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ABSTRACT

We study the phenomenon of a collision of a Hénon-like attractor with a Hénon-like repeller leading to the emergence of mixed dynamics in the model describing the motion of two point vortices in a shear flow perturbed by an acoustic wave. The mixed dynamics is a recently discovered type of chaotic behavior for which a chaotic attractor of the system intersects with a chaotic repeller. In all known systems with mixed dynamics, the difference between the numerically obtained attractor and repeller is small. Unlike these systems, the model under consideration demonstrates another type of mixed dynamics that we call "strongly dissipative." In this case, a strange attractor and a strange repeller have a nonempty intersection but are very different from each other, and this difference does not appear to decrease with increasing computation time.

Dynamics of dissipative systems with chaotic behavior are, in many cases, associated with strange (chaotic) attractors. One of the simplest and most well-known examples of systems with a strange attractor is given by the two-dimensional Hénon map.[1] Its attractor contains a saddle fixed point with a negative unstable multiplier and is formed after a cascade of period-doubling bifurcations[2–4] followed by a cascade of heteroclinic "band-fusion" bifurcations.[5] Such strange attractors, further Hénon-like attractors, appear in many two-dimensional (and also higher-dimensional) maps, as well as in Poincaré maps for various systems of differential equations. In this paper, we show that the crisis of a Hénon-like attractor in a reversible system can lead to the appearance of strongly dissipative mixed dynamics (SDMD), when a strange attractor and the symmetric to it, a strange repeller, have a nonempty intersection but are very different from each other. As an example of a system demonstrating such behavior, we consider the model describing the motion of two point vortices in a shear flow perturbed by an acoustic wave.

I. INTRODUCTION

The crisis of a Hénon-like attractor happens due to the collision with the boundary of its absorbing domain.[5,6] In the Hénon map, after such a crisis, the attractor is destroyed and most of the orbits from its neighborhood go to infinity. In problems demonstrating multistable dynamics, orbits after such a crisis can run to another attractor located in a different region of phase space. It is also important to note that in some cases, especially in systems with symmetries,[7–9] the crisis of a Hénon-like attractor can lead to its collision with another chaotic attractor, e.g., with a Hénon-like one. In more rigorous terms, such phenomena are related to the emergence of heteroclinic intersections between the unstable invariant manifolds forming the "skeleton" of the attractors and those stable manifolds that bound their domains of attraction.

In this paper, we investigate a phenomenon of collision of a Hénon-like attractor with a Hénon-like repeller. This is a completely new effect: instead of a union of two attractors, we observe a merger of an attractor and a repeller. We study it on the example of the reversible two-dimensional map arising in the system describing the motion of two point vortices in a shear flow perturbed by an acoustic wave.[7] After such a merger, dissipative dynamics associated with the existence of isolated Hénon-like attractor and Hénon-like repeller [see Fig. 1(a)] are replaced by another type of chaos—the so-called mixed dynamics.[10–14] Here, the strange attractor of the system increases in size explosively and starts to contain infinitely many area-expanding and area-preserving saddle periodic orbits, in addition to area-contracting ones. The picture of the Poincaré map

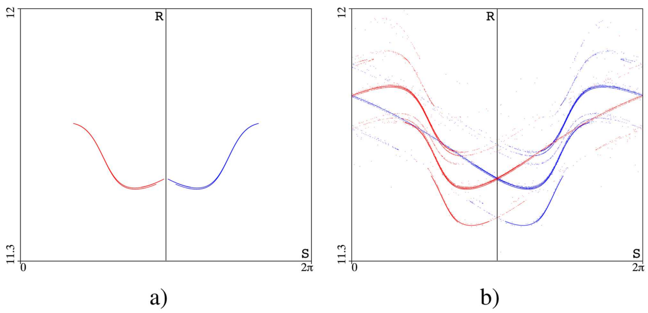


FIG. 1. The phase portraits of the attractor (in blue) and the repeller (in red) in the Poincaré map for the model of two point vortices perturbed by an acoustic wave and a shear flow [see system (1)]. (a) The Hénon-like attractor is separated from the Hénon-like repeller; the parameter values are $A = 0.1$, $\kappa = 4.65$, and $\varepsilon = 0.1463$. (b) Mixed dynamics after the merger of the Hénon-like attractors with the Hénon-like repellers; parameter values: $A = 0.1$, $\kappa = 4.65$, and $\varepsilon = 0.148\,15$.

