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Integrable systems and linear operators connected with two–point Baker–Akhiezer function

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The modern approach to the spectral theory of periodic differential and difference operators is a consequence of the success of the inverse scattering method. In 1967 Gardner, Green, Kruskal and Miura [1] proposed a spectral transformation method as a method for solving the Cauchy problem with rapidly decreasing initial data for the KdV equation. Subsequently, this led to the creation of a new method of mathematical physics — the inverse scattering method. In 1968 Lax [2] generalized this method and discovered the algebraic mechanism lying at the heart of the work of Gardner, Green, Kruskal and Miura: the KdV equation is equivalent to the so–called Lax representation. In 1971 Zakharov and Shabat [3] solved the nonlinear Schrödinger equation by the inverse scattering method. In 1973 the method was applied immediately to several equations in the work of Ablowitz, Kaup, Newell and Segure in [4]. The natural question was how to solve nonlinear equations like KdV for periodic initial data.

The integration scheme of the KdV equation with rapidly decreasing initial data consists in solving the direct problem, the simple evolution of spectral data and solving the inverse problem for the Sturm–Liouville operator on the line. A similar scheme turned out to be unrealizable for the KdV equation with periodic initial data, since both the direct and inverse problems were solved insufficiently efficient.

In the pioneering work of Novikov [5] a class of potentials which are the analogues of multisoliton solutions in the rapidly decreasing case was found. The task of finding a complete and effective description of the whole class of real finite–gap potentials for the one-dimensional Sturm–Liouville operator was solved independently by Dubrovin in [6], Matveev and Its in [7]. The necessity to improve the spectral theory of the Sturm–Liouville operator (see [9], [10], [5], [11], [12], [13], [14]) made it possible to solve the periodic problem for the KdV equation. Subsequently, the results were extended to such fundamental equations of mathematical physics as the sine–Gordon equation, the nonlinear Schrödinger equation, etc.

A key step in the development of the modern spectral theory of periodic operators was the understanding of the fact (now it is self–evident) that Bloch functions (solutions of the periodic Sturm–Liouville problem which are the eigenfunctions of the monodromy operator) are the values of one function on different sheets of a certain Riemann surface, which in the general case may not be smooth and of finite genus. This is true for numerous examples of one-dimensional linear operators with periodic coefficients. Such a curve became known as the spectral curve later. The role of the analytic properties of the Bloch functions on this curve was clarified in the works of Krichever [15], [16], where a general construction of algebro–geometric solutions of equations of mathematical physics was proposed and the key concept of the Baker–Akhiezer function was used.

The exponent e^z is the next in complexity after rational functions among the elementary functions of the complex argument. It is analytic in $\mathbb{C}$ and has an essential singularity at $z = \infty$. If $q(z)$ is rational then $e^{q(z)}$ is analytic in $\overline{\mathbb{C}}$ except poles $q(z)$. Clebsch and Gordon came up with a generalization of functions of exponential type to higher genus Riemann surfaces. If the genus is strictly greater than zero such functions must have poles in contrast to ordinary exponents. Baker noted that simple formulas could be obtained in terms of theta–functions of the corresponding Riemann surfaces for such exponential generalizations. The Baker–Akhiezer function is a function on a Riemann surface uniquely defined by its analytical properties. In [17] it was noted that under certain conditions functions of exponential type on hyperelliptic surfaces are eigen-

values of some linear differential operators of the second order. The main observation of Krichever was that these analytical properties are the key ingredient to the fact that Baker–Akhiezer functions are the eigenfunctions of the wide family of operators.

Initially, the Krichever integration scheme of nonlinear equations using the Baker–Akhiezer function was in a solution of some inverse problem (i.e. the operator was reconstructed from a curve and some algebraic–geometric data on it) without solving the direct problem (in particular, the problem of the constructing the spectral curve). The generality class of the obtained operators remained unclear. In addition, it was completely unclear what is a spectral curve in the case of two–dimensional periodic operators. The connection between the Krichever construction and the spectral theory of two–dimensional periodic operators was obtained in [18] related to the periodic non–stationary Schrödinger operator $\partial_y - \partial_x^2 + u(x,y)$ where the concept of Bloch–Floquet curve was proposed for the first time and an attempt to construct a generalization of the spectral curve for two-dimensional operators was made. The spectral theory of two–dimensional operators is much more difficult then the spectral theory of one–dimensional operators due to the fact that the spectral curve is defined by the characteristic equation of finite monodromy matrix in one–dimensional case. In the two–dimensional case the corresponding operators are infinite–dimensional. To make sense for characteristic equation in this case is complex and not always solvable problem.

