ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2020, Vol. 308, pp. 125–140. © Pleiades Publishing, Ltd., 2020.
Published in Russian in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2020, Vol. 308, pp. 135–151.

Three Types of Attractors and Mixed Dynamics of Nonholonomic Models of Rigid Body Motion

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Received April 23, 2019; revised August 12, 2019; accepted October 26, 2019

Abstract—We survey recent results on the theory of dynamical chaos from the point of view of topological dynamics. We present the concept of three types of dynamics: conservative, dissipative, and mixed dynamics, and also show several simple examples of attractors and repellers of all three types. Their similarities and differences with other known types of attractors and repellers (maximal and Milnor ones) are discussed. We also present elements of the qualitative theory of mixed dynamics of reversible systems. As examples of such systems we consider three nonholonomic models of rigid body motion: the Suslov top, rubber disk, and Celtic stone. It is shown that they exhibit mixed dynamics of different nature; in particular, the mixed dynamics observed in the model of rubber disk is extremely difficult to distinguish from the conservative one.

DOI: 10.1134/S0081543820010101

1. INTRODUCTION

In the recent papers [21, 29], the concept of three types of dynamics in multidimensional systems with compact phase space was proposed. Two of them have been well known for a long time; these are *conservative* and *dissipative* dynamics.

The most known type of conservative dynamics is that demonstrated by Hamiltonian systems or, more generally, systems preserving the phase volume. From the point of view of topological dynamics, the conservative dynamics is characterized by the fact that the entire phase space of the corresponding system is chain transitive, i.e., any two of its points can be connected by ε-orbits for any $\varepsilon > 0$. For more details on ε-orbits and the associated basic concepts of the topological dynamics, see [3] and Section 2.

The dissipative dynamics has a completely different nature; it is associated with the existence of "holes"—absorbing and repelling domains—in the phase space $\mathcal{M}$. Recall that an open domain D_{a} is said to be absorbing if its image under the action of a map T or a flow T_t lies strictly inside it, i.e., $T(\mathrm{cl}(D_{\mathrm{a}})) \subset D_{\mathrm{a}}$ or $T_t(\mathrm{cl}(D_{\mathrm{a}})) \subset D_{\mathrm{a}}$ for $t > 0$, where by $\mathrm{cl}(D_{\mathrm{a}})$ we denote the closure of D_{a}. Similarly, an open domain D_{r} is repelling if $T^{-1}(\mathrm{cl}(D_{\mathrm{r}})) \subset D_{\mathrm{r}}$ or $T_t(\mathrm{cl}(D_{\mathrm{r}})) \subset D_{\mathrm{r}}$ for $t < 0$. By definition, a *dissipative attractor* resides in some absorbing domain D_{a} and all forward orbits of the points from this region tend to it. Analogously, a repeller resides in some repelling domain D_{r} and all backward orbits of the points from this region tend to it. Accordingly, we have here $D_{\mathrm{a}} \cap D_{\mathrm{r}} = \varnothing$.

As regards the third (new) type of dynamics, mixed dynamics, it is characterized first of all by the *principal inseparability* of attractors, repellers, and conservative elements of dynamics [31, 43, 14]. Here, in contrast to the conservative case, the phase space is not chain transitive. It contains real dissipative attractors and repellers, possibly an infinite number of them [31], and then they can be

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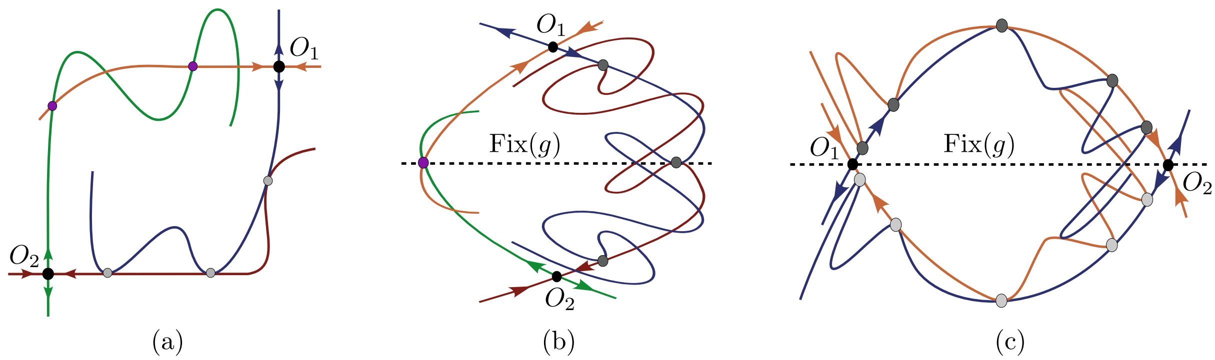

Fig. 1. Examples of nontransversal heteroclinic cycles of two-dimensional diffeomorphisms: (a) in the general case; (b) of a priori non-conservative type in the reversible case; and (c) of conservative type in the reversible case. In each of cases (a) and (b), the Jacobians of the points O_1 and O_2 are, respectively, greater and less than one; in case (c) the Jacobians of the points O_1 and O_2 are both equal to one. Here Fix(g) is the fixed point set of the involution g (see Section 3 below).

inseparable in aggregate from each other.[1] In other words, here, unlike the dissipative case, it is impossible to construct a system of disjoint absorbing and repelling domains. This phenomenon was called mixed dynamics in [25, 28, 14], and its explanation from the topological point of view was given in [29] based on the concept of attractor going back to D. Ruelle [54].

In Section 2 we consider this issue in more detail. We discuss the concept of Conley–Ruelle–Hurley (CRH) attractors (as stable chain transitive closed invariant sets), give examples of such attractors and repellers of all three types, and also compare them with well-known types of attractors such as the maximal and Milnor attractors. The main attention in Section 2 will nevertheless be paid to the mixed dynamics, as a new and little studied type of chaos; to this end we will also give rather simple examples of such dynamics here.

Note that one of the main fundamental properties of systems with mixed dynamics (which can also be considered as a criterion of such dynamics) is the existence of so-called *absolute Newhouse regions* for these systems [31, 59, 60]. Recall that Newhouse regions are open regions in the space of dynamical systems (or in the parameter space) in which systems with homoclinic tangencies are dense (or values of parameters corresponding to systems with homoclinic tangencies are dense) [49, 30, 51, 53]. It was shown by Newhouse himself [48] that, in the dissipative case, there may exist Newhouse regions in which systems with infinitely many stable and saddle periodic orbits are dense and, moreover, generic, i.e., they form subsets of the second Baire category. This is called the Newhouse phenomenon (see also [27]). The absolute Newhouse regions are characterized by the following property: systems with infinitely many periodic orbits of all possible types (sinks, sources, and saddles with Jacobians greater and smaller than 1) are generic in such regions, and these orbits are inseparable in aggregate from each other, i.e., the closures of the sets of orbits of different types have nonempty intersections.