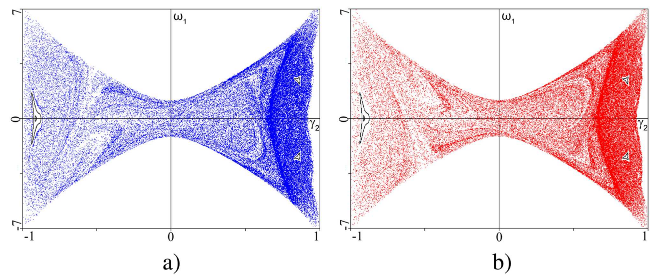


FIG. 2. The phase portraits of (a) the attractor and (b) the repeller in the corresponding two-dimensional Poincaré map for the nonholonomic model of Suslov top.[19] It is seen that the attractor and the repeller almost coincide but differ in some small details.

where the strange attractor intersects with the strange repeller but does not coincide with it is given in Fig. 1(b).

The mixed dynamics phenomenon has been predicted in Ref. 10 and, since then, has been often observed in reversible systems from various applications (see, e.g., Refs. 15–22); recently, an interesting example of this phenomenon was also found in a nonreversible system.[23] The theoretical foundation for mixed dynamics was given recently in Refs. 24 and 25. In all reversible models mentioned in the above papers,[15–22] mixed dynamics was quite similar to the conservative one with small nonconservative (and reversible) perturbations. As a result of such perturbations, periodic sinks and sources appear inside the region with chaotic dynamics due to local[26] and global[11,14] symmetry-breaking bifurcations. As a rule, these sinks and sources have very narrow domains of attraction (repulsion). Therefore, it is difficult to detect them by straightforward numerics; however, there are indirect effective methods for their detection (see, e.g., in Ref. 20). Naturally, the presence of stable and completely unstable periodic orbits (even in very narrow domains) confirms that the system is not conservative, i.e., it does not preserve any measure with a smooth density (a quite delicate example of this phenomenon in the nonholonomic model of a rubber disk is given in Ref. 27). The common phenomenon in near-conservative reversible settings is that the attractor and the repeller intersect and, while remaining different from each other, occupy roughly the same regions in the phase space. In Fig. 2, we present an illustrative example of such an effect by constructing the attractor and the repeller for the nonholonomic model of Suslov top.[19]

This phenomenon is robustly present in reversible systems of various nature. It is worth noting that, in all known reversible models with mixed dynamics, the difference between the numerically obtained attractor and repeller decreases with the increase of the time of computation (however, the asymmetry in the distribution of points in the attractor and the repeller persists). This is in an agreement with Theorem 2 from Ref. 25, which states that if an attractor of any system intersects with a repeller then these two sets must coincide.

In this paper, we present a far from conservative example of reversible mixed dynamics, when a strange attractor and a strange repeller have a nonempty intersection but are very much different from each other, and this difference does not seem to vanish with a reasonable increase in the computation time, in an apparent contradiction with the above mentioned theorem from Ref. 25 [see Fig. 1(b)].

We call such a phenomenon the *strongly dissipative mixed dynamics* (SDMD). In comparison with the previously known cases of mixed dynamics, the phase volume contraction rate (the sum of Lyapunov exponents) on the attractor for SDMD is much less than zero, which makes the system far from conservative and, in our opinion, makes the large difference between the distribution of points in the attractor and the repeller possible. We believe that the computation time needed to see that the intersecting attractor and repeller occupy the same region in the phase space, as prescribed by Ref. 25, is extremely large in this case and is unachievable in realistic simulations.

Beyond demonstrating SDMD in system (1), we also suggest a bifurcation scenario of the transition from conservative to mixed dynamics in one-parametric family of two-dimensional reversible diffeomorphisms. The main part of this scenario is the merger of a Hénon-like attractor with a Hénon-like repeller. This merger occurs due to the appearance of heteroclinic connections between the invariant manifolds of a pair of saddle fixed points, one of which belongs to the attractor, and the other point belongs to the repeller. The corresponding bifurcations in system (1) are studied in detail. Finally, we conclude that the observed phenomenon of the attractor-repeller merger can also occur in multidimensional reversible and nonreversible maps. Moreover, other types of homoclinic attractors and homoclinic repellers can merge in these cases.