This work consists of two Chapters with the general idea of using the modern approach to the spectral theory of periodic operators and its basic concept of the Baker–Akhiezer function (discrete and continuous).

In the Chapter One we investigate the Hamiltonian theory of equations related to the finite dimensional reductions of 2d Toda lattice hierarchy. These equations define the dynamics on the phase space of strictly low triangular operators. The main idea is in using of the spectral theory of such operators to prove that these equations are Hamiltonian.

Initially the 2d Toda lattice hierarchy is the system of the following Zakharov–Shabat equations

$$\begin{cases} \frac{\partial L_m^+}{\partial t_{m'}^+} - \frac{\partial L_{m'}^+}{\partial t_m^+} + [L_m^+, L_{m'}^+] = 0 \\ \frac{\partial L_m^-}{\partial t_{m'}^-} - \frac{\partial L_{m'}^-}{\partial t_m^-} + [L_m^-, L_{m'}^-] = 0 \\ \frac{\partial L_m^-}{\partial t_{m'}^+} - \frac{\partial L_{m'}^+}{\partial t_m^-} + [L_{m'}^+, L_m^-] = 0, \end{cases} \tag{1}$$

where

$$L_m^- = T^{-m} + \sum_{j=0}^{m-1} a_{i,m}^{j,-} T^{-j}, \ L_m^+ = \sum_{j=1}^{m} a_{i,m}^{m,-} T^j. \tag{2}$$

Despite that the every equation in (1) is correctly defined system in finite number of variables the 2d Toda lattice hierarchy is the infinite system in infinite number of variables. In [19] Krichever got algebro–geometric solutions of 2d Toda lattice hierarchy .

In this work we use the equivalent approach to this hierarchy [20], namely it is the

system of equations on the space of coefficients $\{\psi_\pm(i)\}$ which are the formal series

$$\psi_-(i) = z_-^i(1+\sum_{s=1}^{\infty}\xi_s^-(i)z_-^s),\ \psi_+(i) = z_+^{-i}e^{\varphi_i}(1+\sum_{s=1}^{\infty}\xi_s^+(i)z_+^s).$$

Define operators (2) by the equations

$$L_m^-\psi_-(i) = z_-^{-m}\psi^-(i) + O(z)\psi^-(i),\ L_m^+\psi_+(i) = z_+^{-m}\psi^+(i) + O(1)\psi^+(i).$$

Then the dynamics of $\xi_s^\pm(i)$ is defined as follows

$$\begin{cases} (\partial_{t_m^-} - L_m^-)\psi_- = -z_-^{-m}\psi_-,\ (\partial_{t_m^-} - L_m^-)\psi_+ = 0 \\ (\partial_{t_m^+} - L_m^+)\psi_+ = -z_+^{-m}\psi_+,\ (\partial_{t_m^+} - L_m^+)\psi_- = 0. \end{cases}$$

The commutativity of the flows $\partial_{t_m^\pm} - L_m^\pm$ is obvious which means the Zakharov–Shabat equations.

The main type of reductions considered in the theory of KP hierarchy are reductions to stationary points of one of the flows in the hierarchy (or a linear combination of such flows). The corresponding invariant submanifolds have finite functional dimension and correspond to pseudo-differential operators such that their n–th power is a differential operator. In the case of difference equations, the problem of finding such reductions becomes much more difficult. But in the case of a 2d Toda lattice hierarchy it can be solved as follows. Let the conditions hold

$$\begin{cases} L\psi_- = z_-^{-k-1}\psi_- \\ L\psi_+ = z_+\psi_+, \end{cases} \tag{3}$$

where $L = L_{k+1}^-$ is a periodic operator. From the second condition in (3) it follows that L is strictly low triangular. Zakharov–Shabat equations (1) transforms to the Lax equations

$$\left[\partial_{t_m^\pm} - L_m^\pm, L\right] = 0. \tag{4}$$

To understand that the equation (4) is a well-defined dynamical system on the finite-dimensional space $\mathscr{L}_{k+1}$ of n–periodic strictly lower triangular operators of order $k+1$, it is necessary to understand that the coefficients of the operators $L_m^\pm$ are functions on the space $\mathscr{L}_{k+1}$. It is possible if the condition $(n, k+1) = 1$ holds. Briefly, the proof scheme is as follows which we give in order to explain some of the results.