The absolute Newhouse regions were discovered in [31] in the case of two-dimensional diffeomorphisms close to a diffeomorphism with a nontransversal heteroclinic cycle of *mixed type*, which contains two saddle fixed points with the Jacobians greater and less than one. An example of such a cycle is shown in Fig. 1a. Here, for two saddle fixed points O_1 and O_2, one pair of their invariant manifolds, $W^s(O_1)$ and $W^u(O_2)$, intersects transversally at the points of some heteroclinic orbit Γ_{12},

[1] When there are infinitely many (dissipative) attractors and repellers, their closures can intersect. In this case, invariant sets of a neutral type must lie in their intersection, and these sets attract nothing and repel nothing. In the case of general type systems (for example, systems with sign alternating divergence), the dynamical structure of such invariant sets seems very unclear. For systems with additional structures (for instance, reversible systems), the role of invariant sets of a neutral type may well be played by elliptic periodic orbits, KAM curves, etc. [24].

while the other pair of manifolds, $W^{\mathrm{s}}(O_2)$ and $W^{\mathrm{u}}(O_1)$, has a quadratic tangency at the points of some (nontransversal) heteroclinic orbit Γ_{21}. We also note that symmetric nontransversal heteroclinic cycles of mixed type exist for reversible two-dimensional diffeomorphisms as well (Fig. 1b). However, the dynamics here is much richer. In particular, as shown in [43], diffeomorphisms with infinitely many coexisting periodic sinks, sources, and *elliptic orbits* are generic in reversible absolute Newhouse regions. The same statement holds for symmetric nontransversal cycles of other types, including those like the cycle in Fig. 1c (see [14]).

These results have aroused interest in the study of mixed dynamics in reversible systems. Moreover, such systems are often met in applications, and the phenomenon of the existence of phase-space regions with visible conservative and dissipative behavior of orbits was previously mentioned as one of the most interesting properties of reversible systems [52]. We believe that this property should now be considered as a manifestation of reversible mixed dynamics, a universal property of reversible systems with a complicated structure of the set of symmetric orbits. Absolute Newhouse regions and mixed dynamics in reversible systems are discussed in Section 3. One of the most famous classes of such systems consists of nonholonomic models of rigid body motion. In Section 4 we consider some of these models (the Suslov top model, rubber disk model, and Celtic stone model).

2. ATTRACTORS, REPELLERS, AND MIXED DYNAMICS

At present, there are many different definitions of attractor, but it is always implied that an attractor must be a closed, stable, and invariant set. However, say, the type of stability can be understood in different ways. For example, in the case of a *maximal attractor* $A_{\max}$, the stability type is not indicated at all, since such an attractor is defined as an invariant set that belongs to some absorbing region D and is such that

$$A_{\max} = \bigcap_{n=0}^{\infty} T^n(D) \qquad \text{or} \qquad A_{\max} = \bigcap_{t\geq 0} T_t(D)$$

in the case of maps or flows, respectively.

Another well-known notion of attractor is the *Milnor attractor* A_{M} (see [46]), which is defined as *a closed, invariant, and minimal* (*with respect to embedding*) *set in D that contains the ω-limit points of the forward orbits for almost all points* (*i.e., all points of a set of full Lebesgue measure*) *in D.*

Of course, both types of attractors defined in this way have their own advantages and drawbacks. For example, both the maximal attractor and the Milnor attractor may sometimes not be attractors at all. For maximal attractors, this often happens if one chooses an inappropriate absorbing domain. For example, if such a domain contains a completely unstable orbit (source) or a saddle, then the maximal attractor will also contain it (see examples in Figs. 2a and 2b, where U is a large absorbing domain, while U_a and U_b are absorbing domains with unique attractors, the stable points a and b). As for Milnor attractors, even for the simplest systems they may not be attractors. So, for the example of a circle map with a semi-stable fixed point O in Fig. 2c, the point itself is the Milnor attractor, since it is the ω-limit point for all orbits. However, evidently, this point is not an attractor in the usual sense.

Note also that, in contrast to maximal attractors, Milnor attractors may not be topological invariants (in the sense that two topologically equivalent systems may have different Milnor attractors). Examples of such systems were presented by S. Minkov [47]. One of his examples is a C^1-smooth diffeomorphism $\widehat{f}$ of a square Q onto itself which has the following properties (Fig. 2d). The square Q contains a Cantor set $\mathcal{K}$ of vertical segments; all points of the sides of the square and of the set $\mathcal{K}$ are fixed; and the remaining points of the square tend to its lower side $[a,b]$ under forward iterations of $\widehat{f}$. In the case when the measure of $\mathcal{K}$ is zero, the Milnor attractor of $\widehat{f}$ is the segment $[a,b]$. However, when the measure of $\mathcal{K}$ is nonzero, the Milnor attractor is the set $[a,b]\cup\mathcal{K}$.

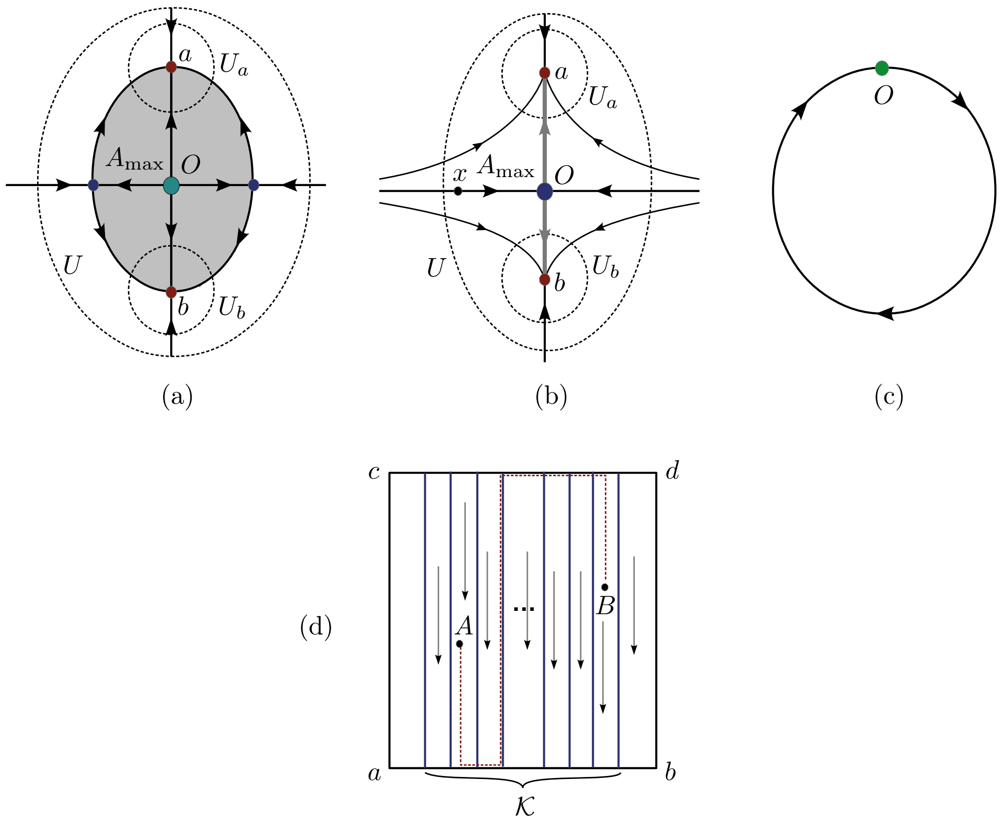

Fig. 2. Examples of different types of attractors that are actually not attractors in the usual sense: (a, b) maximal attractors $A_{\max}$; (c) a Milnor attractor; and (d) a two-dimensional diffeomorphism with a Cantor set $\mathcal{K}$ of segments of fixed points, for which the Milnor attractor A_{M} is either the segment $[ab]$, if $\operatorname{mes}\mathcal{K} = 0$, or $[ab] \cup \mathcal{K}$, if $\operatorname{mes}\mathcal{K} \neq 0$.