The rest of the paper is organized as follows. In Sec. II, we describe the phenomenological bifurcation scenario, which leads to the merger of Hénon-like attractors with Hénon-like repellers and, as a result, to the appearance of mixed dynamics. In Sec. III, we present a model describing the motions of two point vortices in a shear flow perturbed by an acoustic wave. In Sec. IV, we study, in this model, bifurcations leading from conservative to dissipative dynamics (related to the existence of separated Hénon-like attractors and repellers) and, finally, to strongly dissipative mixed dynamics.

II. SCENARIO OF THE MERGER OF A HÉNON-LIKE ATTRACTOR WITH A HÉNON-LIKE REPELLER

Let us consider a one-parameter family of two-dimensional reversible maps $\bar{x} = f(x, \varepsilon)$ defined on a compact manifold and depending on a parameter ε. Suppose that for all ε, these maps are reversible with respect to the same involution h (i.e., $f = h \circ f^{-1} \circ h$, where $h \circ h = id$) for which the set $Fix(h)$ of its fixed points (when $h(x) = x$) is one-dimensional.

Further, let O be a fixed point belonging to the line $Fix(h)$. Suppose that this point is elliptic for $\varepsilon < \varepsilon_0$ and it undergoes a reversible pitchfork bifurcation[26] at $\varepsilon = \varepsilon_0$. After this, the point O becomes a saddle, and a symmetric pair of asymptotically stable, S^a, and completely unstable, S^r, fixed points (one point is symmetric to another with respect to h) appears near O (see Fig. 3 at $\varepsilon = \varepsilon_0$). We also suppose that, with further increase in the parameter, at $\varepsilon = \varepsilon_F$, a *Feigenbaum-like attractor*[28] AF is born via a cascade of period-doubling bifurcations with S^a. By the reversibility, a Feigenbaum-like repeller RF is born from S^r at the same moment. We note that after the first period-doubling bifurcation, points S^a and S^r become saddles.

Recall that immediately after the onset of chaotic dynamics through the cascade of period-doubling bifurcations, the Feigenbaum-like attractor consists of disjoint components.[2,3] With the further increase in ε these components merge pairwise (due to the occurrence of heteroclinic intersections between stable and unstable manifolds of the saddle orbits belonging to different components[5]). Finally, two last components separated by the stable manifold of S^a are merged and the *homoclinic Hénon-like attractor* AH appears [see Fig. 3(a) at $\varepsilon = \varepsilon_H$ and Fig. 3(b)]. By the reversibility, the homoclinic Hénon-like repeller RH containing the fixed point S^r occurs at the same moment. We call these attractors (repellers) homoclinic since they contain a single fixed point (with a pair of negative multipliers) and its unstable manifold.[29]

With a further increase in ε, the Hénon-like attractor AH becomes larger and approaches the boundary of its basin of attraction, which is formed by the stable manifold W^s of the saddle fixed point O (accordingly, the basin for RH is bounded by the unstable manifold W^u of the same point O). Also we note that both stable and unstable manifolds are separated by the point O into pairs of stable and unstable separatrices, and one pair of separatrices already intersects, while another does not [see Fig. 3(b)].

When $\varepsilon = \varepsilon_{MD}$, the crisis of attractor AH and repeller RH occurs (AH collides with the boundary of its absorbing domain W^s, while RH symmetrically collides with the boundary of its repulsing domain W^u), after which both these sets get involved into the same homoclinic structure, the attractor merges with the repeller, mixed dynamics appear [see Fig. 3(c)].

III. THE MODEL

In this section, we briefly describe the model of two point vortices interacting with a shear flow perturbed by an acoustic wave.[7] It is known that the system of two point vortices in the absence of an acoustic forcing is integrable and, moreover, Hamiltonian (see, e.g., Ref. 30). However, the addition of the acoustic forcing breaks