For the restriction of strictly low triangular periodic operator L of order $k+1$ on the space of quasiperiodic solutions $L(w) := L|_{\{\psi|w\psi_{i+n}=\psi_i\}}$ there exists two formal series $\psi_\pm$ with the properties given by the next two Lemmas

Lemma 1. *[21] For $L(w)$, $(n, k+1) = 1$ there exists a unique formal series*

$$E(z) = z_-^{-k-1}\left(1+\sum_{s=1}^{\infty}e_s z_-^s\right) \tag{5}$$

such that the problem $L(w)\psi = E\psi$ has unique solution of the form

$$\psi_-(i) = z_-^i\big(1 + \sum_{s=1}^{\infty} \xi_s^-(i) z_-^s\big),\ \xi_s^-(i) = \xi_s^-(i+n),\ \xi_s^-(0) = 0.$$

Lemma 2. *[21] The equation $L(w)\psi = E\psi$ has a unique formal solution of the following form*

$$\psi_+(i) = e^{\varphi_i} z_+^{-i}\big(1 + \sum_{s=1}^{\infty} \xi_s^+(i) z_+^s\big),\ a_i^{(1)} = e^{\varphi_i - \varphi_{i-1}},\ \xi_s^+(0) = 0.$$

For every point of the spectral curve with two marked points $p_\pm$, defined by the equation

$$\det(E - L(w)) = w^{k+1} - E^n + \sum_{i>0, j\geq 0, ni+(k+1)j<n(k+1)} r_{ij} w^i E^j = 0$$

we get the eigenvector $\psi(p) = \big(\psi_1, \psi_2, \ldots, \psi_n\big)^t$, $p = (w, E) \in \Gamma$. The behavior of $\psi(p)$ at points $p_\pm$ is given by two previous lemmas, the number of poles equals to the genus of Γ, so we claim that $\psi(p)$ is a two point Baker–Akhiezer function. It is clear that the values $\psi(p)$ at $p_\pm$ are particular cases of $\psi_\pm$. Using ψ we could get operators $L_m^\pm$. Thus, the system (4) is a finite dimensional system on the space $\mathscr{L}_{k+1}$.

In 1971 Zakharov and Faddeev proved in [22] that KdV is a Hamiltonian system. Much later in 1978 Magri in [23] showed that KdV is Hamiltonian with another Hamiltonian and Poisson bracket. So there was an understanding that the non–linear equations could be bi–Hamiltonian, and this was remarkable because it allowed to construct the integrals of motion which are in involution by some procedure.

We could introduce the family of 2–forms on the space of periodic operators (for our case it is $\mathscr{L}_{k+1}$) identified with the phase space of dynamical system following two articles [24] and[25] as

$$\omega^{(i)} = -\frac{1}{2}\sum_\alpha \mathrm{res}_{p_\alpha} E^{-i} \langle \psi^+(w)\, \delta L(w) \wedge \delta\psi(w)\rangle\, d\Omega,\ i \in \mathbb{N}.$$

Here the sum in the last formula is over points p_α on spectral curve Γ, where the right hand side has a pole *a priori*: 1) at the marked points $p_\pm$, where the Baker–Akhiezer function and its dual have poles; 2) at zeroes p_ℓ, $\ell = 1, \ldots, k$ of function $E = E(p)$, where $w = w(p)$ is non–zero, i.e. $E(p_\ell) = 0$, if $w(p_\ell) \neq 0$. Due to the general principles (see [25]) the substitution of the vector field defined by the Lax equation into these forms is an exact form. If the 2–forms are non–degenerate it means that the equations are Hamiltonian. Before the appearance of these two articles this statement was directly verified for each equation. In addition, the action–angle variables were not calculated in an efficient way. At the same time, the implementation of this scheme is a complex technical problem depending on the initial form of the operator space. Using the scheme below, firstly, we prove that 2–forms $\omega^{(i)}$, $i = 0, 1$ are non–degenerate on Λ_i^c