2.1. Conley–Ruelle–Hurley attractors. The main attention in the paper will be paid to another type of attractors, whose definition goes back to the works by C. Conley, D. Ruelle, and M. Hurley [12, 54, 32] and is based on the concept of ε-orbits. First, recall some definitions from topological dynamics. Consider a homeomorphism f of a compact metric space $\mathcal{M}$.

Definition 2.1. A sequence of points $x_1, \dots, x_N$ is called an ε-*orbit* (of length N) for a map f if $\operatorname{dist}(f(x_j), x_{j+1}) < \varepsilon$ for all $j = 1, \dots, N-1$. We will say that an ε-orbit $x_1, \dots, x_N$ *connects* the points x_1 and x_N, and that x_N is *attainable* from x_1 by ε-orbits of length N.

Definition 2.2. A closed invariant set Λ is said to be *chain transitive* if, for any $\varepsilon > 0$ and any two points $x, y \in \Lambda$, there exists an ε-orbit that lies in Λ and connects x and y.

Definition 2.3. A closed invariant set A_{tot} is said to be *stable with respect to permanently acting perturbations* (or totally stable) if it is Lyapunov stable with respect to ε-orbits for any sufficiently small $\varepsilon > 0$, that is, if for any $\delta > 0$ there exists a sufficiently small $\varepsilon > 0$ such that any positive ε-orbit of any point of the set A_{tot} does not leave the δ-neighborhood of A_{tot}.[2]

The following definition of attractor, which we will use, goes back to the works of C. Conley [12], D. Ruelle [54], and M. Hurley [32]. Therefore, we will call such sets Conley–Ruelle–Hurley attractors or briefly CRH attractors.

[2]One can also define the stability of A_{tot} with respect to permanently acting perturbations in an equivalent way: for any $\delta > 0$ there exist sufficiently small $\delta_1 > 0$ and $\varepsilon > 0$ such that any positive ε-orbit starting in the δ_1-neighborhood of A_{tot} does not leave its δ-neighborhood. This definition looks essentially like a direct reformulation of the notion of Lyapunov stability for ε-orbits (see, e.g., [15]).

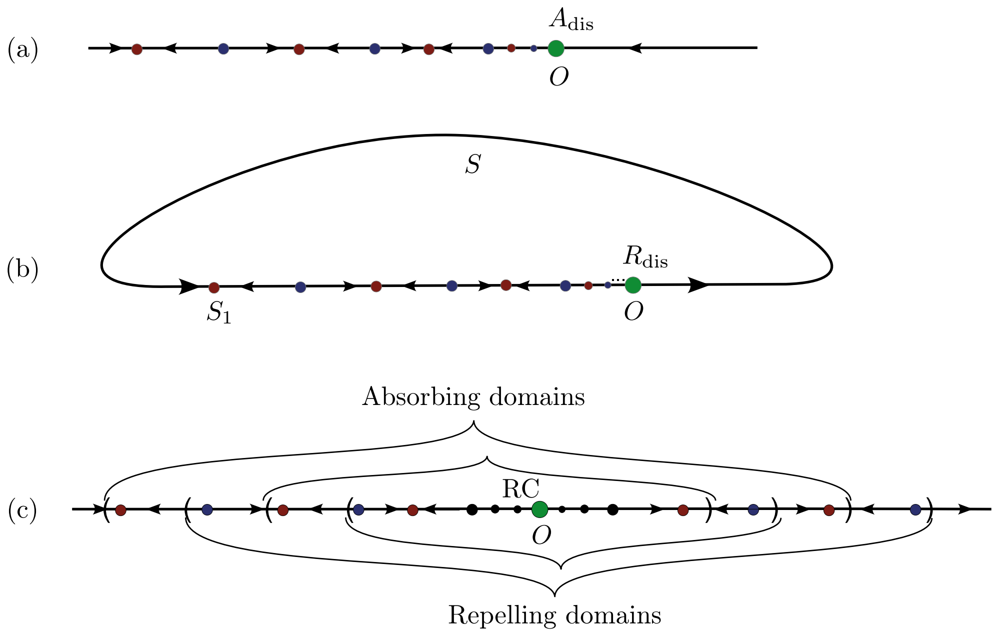

Fig. 3. Examples of one-dimensional maps with (a) a dissipative CRH attractor $A_{\rm dis}$, (b) a dissipative CRH repeller $R_{\rm dis}$, and (c) a reversible core RC, i.e., a CRH attractor which is also a CRH repeller.

Definition 2.4. A *CRH attractor* is a stable (with respect to permanently acting perturbations) chain transitive closed invariant set lying entirely in some absorbing region U. We also define a *CRH repeller* as a CRH attractor for the reverse-time system.

It was shown in [29] that only the following three types of CRH attractors are possible for homeomorphisms of a compact metric space $\mathcal{M}$:

- *conservative* type, when the whole phase space $\mathcal{M}$ is a chain transitive set and, hence, the space $\mathcal{M}$ itself will be simultaneously a CRH attractor and a CRH repeller for the corresponding homeomorphism;
- *dissipative* type, when in any neighborhood of a CRH attractor there are points (different from the points of the attractor itself) whose forward orbits tend to the CRH attractor;
- *mixed* type, when an orbit always stays in a neighborhood of a *reversible core* (i.e., a CRH attractor which is also a CRH repeller) but tends to it neither in the forward nor in the backward time direction. In this case, the core holds all positive and negative semi-orbits in a small neighborhood of it; however, for any point that does not belong to the core, there is an $\varepsilon > 0$ such that no positive or negative ε-orbit connects this point to points of the reversible core.

Note that dissipative CRH attractors may be structured in different ways from the point of view of topological dynamics. There are well-known examples of such attractors to which all orbits from their neighborhoods tend. These include hyperbolic attractors (in particular, simple attractors like periodic sinks), Lorenz attractors, and also so-called pseudohyperbolic attractors [61, 62, 23]. All of them are also maximal attractors in their neighborhoods. Another type of dissipative CRH attractors is given by those to which not all orbits of points from their neighborhoods tend. The simplest example of such an attractor in the case of one-dimensional map is shown in Fig. 3a. Here the point O is a dissipative CRH attractor, since all points to the right of O tend to it, and at the same time O is also the limit of an infinite sequence of sinks and sources. In the case of the example of Fig. 3b, the point O is a CRH repeller (and it is also the limit of an infinite sequence of sinks and sources).