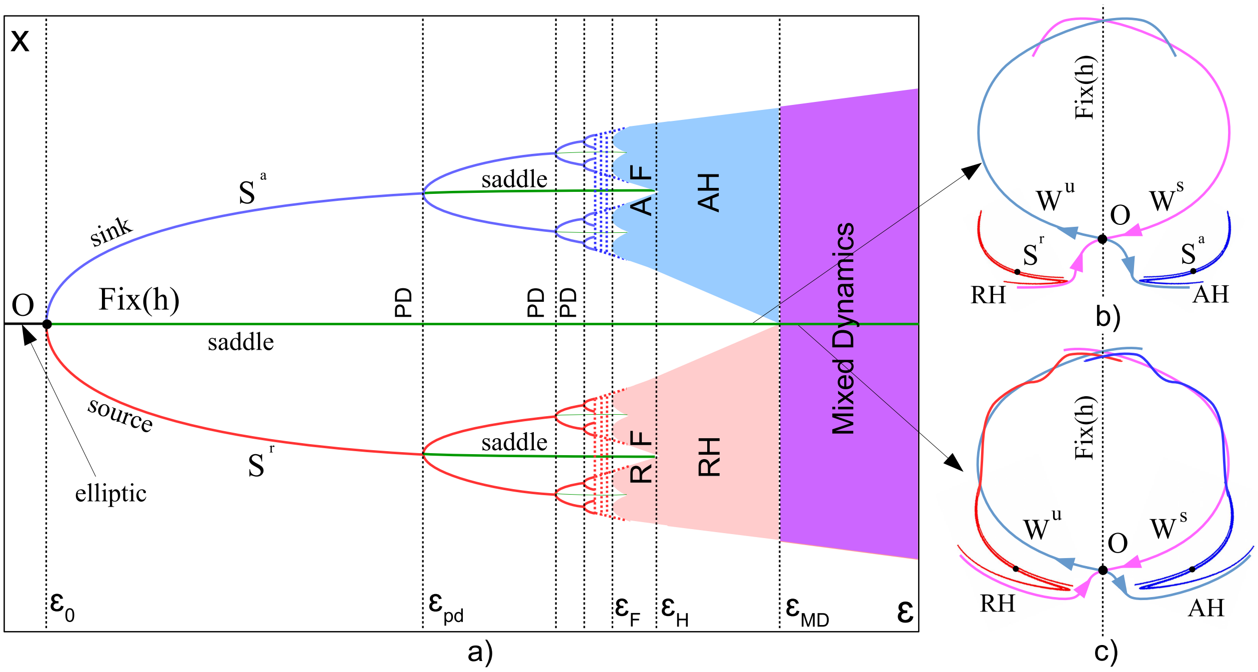


FIG. 3. The scenario of merger of a Hénon-like attractor with a Hénon-like repeller leading to the appearance of mixed dynamics in reversible two-dimensional maps.

integrability and makes the system nonconservative.[31] Simple attractors and repellers (asymptotically stable and completely unstable periodic orbits) can appear in this case via reversible supercritical pitchfork bifurcations.[31,32] If the vortices are also perturbed by an external shear flow with constant vorticity, their dynamics become more complicated. As was shown in Ref. 7, Hénon-like strange attractors can appear in this case. After certain simplifications of the model (see details in Ref. 7), the system takes the following form:

$$\begin{cases} \dot{R} = \dfrac{1}{2} AR \sin 2\varphi - \varepsilon \sin\varphi \sin S \sin(R\sin\varphi), \\ \dot{S} = -1 + \varepsilon \cos S \cos(R \sin\varphi), \\ \dot{\varphi} = \dfrac{\kappa}{R^2} + A\cos^2\varphi - \dfrac{\varepsilon}{R}\cos\varphi \sin S \sin(R\sin\varphi). \end{cases} \tag{1}$$

Here, $R \in (0,\infty), S \in [0,2\pi), \varphi \in [0,2\pi)$ are the phase variables, and $A, \varepsilon,$ and κ are the parameters. Note that this model is a generalization of the system describing the motions of two point vortices that interact only with an acoustic wave.[31,32] In system (1), the parameter ε characterizes the amplitude of the acoustic wave, A is the vorticity of the external shear flow, and κ is the sum of the intensities of the vortices.