Lemma 3. *[34] 2–form $\omega^{(0)}$ is non–degenerate on the submanifold*

$$\Lambda_0^c := \{L|\, e_s(L) = c_s, s = 1, \dots k\}$$

Lemma 4. *[34] 2–form $\omega^{(1)}$ is non–degenerate on the submanifold*

$$\Lambda_1^c := \{L|\, r_{i,0}(L) = c_i, 1 = 1, \dots k\}$$

where $c = (c_1, \dots, c_k)$ are constants, $r_{i,0}(L) = r_{i,0}$ are coefficients of polynomial

$$\det L(w) = w^{k+1} + \sum_{i=1}^{k} r_{i,0} w^i.$$

Next, we prove that the vector fields $\partial_{t_m^\pm}$ defined by the Lax equation (4) on the space $\mathscr{L}_{k+1}$ and restricted to the submanifolds $\Lambda_i^c,\ i = 0, 1$ are Hamiltonian with respect to the forms $\omega^{(i)}$ and calculate Hamiltonians explicitly.

Theorem 1. *[34] The vector field $\partial_{t_m^\pm}$ defined by the Lax equation (4) and restricted to the submanifold Λ_i^c is Hamiltonian for $i = 0, 1$ with respects to the forms $\widehat{\omega}^{(i)} = \omega^{(i)}\big|_{\Lambda_i^c}$ with Hamiltonians*

$$H^{(0)}_{t_m^-} = res_{p_-} z_-^{-m} E(z_-)\, d\ln z_- = e_{m+k+1}, \tag{6}$$

$$H^{(1)}_{t_m^-} = res_{p_-} z_-^{-m} \ln E(z_-)\, d\ln z_- \tag{7}$$

where $E(z_-)$ is a series (5) with coefficients defined by Lemma 1, and

$$H^{(i)}_{t_m^+} = \frac{1}{n} res_{p_+} E^{-m-i} \ln w(E) dE, \quad i = 0, 1. \tag{8}$$

The above results is applied to the case when L is a strictly lower triangular operator of arbitrary order. In the case of low triangular dynamics

$$\partial_{t_m^-} L = [L, L_m^-], \tag{9}$$

if we define $(L_1^-\psi)_i = v_i\psi_i + \psi_{i-1}$, $(L\psi)_i = a_i^1\psi_{i-1} + \ldots + a_i^k\psi_{i-k} + \psi_{i-k-1}$ and $v_i = \partial_{t_1^-}\varphi_i$, the equation (9) is equivalent to the following system in variables a_i^j

$$\begin{cases} \partial_{t_1^-} a_i^1 = a_i^1(v_i - v_{i-j}) \\ \partial_{t_1^-} a_i^j = a_i^j(v_i - v_{i-j}) + a_{i-1}^{j-1} - a_i^{j-1},\ j = \overline{2, k} \\ a_{i-1}^k - a_i^k = v_{i-k-1} - v_i. \end{cases} \tag{10}$$

In particular case if L is of second order then the equations (10) have the simple form

$$\partial_{t_1^-}\varphi_{i-1} - \partial_{t_1^-}\varphi_{i+1} = e^{\varphi_i - \varphi_{i-1}} - e^{\varphi_{i+1} - \varphi_i}. \tag{11}$$

and the following theorem holds

Theorem 2. *[34] Let $\mathscr{L}_2 = \{a_i T^{-1} + T^{-2} \mid a_{i+n} = a_i\}$ is the space of periodic strictly low triangular difference operators of the second order. Then*

1. *the equation $\partial_{t_1^-} L = [L, L_1^-]$ (or (11)) on the space $\mathscr{L}_2$ restricted to the submanifold Λ_0^c is Hamiltonian with respect to the symplectic 2–form $\widehat{\omega}^{(0)} = \langle dx_i \wedge dx_{i-1} \rangle$, where $a_i = x_i - x_{i-2} + e_1$ and the corresponding Hamiltonian is*

$$H^{(0)}_{\partial_{t_1^-}} = \frac{1}{n}\langle x_i^2 (x_{i-1} - x_{i+1})\rangle + \frac{e_1}{n}\langle x_i (x_{i+1} - x_i)\rangle.$$