The most famous example of conservative-type CRH attractors is given by volume-preserving maps of a compact manifold $\mathcal{M}$, which is then a chain transitive set, and so the manifold $\mathcal{M}$ itself is both a CRH attractor and a CRH repeller. There are also similar examples of non-volume-preserving maps, for example, Anosov maps of a torus. Another example is the circle map $\varphi \mapsto \varphi + \sin^2(\varphi/2)$, which has a semi-stable fixed point O (see Fig. 2c). Note that in this case the whole circle is a CRH attractor, because any two of its points can be connected by ε-orbits for any $\varepsilon > 0$.

Note that the two-dimensional diffeomorphism $\widehat{f}$ illustrated in Fig. 2d also has a CRH attractor of conservative type; this is the whole square Q. It is easy to see that Q is a chain transitive set for $\widehat{f}$. Indeed, let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points of Q. Then it is easy to build an ε-orbit connecting the points A and B: from A we go down to the segment $[a, b]$, along this segment (which consists of fixed points) we reach (along an ε-orbit) one of the lines of the set $\mathcal{K}$, along this line we go up to the segment $[c, d]$, along it we arrive at a point with the coordinate $x = x_2$, and then go down to the point B (the path AB shown by the dashed line in Fig. 2d).

There are few examples of reversible cores (CRH attractors of mixed types), since their theory began to develop relatively recently. A reversible core differs from a dissipative attractor in that it does not attract any orbits. On the other hand, the dynamics here is not purely conservative either, since a reversible core, as shown in [29, Theorem 1], is always the limit of a sequence of dissipative attractors and repellers. This fact is well illustrated even by the simplest example of a reversible core for the one-dimensional map shown in Fig. 3c. Here the point O is a reversible core: it is the limit of a countable set of simple dissipative attractors and repellers; however, O itself does not attract any orbits: any orbit from its neighborhood tends to the nearest attractor or, in the backward time, to the nearest repeller. Figures 3a and 3b demonstrate similar examples, for which, however, the point O is not a reversible core: in case (a) it is not a repeller (but is a dissipative CRH attractor), and in case (b) it is not an attractor. Nontrivial examples of reversible cores are given by symmetric elliptic periodic orbits of reversible two-dimensional diffeomorphisms [29, 24] (see also Section 3 of the present paper).

2.2. Full attractor and full Ruelle attractor. It is well known that not for all systems attractors are chain transitive sets. For example, quasiattractors [2, 26], which are often met in applications, can contain stable periodic orbits of very large periods that are not detected in experiments. Therefore, the concept of CRH attractor alone, although very important, is clearly not enough to explain many phenomena observed in dynamical systems. For example, it was shown in [29, Theorem 2] that if a CRH attractor has a nonempty intersection with a CRH repeller, then they coincide. However, when the phenomenon of mixed dynamics is observed, the intersecting attractors and repellers never completely coincide in reality, although they can occupy approximately the same region of the phase space.

Further, we assume that the phase space $\mathcal{M}$ of a homeomorphism f is not chain transitive.

Definition 2.5. A set A_x is called an *attractor of a point* x if A_x is a CRH attractor and is attainable from x by ε-orbits for any $\varepsilon > 0$.

Note that several CRH attractors may exist for one point x. For example, in Fig. 2b, for the point x lying on the stable separatrix of the point O, there are two such attractors, the points a and b. If the number of such attractors is finite, then their union is the *full attractor* of x. If there are infinitely many CRH attractors, then the full attractor of x is the *closure of the union of all attractors of* x. The full attractor thus defined is a closed invariant set $A_0(x)$, but it may not be a stable set. One can show that the minimal (with respect to embedding) stable set containing $A_0(x)$ is the *prolongation* of $A_0(x)$, i.e., the *set of all points attainable from* $A_0(x)$ *for all arbitrarily small* $\varepsilon > 0$. We will call this set the *full Ruelle attractor of* x.

The full attractor of a map is defined in a similar way.

Definition 2.6. For a map f, the *full attractor* is the closure of the union of all CRH attractors of its points, and the *full Ruelle attractor* is the prolongation of its full attractor.

Repellers are defined in the same way, as attractors for the inverse map f^{-1}.

Consider simple examples shown in Fig. 3. In Fig. 3a, the full attractor consists of a countable set of sinks and the point O, to which they accumulate. The point O itself is a dissipative CRH attractor (but not a repeller). Obviously, here the full attractor and the full Ruelle attractor coincide. Similarly, in the example of Fig. 3b, the full attractor consists of a countable set of sinks and the point O. However, the point O itself is not an attractor. The full Ruelle attractor here is obtained by adding the arc $S = \widehat{OS_1}$ to the full attractor. The resulting set is already stable. In the example of Fig. 3c, the full Ruelle attractor (the union of a countable set of sinks and the point O) and the full Ruelle repeller (the union of a countable set of sources and the point O) intersect at the point O, which is the reversible core here. The last example can be considered as the simplest illustration of mixed dynamics.

3. ABSOLUTE NEWHOUSE REGIONS AND MIXED DYNAMICS OF REVERSIBLE SYSTEMS

In this section we consider reversible systems, since for them the existence of mixed dynamics is a rather natural property due to their strong symmetries. In addition, the interest in studying chaotic dynamics of such systems is also explained by the fact that they often arise in applications. First, recall some basic facts of the theory of reversible systems. For definiteness, we will consider only reversible maps.

By definition, a C^r diffeomorphism f of an orientable manifold is said to be *reversible* if it is conjugate to its inverse f^{-1}, that is,

$$f^{-1} = g \circ f \circ g, \tag{3.1}$$

by means of some C^r-smooth involution g (by definition, $g \circ g = \mathrm{id}$).

A periodic orbit $\{x_0, \dots, x_m\}$ of a reversible diffeomorphism f is called *symmetric* if it is invariant with respect to the involution g. This property can be expressed as $g(x_i) = f^j(x_i)$ for $i = 0, \dots, m$ and some $j \le m$. Thus, f^j takes any point of the orbit to another point of the same orbit such that the two points are symmetric with respect to the set $\mathrm{Fix}(g) = \{x\colon g(x) = x\}$ of fixed points of the involution. We will assume that the involution is nondegenerate. In the two-dimensional case, this means that $\dim \mathrm{Fix}(g) = 1$. For a symmetric periodic orbit, the following statement holds:

- If λ is a multiplier of the orbit, then λ^{-1} is also a multiplier of the orbit.