Note that Eq. (1) are invariant with respect to the substitution

$$H: \{R \to R, \quad S \to -S, \quad \varphi \to -\varphi, \quad t \to -t\}. \tag{2}$$

Thus, the system under consideration is reversible. Also, we note that any half-cylinder $\varphi = \text{const}$ can be chosen as a secant. In the case $\varphi = 0$ (or $\varphi = \pi$), relation (2) defines, for the corresponding Poincaré map on $(S \times R)$, the involution

$$h: \{R \to R, \quad S \to -S\} \tag{3}$$

or equivalently

$$\{R \to R, \quad S \to 2\pi - S\}.$$

The set $Fix(h)$ of fixed points of this involution consists of two lines,

$$Fix(h) = \{S = 0\} \cup \{S = \pi\}.$$

Hereafter, we choose $\varphi = 0$ as the secant for the system and perform one-parameter analysis when varying ε, assuming that other parameters are fixed as follows:

$$A = 0.1, \kappa = 4.65.$$

IV. BIFURCATION ANALYSIS

For $\varepsilon = 0$, system (1) describing the motion of unperturbed vortices is integrable[33] and its phase space is foliated into invariant tori. When $\varepsilon > 0$, some tori become resonant and, in the corresponding Poincaré map, pairs of symmetric [with respect to the involution (3)] saddle and elliptic fixed point appear [points s_i^0 belong to the line $S = 0$ and points s_i^π belong to $S = \pi$; see Fig. 4(a)]. With a further increase in ε, some elliptic fixed points undergo reversible pitchfork bifurcations due to which they become symmetric saddles while pairs of an asymptotically stable fixed point s_i^a and a completely unstable fixed point s_i^r are born in their neighborhood [see Fig. 4(b)]. Note that, for sufficiently large values of ε, eight asymptotically stable (and also eight completely unstable) fixed points coexist [see Fig. 4(c)], i.e., the dynamics in the system become significantly multistable.

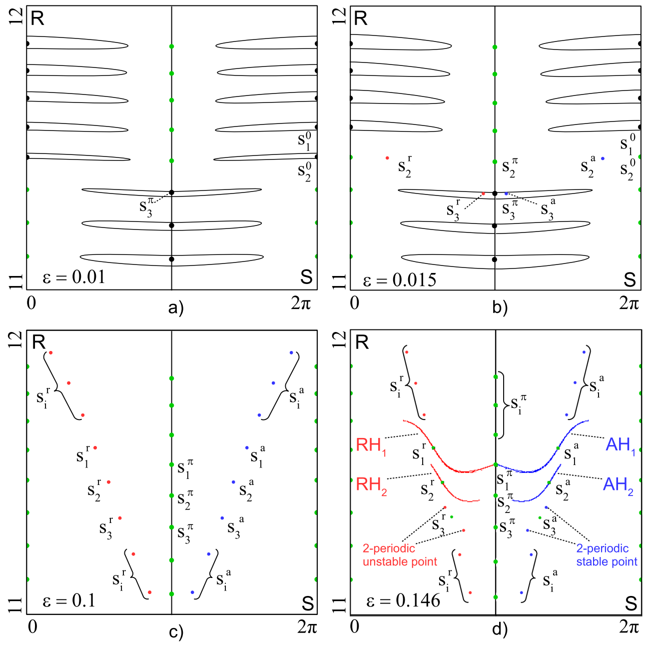
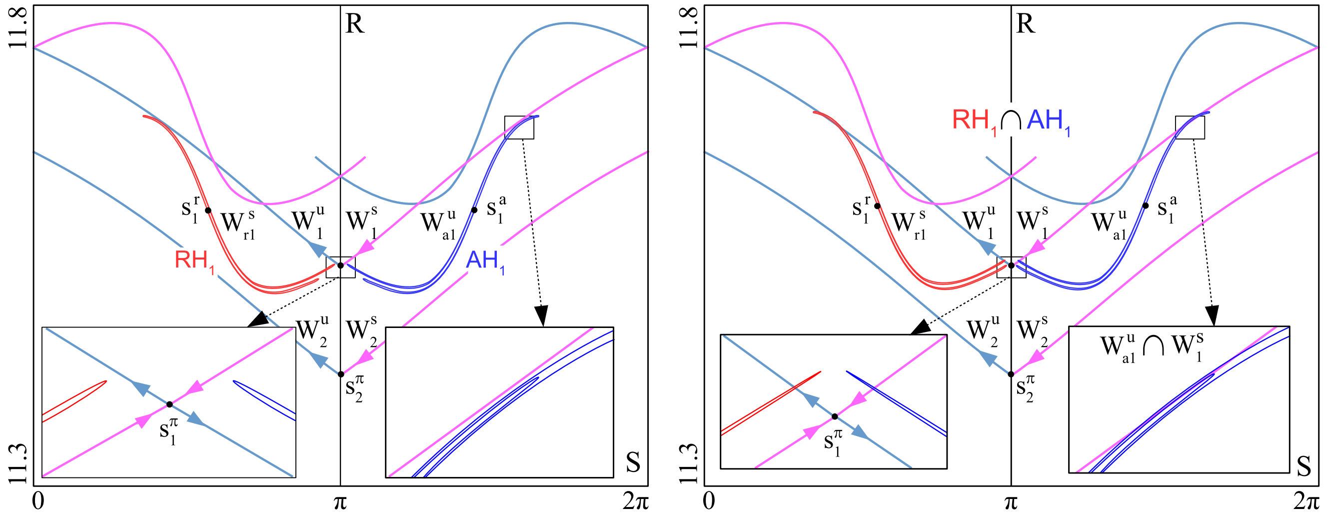