2. *the equation $\partial_{t_1^-} L = [L, L_1^-]$ on the space $\mathscr{L}_2$ restricted to a submanifold Λ_1^c is Hamiltonian with respect to the symplectic 2–form $\widehat{\omega}^{(1)} = \langle d\varphi_i \wedge d\varphi_{i+1} \rangle$, where $a_i = e^{\varphi_i - \varphi_{i-1}}$ and the corresponding Hamiltonian is*

$$H^{1}_{\partial_{t_1^-}} = \langle e^{\varphi_i - \varphi_{i-1}} \rangle.$$

For the case of the third order we have

Theorem 3. *[34] In case $k = 2$ the equation $\partial_{t_1^-} L = [L, L_1^-]$ on the space $\mathscr{L}_3$, restricted to a submanifold Λ_0^c, is Hamiltonian with respect to the symplectic 2–form*

$$\widehat{\omega}^{(0)} = \langle\, dy_i \wedge (dx_{i-1} - dx_{i+2}) + d(x_{i-1}x_{i-2}) \wedge dx_i + e_1 dx_i \wedge dx_{i-1} \rangle,$$

and the corresponding Hamiltonian is

$$H^{(0)}_{\partial_{t_1^-}} = \frac{1}{n}\langle\, y_{i-1}(y_i - y_{i-3})\rangle + \frac{1}{n}\langle x_i x_{i-1} x_{i-2}(x_{i-1} - x_i)\rangle + \frac{e_1}{n}\langle (x_i^2 (x_{i-1} - x_{i+1})\rangle$$

$$+ \frac{e_2}{n}\langle x_{i-1}(x_i - x_{i-1})\rangle + \frac{1}{n}\langle y_i (x_{i+2}^2 - x_{i-1}^2 - x_{i+2}x_{i+1} + x_{i-2}x_{i-1})\rangle.$$

In case of arbitrary order of L we have

Theorem 4. *The equation (9) restricted to a leaves with constant w_ℓ is Hamiltonian with respect to the form*

$$\omega^{(1)} = \frac{1}{2}\langle d\varphi_{i-1} \wedge d\varphi_i \rangle - \langle (-1)^{(i-1)k} e^{\varphi_{i-1}} \sum_{j=1}^{k} da_i^{(j)} \wedge |\phi_{i-2}, \ldots, \phi_{i-k}, d\phi_{i-j}| \rangle, \tag{12}$$

where $\phi_i^\ell = \psi_i(p_\ell)$ are the values of Baker–Akhiezer function at the preimages $E = 0$ where $w \neq 0$, i.e. $p_\ell = (w_\ell, 0)$, $a_i^{(j)} = (-1)^{ik+1} e^{\varphi_i} |\phi_{i-1}, \ldots, \phi_{i-j+1}, \phi_{i-k-1}, \phi_{i-j-1}, \ldots, \phi_{i-k}|$, and $e^{-\varphi_i} := (-1)^{ik} |\phi_{i-1}, \ldots, \ldots, \phi_{i-k}|$. The corresponding Hamiltonian is

$$H^{1}_{\partial_{t_1^-}} = \langle a_i^{(k)} \rangle.$$

In case of upper triangular dynamics the following theorem holds

Theorem 5. *[34] The equation*

$$\partial_{t_1^+} L = [L, L_1^+], \tag{13}$$

restricted to the leaves with fixed w_ℓ is Hamiltonian with respect to the form (12). The corresponding Hamiltonian is

$$H_{\partial_{t_1^+}} = \frac{1}{n} res_{E=0} \ln w(E) E^{-2} dE = w_1 = -\langle a_i^{(2)} e^{\varphi_{i-2}-\varphi_i} \rangle,$$

where w_1 is the first coefficient of w in the neighborhood of p_+.

If L is of second order and $(L_1^+\psi)_i = c_i\psi_{i+1}$, $c_i = e^{\varphi_i - \varphi_{i+1}}$, $(L\psi)_i = a_i^1\psi_{i-1} + \ldots + a_i^k\psi_{i-k} + \psi_{i-k-1}$ then the equations of motion (13) are simple

$$\partial_{t_1^+} a_i^{j-1} = c_i a_{i+1}^j - c_{i-j} a_i^j, \ j = \overline{2,k},$$

and in particular cases $k = 1$

$$\partial_{t_1^+}\varphi_i - \partial_{t_1^+}\varphi_{i-1} = e^{\varphi_{i-1}-\varphi_{i+1}} - e^{\varphi_{i-2}-\varphi_i}.$$