Then, in the two-dimensional case, a symmetric periodic orbit of an orientable diffeomorphism f has multipliers λ and λ^{-1}. In the case $\lambda = \lambda^{-1}$, such parabolic orbits (with $\lambda = \pm 1$) are obviously nonrough (structurally unstable). For $\lambda \ne \lambda^{-1}$ symmetric periodic points can be divided into two types: saddle points, if λ is real, and elliptic points, if $\lambda_{1,2} = e^{\pm i\varphi}$ with some $\varphi \in (0, \pi)$. The saddle points are rough (structurally stable). As for elliptic points, although they are very similar to conservative elliptic points, they can differ greatly from the latter, as shown by the following result.

Theorem 1 [24, 29]. *All symmetric elliptic periodic orbits of a C^r-generic ($r = 1, \dots, \infty$) two-dimensional reversible map are limits of periodic sinks and sources.*[3]

Thus, the mixed dynamics manifests itself locally but wherever symmetric elliptic points exist. This important circumstance, of course, testifies to the fact that mixed dynamics should be viewed as one of the fundamental properties of reversible systems.

[3] The genericity is understood here in the sense that reversible maps with the indicated properties form a subset of the second Baire category in the space of C^r-smooth reversible maps having symmetric elliptic periodic orbits.

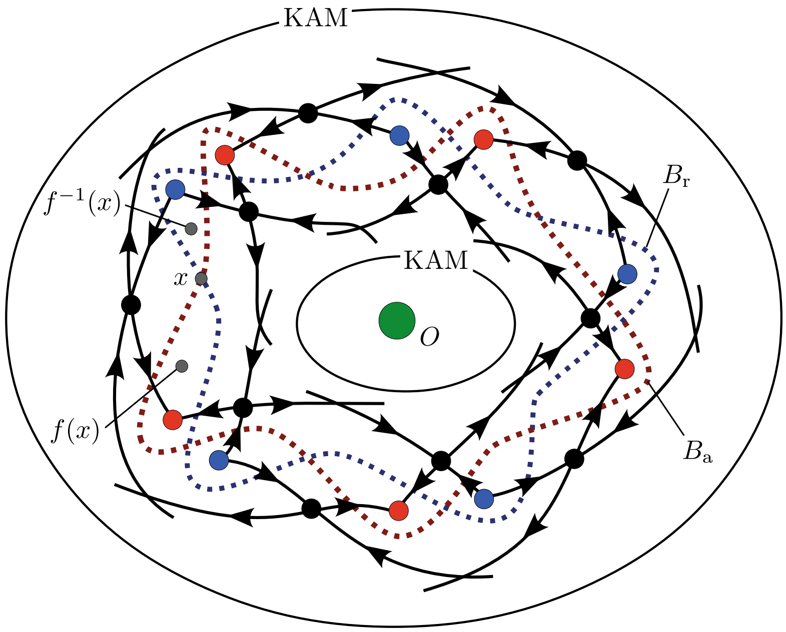

Fig. 4. Typical behavior of orbits in a resonance zone around an elliptic symmetric point of a reversible two-dimensional map. Here the absorbing and repelling domains intersect, and all of them contain the elliptic point O. A point x is shown that belongs to the intersection of the boundaries B_{a} and B_{r} of the absorbing and repelling regions: the point $f(x)$ lies inside the absorbing region and leaves the repelling region; the point $f^{-1}(x)$, on the contrary, enters the repelling region and leaves the absorbing region.

We also note that, as is well known [55], symmetric elliptic periodic orbits of two-dimensional reversible diffeomorphisms are structured in much the same way as in the conservative case: there also exists a continuum of KAM curves that form a subset of positive measure (and their relative measure tends to 1 as a neighborhood shrinks to a zero), and these KAM curves are separated by resonance zones. However, for reversible diffeomorphisms, the orbit behavior in these zones can be completely different. Here, by virtue of Theorem 1, it is generic when, on resonant levels, periodic saddles alternate with symmetric pairs of periodic points consisting of sinks and sources (Fig. 4), and this means that

- all symmetric elliptic periodic orbits of a C^r-generic ($r = 1, \dots, \infty$) two-dimensional reversible map are reversible cores;
- the generic elliptic periodic point of a reversible diffeomorphism is *stable with respect to permanently acting perturbations*, while in the case of area-preserving maps any such orbit is unstable (although it is Lyapunov orbitally stable).

Remark 3.1. Theorem 1 was proved in [24, 29] (see [29, Sect. 3]) only for C^∞-generic perturbations preserving reversibility. By means of these perturbations, one can create resonance levels (for all sufficiently large q) containing a degenerate elliptic periodic point near which its Poincaré map can be embedded in an autonomous flow of the form

$$\dot{z} = -i\mu z + i\Omega(|z|^2)z + i\delta(z*)^{q-1} + iBz^{q+1} + iCz(z^*)^q,$$

where z and z^* are complex conjugate coordinates, μ and δ are small real parameters, and B and C are some real coefficients (note that all coefficients on the right-hand side are purely imaginary; therefore, this equation is time reversible with respect to the involution $z \to z^*$). For $C = B(q+1)$ the equation is Hamiltonian; however, if this equality is violated, then, as shown in [29], stable and completely unstable equilibrium states can arise here. In the analytical case, it is totally unclear whether such resonance levels exist near any symmetric elliptic point, although this phenomenon is obviously not too degenerate: it has codimension 1.

The second important circumstance testifying to the universality of mixed dynamics in reversible systems is what can be called the *conjecture on reversible mixed dynamics*:

- Near any reversible diffeomorphism with a symmetric homoclinic tangency or a symmetric nontransversal heteroclinic cycle, there are absolute reversible Newhouse regions.

This conjecture was formulated in [14] and was almost immediately proved in [24] for Newhouse regions in the C^r topology with $2 \leq r \leq \infty$. In the analytical case, as well as in the case of parametric families, it was proved only for the so-called a priori non-conservative diffeomorphisms [43, 13, 22], when the (heteroclinic) cycle contains non-conservative elements (for example, saddles with the Jacobians greater and less than one, or pairs of nonsymmetric homoclinic tangencies of a symmetric saddle point, as in [13]), as well as for diffeomorphisms with heteroclinic cycles of conservative type [14], such as in Fig. 1c. Essentially, it remains to consider only two most difficult cases: reversible diffeomorphisms with symmetric quadratic and cubic homoclinic tangencies.

The third, global, circumstance confirming the widespread adoption of mixed dynamics in reversible systems is associated with the phenomenon of collision of attractors and repellers, which is often observed in experiments. If a reversible system has a dissipative attractor, then it also has a symmetric repeller. If these attractor and repeller do not intersect, then the dynamics on them will be quite usual and typical for dissipative systems. However, when they collide on the line of symmetry (this is always a crisis of both the attractor and the repeller[4]), one can often observe that the attractor and repeller almost coincide: they begin to occupy approximately the same domain of the phase space and become difficult to distinguish in numerical experiments as phase portraits obtained when calculating in the forward and backward time (see, e.g., the corresponding illustrations in Section 4).