FIG. 4. Phase portraits in the Poincaré map for system (1) for different ε. (a) Resonant tori for small ε; (b) elliptic points s_2^0 and s_3^π undergo reversible pitchfork bifurcation after which they become symmetrical saddles, and the pairs (s_2^a, s_2^r) and (s_3^a, s_3^r) of asymptotically stable and completely unstable points are born; (c) simple multistability: eight stable (s_i^a) and eight completely unstable (s_i^r) fixed points coexist in the system; (d) Hénon-like attractors AH_1 and AH_2 (and also Hénon-like repellers RH_1 and RH_2) develop out of stable points s_1^a and s_2^a (from completely unstable points s_1^r and s_2^r).

With the increase of ε, the Hénon-like attractors AH_i and the Hénon-like repellers RH_i develop out of the points s_i^a and s_i^r via cascades of period-doubling bifurcations followed by a cascade of heteroclinic "band-fusion" bifurcations. Figure 4(d) shows the coexistence of two Hénon-like attractors AH_1 and AH_2 with a pair of stable fixed and 2-periodic points for $\varepsilon = 0.146$. In this figure, s_i^a and s_i^r are fixed points that become saddle after the period-doubling bifurcation of the corresponding stable and completely unstable fixed points. We note that the saddles s_i^a are area-contracting (have the Jacobian less than 1, $J < 1$), while the saddles s_i^r are area-expanding (have the Jacobian greater than 1, $J > 1$).

When $\varepsilon = \varepsilon_{cris1} \approx 0.146\,35$, the attractor AH_1 and the repeller RH_1 undergo crisis due to which these two sets begin to intersect. It is important to note that the intersections of attractors and repellers in the system appear due to the heteroclinic bifurcations. Further, we will describe such bifurcations in detail, but first, let us recall two well-known facts related to homoclinic attractors:[29]

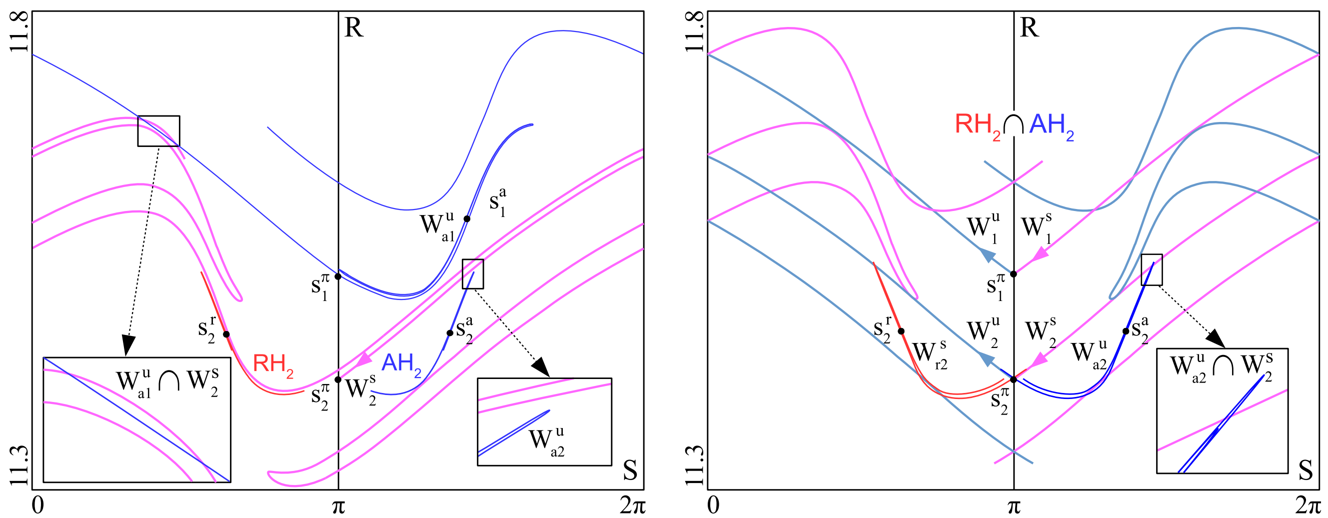
- in many cases, the boundary of absorbing domains for homoclinic attractors is formed by stable manifolds of some saddle points;
- a homoclinic attractor contains the closure of the unstable manifold of one of its saddle points.