The corresponding Hamiltonian equals

$$H_{t_1^+}^{(1)} = -\langle e^{\varphi_{i-2}-\varphi_i} \rangle.$$

In addition we find the coordinates such that the 2–forms $\omega^{(i)}$, $i = 1, 2$ are local (in terms of coefficient of operator which are the natural coordinates on the space $\mathscr{L}_{k+1}$ the forms $\omega^{(i)}$ are non–local). We prove that $\omega^{(0)}$ is local in terms of $\{\xi_s^-(i), e_s\}$, $s = \overline{1,k}$ and $\omega^{(1)}$ is local in terms $\phi_i^\ell = \psi_i(p_\ell)$, $p_\ell = (w_\ell, 0)$. Further we consider the question of symplectic structure on the space of superperiodic strictly low triangular operators $\mathcal{E}_{k+1,n}$ of order $k+1$. $\mathcal{E}_{k+1,n}$ is the space of n–periodic operators of $k+1$–order acting on the discrete functions by the following rule

$$L'\psi_i' = \psi_{i-k-1}' + a_i^k\psi_{i-k}' + \ldots a_i^1\psi_{i-1}', \ a_{i+n}^j = a_i^j$$

and such that all solutions of $L\psi_i' \equiv -\psi_i'$ are n–(anti)periodic functions, i.e. $\psi_{i+n}' = (-1)^{n+k}\psi_i'$. The space $\mathcal{E}_{k+1,n}$ appears naturally in the theory of cluster algebras [26], the representation theory [27] and theory of pentagram maps [28], [29]).

In Chapter Two the main results are related to the direct construction of the spectral theory of the two-dimensional periodic Schrödinger operator and the generalization of the well–known Veselov–Novikov construction.

For the two-dimensional Schrödinger operator the inverse problem was solved firstly and the spectral theory was constructed later. In [30] Dubrovin, Krichever and Novikov proposed a construction that allows one to reconstruct a two–dimensional Schrödinger operator with a magnetic field

$$(i\partial_x + A_1(x,y))^2 + (i\partial_y + A_2(x,y))^2 + u(x,y). \tag{14}$$

Using non–singular algebraic curve $\mathscr{T}$ of genus g, two points $P_\pm$, local coordinates $k_\pm^{-1}$ in the neighborhoods of $P_\pm$, the general divisor D of degree g the Baker–Akhiezer function is constructed with prescribed exponential singularities at $P_\pm$. It is a eigenfunction of (14) on the zero energy level. Note that the resulting magnetic field has a zero flux.

In [31], [32] Veselov and Novikov found the conditions on algebraic–geometric data of previous construction corresponding to the so–called potential Schrödinger operators

$$\mathcal{H} = -\Delta + u(x,y),\ \Delta = \partial_x^2 + \partial_y^2. \tag{15}$$

If the extra conditions holds such as 1) the existence of an involution σ on $\mathscr{T}$ with property $\sigma(P_\pm) = P_\pm$; 2) the transformation rule of local coordinates $k_\pm^{-1}(\sigma(p)) = -k_\pm^{-1}(p)$; 3) the existence of meromorphic differential $d\Omega$ with zeroes at $D+\sigma(D)$ and simple poles at $P_\pm$, then the operator of the form (15) is uniquely reconstructed from this algebraic–geometric data and the corresponding potential has the form of the second logarithmic derivative of Prym theta–function of the the covering $\mathscr{T} \to \mathscr{T}_0 = \mathscr{T}/\sigma$. The sufficient condition of reality of the potential $u(x,y)$ is the existence of antihilomorphic involution τ commuting with σ i.e. $\sigma\tau = \tau\sigma$ with properties $\tau(P_\pm) = P_\mp$, $\tau^*(k_\pm^{-1}) = \bar{k}_\mp^{-1}$, $\tau(D) = D$. In addition there are two types of conditions on spectral data corresponding to the nonsingularity of $u(x,y)$: 1) $\mathscr{T}$ must be a M–curve and for $g+1$ fixed ovals the property $\sigma(a_{j+g_0}) = \tilde{a}_j, j = \overline{1, g_0}$, $g_0 =$ genus $\mathscr{T}/\sigma$ holds; 2) the anti–involution $\sigma\tau$ is of the separating type and the differential $d\Omega$ is positive on fixed ovals with respect to some fixed orientation.