There are other mechanisms for the emergence of mixed dynamics. For example, there is a mechanism that can be called reversible intermittency, when simple symmetric attractor and repeller (for instance, periodic sink and source) collide and both disappear, but after that a large chaotic set arises that has all features of mixed dynamics. Such a phenomenon was first discovered in [19] in the Pikovsky–Topaj model of four rotators [58]; it was also observed in a nonholonomic Celtic stone model [18].

4. MIXED DYNAMICS OF NONHOLONOMIC MODELS OF RIGID BODY MOTION

One of the best known classes of applied problems that exhibit mixed dynamics is formed by the problems of nonholonomic mechanics associated with the study of motion of rigid bodies on surfaces in the presence of nonintegrable (nonholonomic) constraints. Examples of such constraints are the absence of slipping at the point of contact of the body with the surface, the absence of spinning, the vanishing of one of the components of the angular velocity of the body, etc. Usually, nonholonomic models are described by systems of five ordinary differential equations in the variables $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ and (φ, θ), where $\vec{\omega}$ is the angular velocity vector in the moving coordinate system rigidly connected with the body and (φ, θ) are two Euler angles describing the orientation of the body [45]. Very often, instead of the Euler angles, one uses the projections $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ of the stationary vertical unit vector onto the axes of the moving coordinate system to describe the body position [10]. Here, for the components of vector $\vec{\gamma}$, one has the relation $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, the

[4]Bifurcations of attractors (repellers) are usually divided into two types: internal bifurcations and crises [4]. The latter include bifurcations that result either in the disappearance of the attractor or in a sharp (discontinuous) decrease or increase in the region of its existence in the phase space.

so-called geometric integral, and, accordingly, the system becomes five-dimensional at the level of this integral.

The presence of nonholonomic constraints leads to the appearance of integrals of motion, which allow one to reduce the dimension of system. So, for example, the assumption that there is no slipping between the body and the surface (the speed of the body at the point of contact is zero) means that the friction force does no work and, hence, the full energy E of the system is preserved. Similarly, the assumption that there is no rotation of the body around the vertical axis leads to the appearance of the integral of motion $(\vec{\omega}, \vec{\gamma}) = 0$.

According to Euler and Jacobi, an N-dimensional system of ordinary differential equations is integrable in quadratures if it has $N - 2$ independent first integrals and also a smooth invariant measure (see, e.g., [40]). Except for some specific cases, nonholonomic models of mechanics do not have three first integrals, and a smooth invariant measure does not exist for them either. Thus, at level sets of the first integrals, such systems become non-conservative, and attractors, including chaotic ones [11], may appear in them.

Another remarkable property of nonholonomic mechanical systems is their reversibility, i.e., the invariance with respect to a certain coordinate transformation (which is an involution) and to the time reversal. All such systems are reversible with respect to the transformation $\vec{\omega} \to -\vec{\omega}$, $t \to -t$. Depending on the shape of the body and the type of constraints, additional involutions may also arise here.

The reversibility of non-conservative systems leads to the fact that, in the phase space of the system, for each periodic orbit with Jacobian $J \neq 1$ there exists a periodic orbit symmetric to it with the Jacobian J^{-1}, for each attractor there exists a repeller, and so on. Such a symmetry of phase orbits and nonintegrability can naturally lead to the appearance of mixed dynamics. For the first time, mixed dynamics in such models was found in the nonholonomic model of Celtic stone [16, 17] as well as in the nonholonomic model of rubber Chaplygin top [34, 35].

In this section, we consider three nonholonomic models in which mixed dynamics has been found and investigated: the model of Suslov top, the model of a rubber disk moving on a plane, and the Celtic stone model. For the sake of brevity, we will not present the equations of these models; they can be found in the corresponding papers cited here. Basic information about the dynamics of nonholonomic models can be found, for example, in the books [45, 10].

4.1. Nonholonomic model of Suslov top. As the first example of a nonholonomic model that demonstrates mixed dynamics, we consider a system describing the motion of a Suslov top, a heavy rigid body with a fixed point subject to the nonholonomic constraint $(\vec{\omega}, \vec{e}) = 0$, which forbids its rotation around some axis $\vec{e}$. Such a constraint was introduced by G. K. Suslov in [57] (see also [56, 50]). As an illustrative example of the model in which the indicated constraint arises naturally, we point out the construction proposed in [63], in which a rigid body with a fixed point rolls on sharp wheels inside a fixed sphere (Fig. 5a). The sharp edges of the wheels forbid motions of the body in the direction perpendicular to their plane. At a level of the energy integral, since one of the components of the angular vector is zero, the dynamics of the Suslov top is described in the general case by a three-dimensional flow or a two-dimensional Poincaré map (see [50]).

In the general case, the system describing the motion of the Suslov top does not have a smooth invariant measure [38]. In the phase space of this system, both regular and chaotic attractors can appear [6]. For some parameter values, as was recently shown in [36], a Hénon-like attractor of the system merges with a Hénon-like repeller symmetric to it (under the involution). As a result of such a merger, both the attractor and repeller instantly increase in size (Figs. 5b and 5c) and intersect. Thus, mixed dynamics in the system under consideration occurs explosively. A similar phenomenon was also found in a model of vortex dynamics [37].

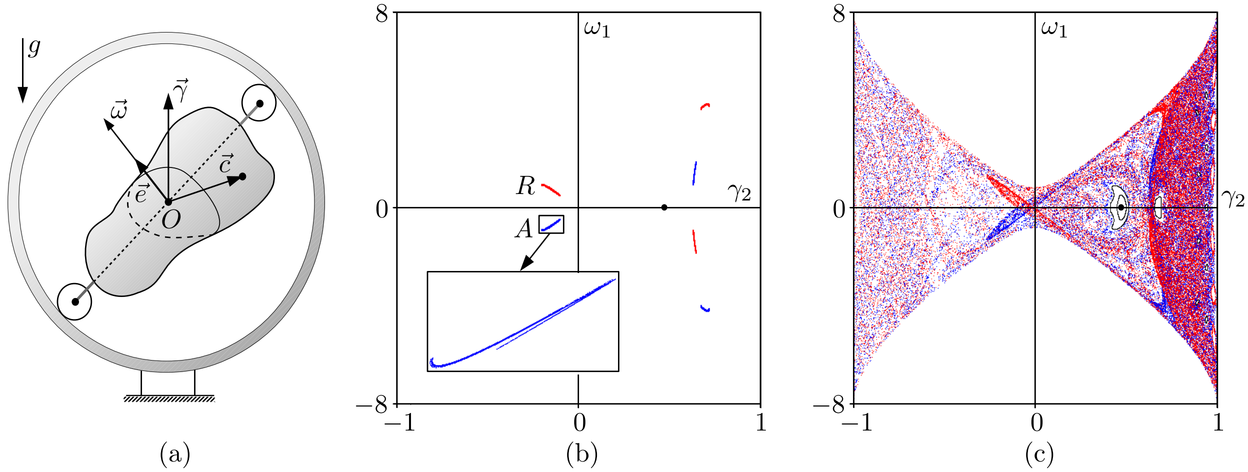

Fig. 5. Dynamics of a Suslov top: (a) the construction proposed in [63]; (b) the three-component Hénon-like attractor A and repeller R are symmetric and separated (an enlarged fragment of the attractor is also shown); (c) mixed dynamics after their merger: the full attractor and full repeller of the model almost coincide, but it is clearly seen that they differ in some small details.