a) $\varepsilon = 0.1463$ b) $\varepsilon = 0.1464$

FIG. 5. Invariant manifolds forming the attractor, the repeller, and their basins. Here, W^u_{a1} (in blue) is the unstable manifold of the saddle fixed point s^a_1, and W^s_{r1} (in red) is the stable manifold of the saddle fixed point s^r_1; also, W^s_i and W^u_i, $i = 1, 2$ are the stable and unstable separatrises of the symmetric saddles s^π_i. (a) The Hénon-like attractor AH_1 and Hénon-like repeller RH_1 are separated. (b) The merger of AH_1 and RH_1 due to the heteroclinic intersections $W^u_{a1} \cap W^s_1$ and $W^s_{r1} \cap W^u_1$.

In the case under consideration, the Hénon-like attractor AH_1 is formed by the closure of the unstable manifold W^u_{a1} of the saddle fixed point s^a_1. The absorbing domain of AH_1 is bounded from above by the stable separatrix W^s_1 of the saddle fixed point s^π_1, and from below—by the stable separatrix W^s_2 of the saddle fixed point s^π_2 [see Fig. 5(a)]. We also note that the separatrices W^s_1 and W^u_1 intersect transversally, like in the scheme presented in Fig. 3(b).



a) $\varepsilon = 0.1465$ b) $\varepsilon = 0.1482$

FIG. 6. Here, W^u_{a1} and W^u_{a2} (in blue) are the unstable manifolds of the saddle points s^a_1 and s^a_2, and W^s_{r2} (in red) is the stable manifold of the saddle point s^a_2; also, W^s_i and W^u_i, $i = 1, 2$ are the stable and unstable separatrices of the symmetrical saddles s^π_i. (a) Since $W^u_{a1} \cap W^s_2 \neq \emptyset$, the orbits from the neighborhood of AH_1 run to another Hénon-like attractor AH_2. (b) Merger of the Hénon-like attractor AH_2 and the Hénon-like repeller RH_2 due to the appearance of heteroclinic intersections $W^u_{a2} \cap W^s_2$ and $W^s_{r2} \cap W^u_2$, respectively.

When $\varepsilon > \varepsilon_{cris1}$, intersections between W^u_{a1} and W^s_1, as well as W^s_{r1} and W^u_1, appear [see Fig. 5(b)], and, as a result, the attractor AH_1 collides with the upper boundary of the absorbing domain, while the repeller RH_1 collides with the upper boundary of its repulsing domain. After this collision, the attractor merges with the repeller and increases explosively; the invariant manifold $W^u_{a1}(S^a_1)$, which used to form AH_1, can pass arbitrarily near the points S^π_1 and S^r_1. But in this case such a merger is not visible due to the multistability: another homoclinic Hénon-like attractor AH_2 attracts almost all orbits from a neighborhood of AH_1 and, symmetrically, RH_2 attracts orbits from a neighborhood of RH_1 in backward time (here, after the collision of AH_1 and RH_1, the unstable manifold W^u_{a1} also intersects with the stable manifold W^s_2 of the symmetric saddle point s^π_2 [see Fig. 6(a)], and this intersection allows for a transition from AH_1 to AH_2).