With the development of the spectral theory of (15) the question of comleteness of the Veselov–Novikov conditions arised naturally (for example, is it possible to construct a nonsingular (15) from a singular curve), the solution of the direct problem of spectral theory should answer the question. The constructive approach of [33] establishing some specific properties of spectral curve is important. In general case there exists holomorphic involution σ :

$$\sigma :\ (w_1, w_2) \to (w_1^{-1}, w_2^{-1})$$

and, if the potential is real, then there exists antiholomorphic involution τ :

$$\tau :\ (w_1, w_2) \to (\bar{w}_1, \bar{w}_2),$$

commuting with σ preserving the pole divisor of Bloch solution. Moreover, if the potential is finite–zone then there exists two marked points $P_\pm$ which are fixed by σ and the Bloch–solution has a form of two–point Baker–Akhiezer function in their neighborhoods. So, the results obtained in [33] gave the necessity of some Veselov–Novikov conditions.

The spectral theory of the two-dimensional periodic Schrödinger operator is not fully constructed. Despite the non-stationary Schrödinger operator $\partial_y - \partial_x^2 + u(x,y)$ with a smooth real periodic potential, the spectral curve of the two-dimensional Schrödinger operator $\mathcal{H}$ with smooth and real periodic potential may have a finite number of singular points. The questions about types of singularities remains open.

One of the main purpose of the Chapter Two is to clarify some problems of the spectral theory of Schrödinger operator. As in [33] the Fermi curve of $\mathcal{H}$ is considered as the perturbation of spectral curve of $-\Delta - \lambda$, where $u(x,y) = -\lambda + v(x,y)$, λ is constant, $v(x,y)$ is sufficiently small and periodic, the condition $\lambda \neq \frac{m^2\ell_1^2 + n^2\ell_2^2}{4\ell_1\ell_2}$, $m, n \in \mathbb{Z}$ holds. In case of positive potentials, i.e. $\lambda < 0$ we present the new construction allowing to simplify

the description of the Fermi curve in terms of some data $\mathcal{P}$, consisting of the positive number and finite (or infinite) number of pairs satisfying the given constraints. We prove that the spectral curve of two–dimensional Schrödinger operator with positive potential is M–curve with respect to the antiinvolution τ, in addition the poles of the Bloch function locate one on each of the fixed ovals. Note that the spectral curve of the two-dimensional periodic Schrödinger operator with a nonsingular potential is an M–curve necessarily. The topological type is stable until, at some value of the deformation parameter, the zero level becomes an eigenlevel for the Schrödinger operator on the space of (anti)periodic functions.

Theorem 6. *[35] For any real positive periodic potential $u(x,y) > 0$ analytically continued to a neighborhood of real x, y, the Bloch solutions of the equation $(-\Delta - \lambda + v)\tilde{\phi} = 0$ are parameterized by the points of the Riemann surface $\Gamma(\mathcal{P})$ for some admissible set $\mathcal{P}$. The corresponding function $\tilde{\phi}$ is meromorphic and has one simple pole on each of the cycles a_s.*

One of the main results is the generalized Veselov–Novikov construction allowing to construct Schrödinger operators such that the zero level is eigenlevel of the periodic problem. In fact, the inverse problem is never solved with a singular curve (Fermi curve could be singular), instead, the Baker–Akhiezer function is defined on its normalization. Instead of the usual Veselov–Novikov construction, the generalized Veselov–Novikov construction proposed in Chapter Two uses an involution on curve having $n+1$–pairs of fixed points $(P_\pm, p^1_\pm, \dots p^n_\pm)$ for arbitrary $n > 0$. In this case the Baker–Akhiezer function is defined by divisor of degree $g+n$, analytical properties in the neighborhoods of infinities and n conditions of gluing at $p^i_\pm$. The formulas for the Baker–Akhiezer function and the corresponding potential of the two-dimensional Schrödinger operator in terms of proper Prym theta–function θ are found.