4.2. Nonholonomic model of rubber disc. The model of Suslov top considered above exhibits mixed dynamics in which the full attractor of the system is clearly different from its full repeller. However, in some nonholonomic systems, such a difference is hardly noticeable. It can be only detected under careful investigation of the phase portrait of the system, after a preliminary study of bifurcations of emergence of stable periodic orbits.

Among the models of this type, there is a nonholonomic model of rubber disk, a balanced round solid body of zero thickness that has different principal moments of inertia [1, 39] (Fig. 6a). The motion of such a disk along the plane is subject to a pair of nonholonomic constraints: one forbids slipping (the velocity at the point of contact K is zero), while the other prevents spinning around the vertical axis $\vec{\gamma}$. Thus, as in the case of the Suslov top, the dynamics of the rubber disk on a plane is described by a three-dimensional flow or a two-dimensional Poincaré map.

In numerical simulations, the behavior of the orbits in this map is very similar, even in small details, to the conservative one (see, e.g., Fig. 6b). Therefore, the question of the existence of a smooth invariant measure naturally arises for this model. However, instead of searching for an invariant measure, it is reasonable first to answer another, even more natural, question about the existence of mixed dynamics. This question has been answered in the affirmative. Inside the seemingly conservative chaos, on small scales (on the order of $10^{-3} \times 10^{-3}$), it is possible to detect coexisting elliptic, asymptotically stable, and completely unstable periodic points located inside zones with chaotic behavior of orbits (Figs. 6c and 6d). Thus, the approach to studying the model from the point of view of the concept of mixed dynamics turned out to be very successful in this case as well.

4.3. Nonholonomic model of Celtic stone. Finally, we consider a model of Celtic stone on a plane (Fig. 7a), which is an example of a nonholonomic system whose Poincaré map is three-dimensional. A Celtic stone is a rigid body with a rounded symmetric surface that possesses a dynamical asymmetry. If one puts such a stone on a flat surface and twists it around a vertical axis in a certain direction, say, counterclockwise, then it will stably continue its rotation, like any ordinary rounded body. However, if one starts to twist such a stone in the clockwise direction, then, for no apparent reason, it will soon slow down its rotation, begin to oscillate, then change the direction of rotation to the opposite one, and, finally, continue to rotate stably counterclockwise.

The nonholonomic model of Celtic stone gives a simple mathematical explanation for this phenomenon (see, e.g., [5, 33, 44]). The counterclockwise rotation corresponds to the asymptotically

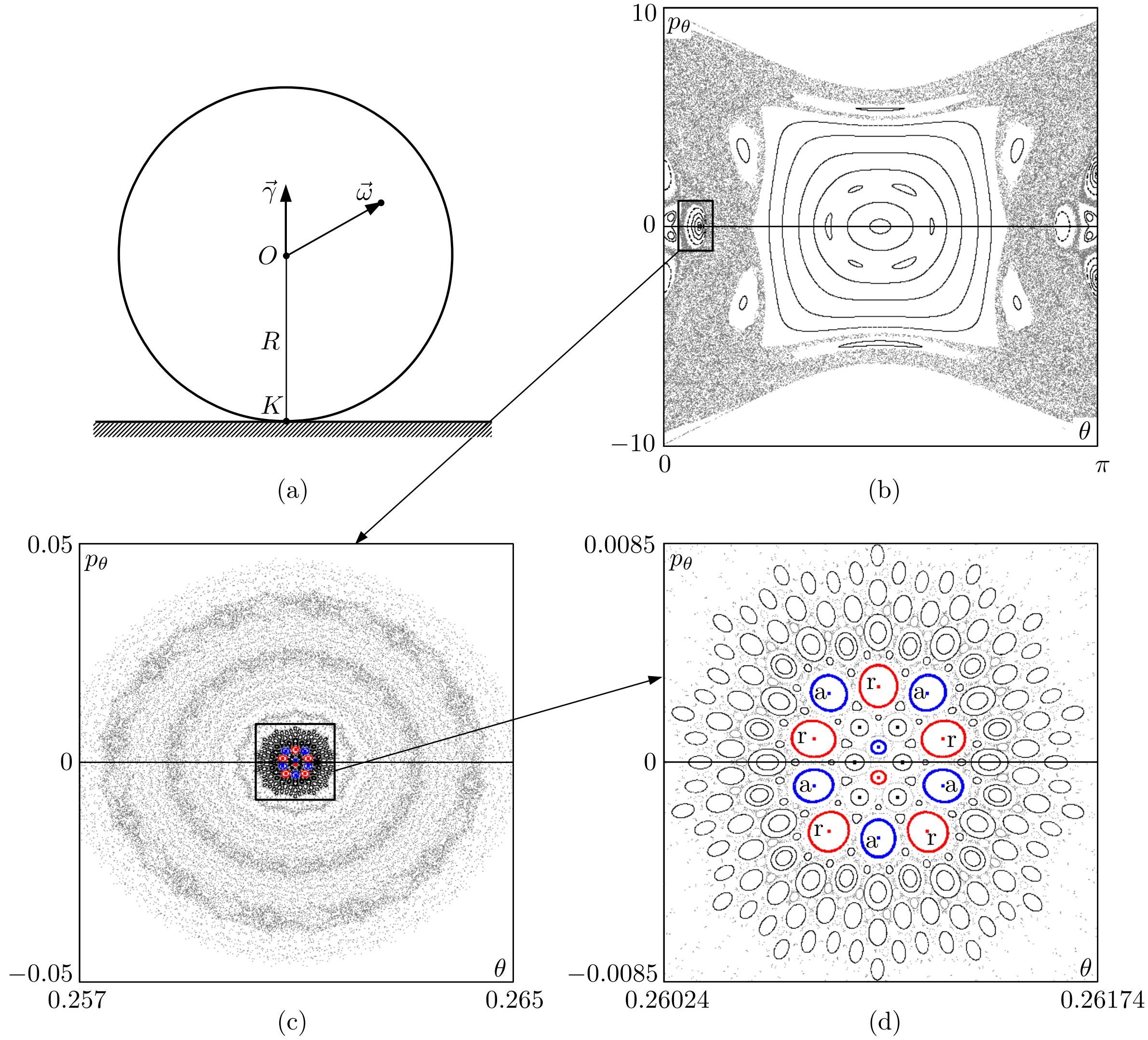

Fig. 6. Dynamics of a rubber disc on the phase plane (θ, p_θ), where θ is the deflection angle of the disc from the vertical axis and p_θ is its generalized momentum: (a) scheme of the disc; (b) phase portrait in the Poincaré map; (c, d) enlarged fragments (with 1000× magnification from (b) to (d)). In the phase portrait of part (d), attractors (a), repellers (r), and their absorbing/repulsing domains are marked (the thick circles around the attractors/repellers are actually pieces of extremely weakly twisting spirals).

stable equilibrium O_s. Since the nonholonomic model of the stone is reversible with respect to the involution $\vec{\omega} \to -\vec{\omega}$ and time reversal, the system also has a completely unstable equilibrium O_u corresponding to the rotation of the stone around the same axis but in the opposite direction. If there are no other attractors in the system (in addition to the stable equilibrium), then the orbits from a neighborhood of O_u will tend to the stable equilibrium O_s.