When $\varepsilon = \varepsilon_{cris2} \approx 0.148\,13$, the unstable manifold W^u_{a2} of s^a_2 belonging to the attractor AH_2 touches the stable separatrix W^s_2, which forms the boundary of the absorbing domain for AH_2. The same nontransversal heteroclinic tangency appears between W^s_{r2} and W^u_2, one of which belongs to the repeller RH_2, while another forms the boundary of its basin. Thus, for $\varepsilon > \varepsilon_{cris2}$, the attractor AH_2 merges with the set $AH_1 \cap RH_1$ and also with a repeller RH_2 [see Fig. 6(b)]. Immediately after this transition, we observe mixed dynamics due to the intersection of two attractors (AH_1 and AH_2) with two repellers (RH_1 and RH_2). The corresponding phase portrait of the attractor and the repeller is presented in Fig. 1(b).

However, the observed mixed dynamics exist for a quite narrow region of the parameters. At $\varepsilon = \varepsilon_{cris3} \approx 0.148\,25$, orbits in forward time tend to the Hénon-like attractor AH_3, which develops out of the fixed point s^a_3 and in backward time—to the Hénon-like repeller RH_3, which develops out of s^r_3. In their turn, AH_3 and RH_3 merge in the same way as AH_1 with RH_1 and AH_2 with RH_2, giving more complicated global connection between different homoclinic attractors and repellers.

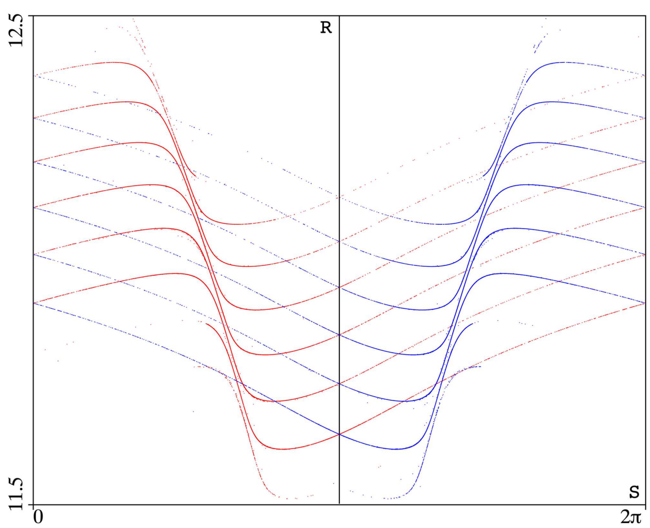


FIG. 7. An illustration for strongly dissipative mixed dynamics after the merger of all eight Hénon-like attractors AH_i and repellers RH_i, $i = 1, \ldots 8$ at $\varepsilon = 0.23$. The attractor is presented in blue color and the repeller in red.

Such a merger of attractors and repellers is terminated at $\varepsilon > \varepsilon_{cris_8} \approx 0.206$, when all 8 Hénon-like attractors AH_i and 8 Hénon-like repellers RH_i merge. Since there are no other attractors in the neighborhood of this intersection, the mixed dynamics become visible [see Fig. 7] and it exists for a quite large interval of parameter values. Moreover, due to the heteroclinic connections, the attractor of the system contains area-expanding and area-preserving saddle fixed points in addition to the area-contracting ones.

One can see in Fig. 7 that the strange attractor and the strange repeller have a nonempty intersection but are very different from each other. It is interesting that this difference does not visually diminish with a reasonable increase in the computation time. To the best of our knowledge, such a type of reversible mixed dynamics is observed for the first time. We call this phenomenon *strongly dissipative mixed dynamics*.

V. CONCLUSION

In this paper, we discuss the phenomenon of merger of Hénon-like attractors with Hénon-like repellers leading to the occurrence of mixed dynamics in the model describing the motion of two point vortices in a shear flow perturbed by an acoustic wave. We show that in this model, such a mechanism leads to the mixed dynamics of a new type, when, after the merger, a chaotic attractor and a chaotic repeller have a nonempty intersection but are very different from each other. Also, we propose a phenomenological bifurcation scenario for this phenomenon for two-dimensional reversible maps. The key part of this scenario is the appearance of heteroclinic intersections between the invariant manifolds of a pair of saddle fixed points, one of which belongs to the attractor, while another point belongs to the repeller. It is worth noting that the proposed scenario can be extended to the case of nonreversible two-dimensional maps and also to multidimensional diffeomorphisms where other types of chaotic homoclinic attractors are possible (see, e.g., Refs. 18 and 34).

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