Theorem 7. *[35] The Baker-Akhiezer function of (15) is*

$$\phi(x,y,p) = \frac{\theta(A(p) + zU_+ + \bar{z}U_- + Z|\Pi)\theta(Z|\Pi)}{\theta(zU_+ + \bar{z}U_- + Z|\Pi)\theta(A(p) + Z|\Pi)} e^{z\Omega_+(p) + \bar{z}\Omega_-(p)}, \tag{16}$$

where U_+ and U_- are the vectors with coordinates $U^j_\pm = \frac{1}{2\pi i}\oint_{b_j} d\Omega_\pm$ and $Z = -\sum_s A(\gamma_s) + \mathcal{K}$, where $\mathcal{K}$ is a constant vector.

Theorem 8. *[35] The Baker-Akhiezer function ϕ given by formula (16) where Z is a generic vector is a solution of $\mathcal{H}\phi = 0$ with potential*

$$u(x,y) = -2\Delta \ln \theta(zU_+ + \bar{z}U_- + Z|\Pi) + E, \quad E := 4\frac{d\Omega_-}{d(k_+^{-1})}(P_+).$$

If

$$2\pi\ell_1(U_+ + U_-) = N^a + \Pi N^b, \quad 2\pi i\ell_2(U_+ - U_-) = M^a + \Pi M^b$$

for some integer vectors N_1, N_2 and M_1, M_2, then the function $u(x,y)$ is $(2\pi\ell_1, 2\pi\ell_2)$-periodic and the functions $\phi_i := \phi(x,y,p_i^\pm)$ are the eigenfunctions of the operator $\mathcal{H}$ on the space of (anti)periodic functions.

The answers to the question of the non-singularity are obtained using the generalized Veselov–Novikov construction and given by the following two theorems

Theorem 9. *[35] Suppose that $\mathscr{T}$ is an M-curve whose antihilomorphic involution has fixed ovals $a_0, a_1, \dots, a_{g_0}$, $\tilde{a}_1, \dots a_{g_0}$ and holomorphic involutions acts as*

$$\sigma(a_i) = \tilde{a}_i, \quad \sigma(b_i) = \tilde{b}_i, \ i = 1, \dots, g_0$$

and

$$\sigma(a_{i'}) = -a_{i'}, \quad \sigma(b_{i'}) = -b_{i'}, \ i' = \overline{g_0 + 1, g_0 + n}.$$

Suppose also that $p^i_\pm \in a_{2g_0+i}$. Let the points γ_s of an admissible divisor D of degree $g + n$ be such that each of the fixed ovals $a_1 \dots, a_{g_0}$, $\tilde{a}_1, \dots \tilde{a}_{g_0}$ and each of the segments into which the ovals a_{2g_0+i} are partitioned by the points $p^i_\pm$ contains precisely one of these points. Then the corresponding potential is real and nonsingular.

Theorem 10. *[35] Suppose that the antiholomorphic involution $\sigma\tau$ is of separating type, i.e., the complement to its fixed ovals is a disjoint union of two domains $\mathscr{T}^\pm$*

$$\sigma\tau(\mathscr{T}^+) = \mathscr{T}^-.$$

Suppose also that $p^i_\pm \in \mathscr{T}^\pm$, the differential $d\Omega$ defining an admissible divisor D is positive on these ovals with respect to the orientation induced from the domain $\mathscr{T}^+$, and

$$res_{p_i^+} d\Omega < 0.$$

Then the corresponding potential of the Schrödinger operator is real and nonsingular.

We give an example of generalized Veselov–Novikov construction in case of hyperelliptic curves. Considering special choice of pairs $p_i^\pm$, we obtain that the Prym matrix Π of normalized holomorphic differentials is doubled period matrix, so the values of Baker–Akhiezer function at gluing points are simple. Note that the corresponding Schrödinger operators have n eigenfunctions. Due to the main property of the hyperelliptic curve we obtain special type of self-consistent conditions (the essence of self-consistency is in dependency of potential from solution). In addition, we prove that the system of Schrödinger equations connecting with this self–consistency condition is Lagrangian. Explicit form of Lagrangian is obtained and Euler–Lagrange equations of motions coincide with original equations.

The results of the thesis are published in two papers

- Ilina, A.V., Krichever, I.M., *Triangular reductions of the 2D toda hierarchy,* Funct. Anal. Its Appl. 51, 48–65 (2017).
- Ilina A.V., Krichever I.M., Nekrasov N.A., *Two–dimensional periodic Schrödinger operators integrable on an energy "eigenlevel"*, Funktsional. Anal. i Prilozhen., 53:1 (2019), 31–48

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