Young Researchers in Algebraic Number Theory

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Chow dilogarithm and reciprocity laws

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Conventions

- Fix some algebraically closed field k of characteristic zero.
- I work modulo torsion. So implicitly any abelian group is tensored by $\mathbb{Q}$. I write $F^\times$ instead of $F^\times \otimes \mathbb{Q}$ etc.

Definition of Bloch-Wigner dilogarithm

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It satisfies the following equation called Abel five-term relation:

$$\mathcal{L}_2(x) - \mathcal{L}_2(y) + \mathcal{L}_2(y/x) +$$
$$+\mathcal{L}_2((1-x^{-1})/(1-y^{-1})) - \mathcal{L}_2((1-x)/(1-y)) = 0.$$

Definition of pre-Bloch group

Definition

For any field F the pre-Bloch group $\mathcal{P}(F)$ is defined as the quotient of the free vector space generated by symbols $\{x\}_2, x \in F\backslash\{0,1\}$, by the following Abel five-term relations:

$$\{x\}_2 - \{y\}_2 + \{y/x\}_2 + \{(1-x^{-1})/(1-y^{-1})\} - \\ -\{(1-x)/(1-y)\}_2 = 0,$$

where $x, y \in F\backslash\{0,1\}, x \neq y$.

Properties of pre-Bloch group

- There is a canonical map $\widetilde{\mathcal{L}}_2 \colon \mathcal{P}(F) \to \mathbb{R}$ given by the formula $\{x\}_2 \mapsto \mathcal{L}_2(x)$.

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- When F is a number field this map is believed to be injective.

Polylogarithmic complex: definition

Let F be an arbitrary field. Consider the following complexes placed in degrees $[1, 2]$:

- $$\Gamma(F, 2)\colon \mathcal{P}(F) \xrightarrow{\delta_2} \Lambda^2 F^\times.$$

  Here the differential is given by the formula $\delta_2(\{x\}_2) = x \wedge (1 - x)$.

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- $$\Gamma_2(F, 3)\colon \mathcal{P}(F) \otimes F^\times \xrightarrow{\delta_3} \Lambda^3 F^\times.$$

  Here the differential is given by the formula $\delta_2(\{x\}_2 \otimes y) = x \wedge (1 - x) \wedge y$.

Tame-symbol map

Let (F, ν) be a discrete valution field. A. Goncharov defined a moprphism of complexes $\partial_\nu : \Gamma_2(F,3) \to \Gamma(\overline{F}_\nu, 2)$ called tame-symbol map.

$$\begin{array}{ccc} \mathcal{P}(F) \otimes F^\times & \xrightarrow{\delta_3} & \Lambda^3 F^\times \\ \partial_\nu \downarrow & & \downarrow \partial_\nu \\ \mathcal{P}(\overline{F}_\nu) & \xrightarrow{\delta_2} & \Lambda^2 \overline{F}_\nu^\times \end{array}$$

Total residue map

Denote by $\mathbf{Fields}_d$ the category of finitely generated extensions of k of transcendent degree d. For a field $F \in \mathbf{Fields}_1$ denote by $val(F)$ the set of all discrete valuations of F. Denote $TotRes_F = \sum_{\nu \in val(F)} \partial_\nu$.

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- D. Rudenko has proved that $TotRes_F$ is null homotopic. [Rudenko, 2021]

Strong reciprocity law: definition

Definition

Strong reciprocity law on a field $F \in \mathbf{Fields}_1$ is a homotopy h between $TotRes_F$ vanishing on the subgroup $\Lambda^2 F^\times \wedge k^\times$.

$$\begin{array}{ccc}
\mathcal{P}(F) \otimes F^\times & \xrightarrow{\delta_3} & \Lambda^3 F^\times \\
{\scriptstyle TotRes_F}\downarrow & \overset{h}{\swarrow} & \downarrow{\scriptstyle TotRes_F} \\
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Strong reciprocity laws: properties

- For a field $F \in \mathbf{Fields}_1$ denote by $SRL(F)$ the set of all strong reciprocity laws on F.

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- If $j\colon F_1 \to F_2$ is an embedding, there is a natural map $j^*\colon SRL(F_2) \to SRL(F_1)$ given by the formula $j^*(h_2)(a) = h_2(a)/[F_2 : F_1]$, where $h \in SRL(F_2)$ and $a \in \Lambda^3 F_1^\times$.

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- One can show that in this way get a functor $SRL\colon \mathbf{Fields}_1 \to \mathbf{Set}$.

Strong reciprocity law: the case of $k(t)$

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- Strong reciprocity law is unique.
- It is given by the following formulas:

$$h_{k(t)}((t-a)\wedge(t-b)\wedge(t-c)) = \left\{\frac{c-a}{c-b}\right\}_2$$

$$h_{k(t)}((at+b)\wedge(ct+b)\wedge e) = 0,$$

  where $a, b, c, d, e \in k$.
- We denote it by $\mathcal{H}_{k(t)}$.

Canonical strong reciprocity law

Main theorem

On any field $F \in \mathbf{Fields}_1$ one can choose a strong reciprocity law $\mathcal{H}_F$ such that for any embedding $j\colon F_1 \hookrightarrow F_2$ we have $j^*(\mathcal{H}_{F_2}) = \mathcal{H}_{F_1}$.

Moreover, the family of strong reciprocity laws $\mathcal{H}_F, F \in \mathbf{Fields}_1$ is uniquely determined by the propriety stated above.

