbb{Q}}\mathscr{W}$ is hereditarily finitely based as a monoid if and only if it satisfies the identity system in Theorem 29 or its dual system.*

2) Cross varieties and inherently non-finitely generated varieties. For each $n \geq 2$, let $\mathbb{A}_n^{\mathsf{zen}}$ denote the variety of monoids defined by the identities

$$x^{n+1} \approx x^n, \quad x^n y \approx y x^n$$

and let $\mathbb{S}_n$ denote the variety of monoids defined by the identity

$$x \prod_{i=1}^{n-1} (h_i x) \approx x^n \prod_{i=1}^{n-1} h_i.$$

Let $\mathbb{K} = \mathbb{A}_2^{\mathsf{zen}} \cap \mathbb{O} \cap \mathbb{O}^{\triangleleft}$.

Theorem 33 (Theorem 1.57)**.** *The following conditions on any subvariety $\mathbb{V}$ of $\mathbb{A}^{\mathsf{zen}}$ are equivalent:*

(a) $\mathbb{V}$ *is Cross*;

(b) $\mathbb{J}_1, \mathbb{J}_2, \mathbb{K} \nsubseteq \mathbb{V}$;

(c) $\mathbb{V} \subseteq \mathbb{A}_n^{\mathsf{zen}} \cap \mathbb{O} \cap \mathbb{S}_n$ *or* $\mathbb{V} \subseteq \mathbb{A}_n^{\mathsf{zen}} \cap \mathbb{O}^{\triangleleft} \cap \mathbb{S}_n$ *for some* $n \geq 2$.

Therefore the subvarieties of $\mathbb{A}_n^{\mathsf{zen}} \cap \mathbb{O} \cap \mathbb{S}_n$ and $\mathbb{A}_n^{\mathsf{zen}} \cap \mathbb{O}^{\triangleleft} \cap \mathbb{S}_n$, where $n \in \{2, 3, 4, \ldots\}$, are precisely all Cross subvarieties of $\mathbb{A}^{\mathsf{zen}}$, and that $\mathbb{J}_1$, $\mathbb{J}_2$, and $\mathbb{K}$ are the only almost Cross subvarieties of $\mathbb{A}^{\mathsf{zen}}$. Specifically, $\mathbb{J}_1$ and $\mathbb{J}_2$ are the only finitely generated almost Cross subvarieties of $\mathbb{A}^{\mathsf{zen}}$, while $\mathbb{K}$ is the unique non-finitely generated almost Cross subvariety of $\mathbb{A}^{\mathsf{zen}}$. The solution to Problem 4 is thus negative.

The almost Cross variety $\mathbb{K}$ is not only non-finitely generated but also *inherently non-finitely generated within* $\mathbb{A}^{\mathsf{zen}}$ in the sense that any subvariety of $\mathbb{A}^{\mathsf{zen}}$ containing $\mathbb{K}$ is non-finitely generated.

Theorem 34 (Theorem 1.58)**.** *The following conditions on any subvariety* $\mathbb{V}$ *of* $\mathbb{A}^{\mathsf{zen}}$ *are equivalent*:

(a) $\mathbb{V}$ *is inherently non-finitely generated within* $\mathbb{A}^{\mathsf{zen}}$;

(b) $\mathbb{K} \subseteq \mathbb{V}$;

(c) $\mathbb{V} \nsubseteq \mathbb{S}_n$ *for all* $n \geq 2$.

Consequently, the variety $\mathbb{K}$ is the unique minimal variety that is inherently non-finitely generated within $\mathbb{A}^{\mathsf{zen}}$, and the subvarieties of $\mathbb{A}^{\mathsf{zen}}$ containing $\mathbb{K}$ are precisely those that are inherently non-finitely generated within $\mathbb{A}^{\mathsf{zen}}$. However, $\mathbb{K}$ is not inherently non-finitely generated within the class of all monoids since it is a subvariety of $\mathsf{V}_{\mathsf{mon}}\{\mathcal{B}_2^1\}$ (Corollary 12.19).

The subvariety $\mathbb{W}$ of $\mathbb{A}_2^{\mathsf{zen}} \cap \mathbb{S}_3$ defined by the identities

$$xyhxty \approx yxhxty, \quad xhytxy \approx xhytyx$$

is the first example of a non-finitely generated subvariety of $\mathbb{A}^{\mathsf{zen}}$ that is not inherently non-finitely generated within $\mathbb{A}^{\mathsf{zen}}$ (Section 12.5).

Let $\mathbb{A}^{\mathsf{com}}$ denote the class of aperiodic monoids with commuting idempotents and let $\widetilde{\mathbb{A}}^{\mathsf{com}}$ denote the class of monoids from $\mathbb{A}^{\mathsf{com}}$ that satisfy the identity $x^2yx \approx xyx^2$.

Theorem 35 (Theorem 1.60). *A subvariety of $\widetilde{\mathbb{A}}^{\mathsf{com}}$ is Cross if and only if it excludes $\mathbb{J}_1$, $\mathbb{J}_2$, and $\mathbb{K}$.*

Proposition 36 (Proposition 1.61). *The varieties $\mathbb{J}_1$, $\mathbb{J}_2$, and $\mathbb{K}$ are precisely all almost Cross subvarieties of $\mathsf{V}_{\mathsf{mon}}\{\mathcal{B}_2^{\mathtt{1}}\}$.*

3) Rees quotients of free monoids: further examples.

Theorem 37 (Theorem 1.62). *For any group G of finite exponent, the direct product $\mathcal{R}_{\mathcal{Q}}\{xyx\} \times G$ is finitely based if and only if G is commutative.*

Recall that the monoid $\mathcal{R}_{\mathcal{Q}}\{xyx\}$ [30] and all finite groups [75] generate Cross varieties of monoids. Therefore Theorem 37 shows that the join of two Cross varieties of monoids need not be Cross.

The Brandt semigroup $\mathcal{B}_2$ is an example of a finitely based finite semigroup S such that the monoid $S^{\mathtt{1}}$ is non-finitely based [76,109]. Examples of the "opposite" type—non-finitely based semigroups S such that the monoids $S^{\mathtt{1}}$ are finitely based—are called *conformable*. Shneerson [105] exhibited the first example of a conformable semigroup, but his example is infinite. Finite conformable semigroups can be constructed by the following recipe.

Theorem 38 (Theorem 1.64). *Suppose that S and N are any semigroups such that $S^{\mathtt{1}}$ is non-finitely based, N is nilpotent, and $S^{\mathtt{1}} \times N^{\mathtt{1}}$ is finitely based. Then the direct product $P = S^{\mathtt{1}} \times N$ is a conformable semigroup.*

By Jackson and Sapir [31, Corollary 5.2], finite semigroups S and N that satisfy the hypotheses of Theorem 38 can be systematically and easily selected from the class of Rees quotients of the free monoids. For example, if

$$S = \mathcal{R}_{\mathcal{Q}}\{xyxy\}\backslash\{\mathtt{1}\} \quad \text{and} \quad N = \mathcal{R}_{\mathcal{Q}}\{x^2y^2, xy^2x\}\backslash\{\mathtt{1}\},$$

then $P = S^1 \times N$ is a conformable semigroup of order $9 \times 12 = 108$. In fact, the subsemigroup

$$\{(a, 0) \mid a \in S^1\} \cup \{(0, b) \mid b \in N\}$$

of P is a conformable semigroup of order 20. Presently, the order of the smallest conformable semigroup is unknown, but such a semigroup is of order at least seven (Section 14.2).

Publications

Results of thesis submitted for defense are published in [121]–[130].

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