It is important to note that, along with the stable equilibrium, the nonholonomic model of Celtic stone may possess other regular and chaotic attractors. For the first time, strange attractors in the dynamics of Celtic stones were discovered in [11]. The papers [42, 17, 8, 18, 20] are also devoted to the study of strange attractors in this model.

Concerning mixed dynamics, this phenomenon in the nonholonomic model of Celtic stone was first found in [17], where it was shown that the attractor of the system may intersect but not coincide with the repeller. Figures 7b–7d show phase portraits of the attractor A and repeller R

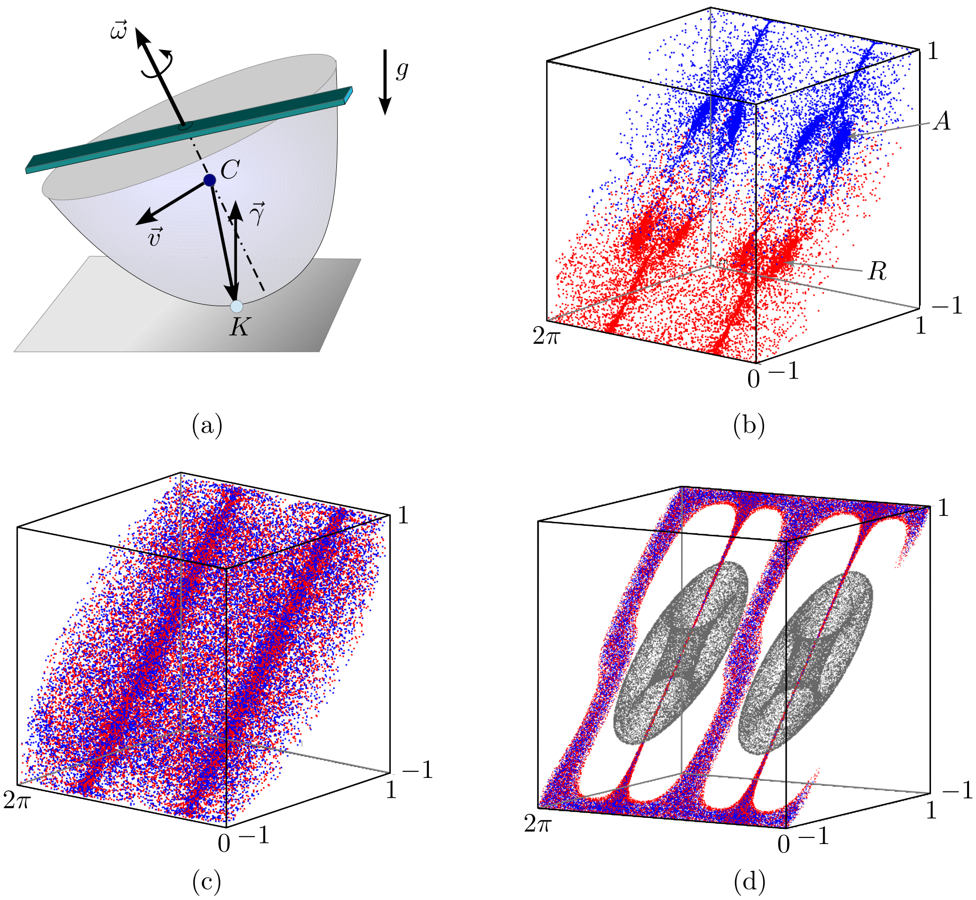

Fig. 7. Dynamics of a Celtic stone on a plane: (a) representation of a Celtic stone as a homogeneous body with a convex surface symmetric with respect to the axes Ox and Oy, with a flat platform $z = z_0$ to which a heavy bar is attached at the center (turning it, one can achieve the corresponding dynamical asymmetry of the stone); here K is the point of contact with the plane, C is the center of mass, and $\vec{v}$ and $\vec{\omega}$ are the velocity and angular velocity of the stone; (b) strange attractor A and strange repeller R for $E = 560$; (c) mixed dynamics for $E = 320$; (d) coexistence of mixed dynamics with the behavior of orbits which is almost indistinguishable from the conservative one (a pair of gray ovals in the middle of the cube) for $E = 150$. All phase portraits are constructed in the Andoyer–Deprit variables (see [10]).

in the nonholonomic model of Celtic stone for various values of the parameter E, the full energy of the stone. When the stone is twisted quite strong (when the energy is high enough), the full attractor of the system is separated from the full repeller (see Fig. 7b). However, when the rotation energy decreases, the attractor and repeller increase in size and, at some moment, begin to intersect; then mixed dynamics arises (see Fig. 7c), which can also coexist with orbit behavior that is almost indistinguishable in numerical simulations from the conservative one (see Fig. 7d). Note that in [18] bifurcations leading to the appearance of mixed dynamics were also studied. In particular, it was shown that this phenomenon can occur explosively, as a result of the collision of stable and completely unstable fixed points on the line of fixed points of the involution.

In conclusion, we note that the list of nonholonomic mechanical systems exhibiting mixed dynamics is not limited by the three examples presented above. Moreover, since most such models are nonintegrable and do not possess a smooth invariant measure, the appearance of mixed dynamics is a typical phenomenon here. As a rule, in such systems, it is more difficult to find cases with the existence of conservative chaos (when there exists a smooth invariant measure) than cases with mixed dynamics. To detect mixed dynamics in most nonholonomic models, it is enough to just construct the full attractor and repeller of the system and then compare these two sets. In this way, the existence of mixed dynamics can also be established in the nonholonomic models of the Chaplygin sleigh [41], the Suslov top under periodic control [7], the Chaplygin top [9], etc. However, in some cases, in order to detect mixed dynamics, more detailed studies are required, for example, those related to the study of local and global symmetry-breaking bifurcations.

FUNDING

This work is supported by the Russian Science Foundation under grants 17-11-01041 (Section 1 and Subsections 4.1, 4.2) and 18-71-00127 (Subsection 4.3) and by the Russian Foundation for Basic Research, project nos. 19-01-00607 (Section 3), 18-31-20052 (Subsection 2.1), and 18-29-10081 (Subsection 2.2). The first two authors also acknowledge the support of the Ministry of Science and Higher Education of the Russian Federation (project no. 1.3287.2017). The third author was also supported in part by the HSE Basic Research Program (2019).

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This article was submitted by the authors simultaneously in Russian and English