Norm map

Theorem (about existence of norm map)

For any field extension $F_1 \subset F_2$ there is a canonical norm map $N_{F_2/F_1} \colon SRL(F_1) \to SRL(F_2)$ satisfying the following conditions:

- If $F_1 \subset F_2 \subset F_3$, then $N_{F_3/F_1} = N_{F_3/F_2} \circ N_{F_2/F_1}$
- The composition $SRL(F_1) \xrightarrow{N_{F_2/F_1}} SRL(F_2) \xrightarrow{j^*} SRL(F_1)$ is identical.
- If $j \colon k(t) \to F$ an embedding then the strong reciprocity law $N_{F/k(t)}(\mathcal{H}_{k(t)})$ does not depend on j.

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- Define the strong reciprocity law $\mathcal{H}_F$ by the formula $N_{F/k(t)}(\mathcal{H}_{k(t)})$.
- It does not depend on j.
- The property $j^*(\mathcal{H}_{F_2}) = \mathcal{H}_{F_1}$ is easy follows from the previous theorem.

Definition of a system of strong reciprocity laws

Let $L \in \mathbf{Fields}_2$. Denote by $dval(L)$ the set of all divisorial discrete valuations.

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Definition

A system of a strong reciprocity laws σ on L is a choose of $\sigma_\nu \in SRL(\overline{L}_\nu)$ for any $\nu \in dval(L)$, such that for any $b \in \Lambda^4 L^\times$, the following formula holds:

$$\sum_{\nu \in dval(L)} \sigma_\nu(\partial_\nu(b)) = 0.$$

Main lemma: preliminaries 1

- For a field $L \in \mathbf{Fields}_2$ denote by $SOSRL$ the set of all systems of strong reciprocity laws on L. It can be extended to a contravariant functor $SOSRL\colon \mathbf{Fields}_2 \to \mathbf{Set}$.

Main lemma: preliminaries 1

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- Denote by $rat \colon \mathbf{Fields}_1 \to \mathbf{Fields}_2$ a functor given by the formula $F \mapsto F(t)$.

Main lemma

Denote by *res* the natural transformation associating to $\sigma \in SOSRL(F(t))$ a strong reciprocity law σ_{ν_∞}, where ν_∞ is the valuation of the field $F(t)$ corresponding to the point $\infty \in \mathbb{P}^1_F$.

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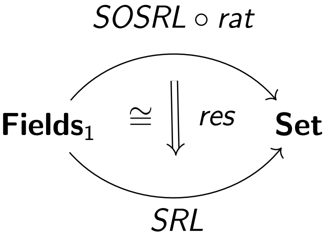
Main lemma

res is an isomorphism of functors.



Definition of norm map

- Let $j\colon F_1 \to F_2$ be an extension. Choose some generator a of this extension. Let $p_a \in F_1[t]$ be its minimal polynomial. Denote by $\nu_a \in davl(F_1(t))$ the corresponding divisorial valuation.

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- Define the map $N_{F_2/F_1,a}\colon SRL(F_1) \to SRL(F_2)$ by the formula $N_{F_2/F_1,a}(h) = (res_{F_1}^{-1}(h))_{\sigma_a}$.

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- The fact that $N_{F_2/F_1,a}$ does not depend on a and all the properties from the theorem about norm map can be proved similarly to the construction of norm map on Milnor k-theory.

Two-dimensional reciprocity law

Corollary 1

Let $L \in \mathbf{Fields}_2$. The association to any $\nu \in dval(L)$ the canonical strong reciprocity law $\mathcal{H}_{\bar{L}_\nu}$ is a system of strong reciprocity laws.

Two-dimensional reciprocity law

Corollary 1

Let $L \in \mathbf{Fields}_2$. The association to any $\nu \in dval(L)$ the canonical strong reciprocity law $\mathcal{H}_{\bar{L}_\nu}$ is a system of strong reciprocity laws.

In other words, for any $b \in \Lambda^4 L^\times$ the following equality holds:

$$\sum_{\nu \in dval(L)} \mathcal{H}_{\bar{L}_\nu} \partial_\nu(b) = 0.$$

Definition of Chow dilogarithm

Let X be a smooth projective curve over $\mathbb{C}$ and $f_1, f_2, f_3 \in \mathbb{C}(X)^\times$ be three non-zero rational functions on X. Chow dilogarithm is defined by the following formula:

$$\mathcal{CL}_2(X|f_1, f_2, f_3) = \frac{1}{2\pi i} \int_{X(\mathbb{C})} r_2(f_1, f_2, f_3),$$

where $r_2(f_1, f_2, f_3)$ is some explicitly defined 2-distribution.

Chow dilogarithm is expressible in terms of Bloch-Wigner dilogarithm

Let $k = \mathbb{C}$. For any smooth projective curve over $\mathbb{C}$ we get the map $\mathcal{H}_{\mathbb{C}(X)} \colon \Lambda^3 \mathbb{C}(X)^\times \to \mathcal{P}(\mathbb{C})$. From the other side we have the map $\widetilde{\mathcal{L}}_2 \colon \mathcal{P}(\mathbb{C}) \to \mathbb{R}$.

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Corollary 2

For any smooth projective curve X over $\mathbb{C}$ and $f_i \in \mathbb{C}(X)^\times$, the following formula holds:

$$\mathcal{CL}_2(X|f_1, f_2, f_3) = \widetilde{\mathcal{L}}_2(\mathcal{H}_{\mathbb{C}(X)}(f_1 \wedge f_2 \wedge f_3)).$$

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Thank you for your attention!