International Journal of Bifurcation and Chaos, Vol. 28, No. 11 (2018) 1830036 (29 pages)
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DOI: 10.1142/S0218127418300367

# Elements of Contemporary Theory of Dynamical Chaos: A Tutorial. Part I. Pseudohyperbolic Attractors

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Received May 11, 2018

The paper is devoted to topical issues of modern mathematical theory of dynamical chaos and its applications. At present, it is customary to assume that dynamical chaos in finite-dimensional smooth systems can exist in three different forms. This is *dissipative chaos*, the mathematical image of which is a strange attractor; *conservative chaos*, for which the entire phase space is a large "chaotic sea" with randomly spaced elliptical islands inside it; and *mixed dynamics*, characterized by the principal inseparability in the phase space of attractors, repellers and conservative elements of dynamics. In the present paper (which opens a series of three of our papers), elements of the theory of pseudohyperbolic attractors of multidimensional maps and flows are presented. Such attractors, as well as hyperbolic ones, are genuine strange attractors, but they allow the existence of homoclinic tangencies. We describe two principal phenomenological scenarios for the appearance of pseudohyperbolic attractors in one-parameter families of three-dimensional diffeomorphisms, and also consider some basic examples of concrete systems in which these scenarios occur. We propagandize new methods for studying pseudohyperbolic attractors (in particular, the method of saddle charts, the modified method of Lyapunov diagrams and the so-called LMP-method for verification of pseudohyperbolicity of attractors) and test them on the above examples. We show that Lorenz-like attractors in three-dimensional generalized Hénon maps and in a nonholonomic model of Celtic stone as well as figure-eight attractors in the model of Chaplygin top are genuine (pseudohyperbolic) ones. Besides, we show an example of four-dimensional Lorenz model with a wild spiral attractor of Shilnikov–Turaev type that was found recently in [Gonchenko *et al.*, 2018].

*Keywords*: Strange attractor; bifurcations; dynamical chaos; homoclinic orbit; Lorenz attractor.

## 1. Introduction

At present, one can distinguish three relatively independent and different forms of dynamical chaos of smooth multidimensional systems — "dissipative chaos", "conservative chaos" and "mixed dynamics". The mathematical image of the dissipative chaos is *strange attractor* — a nontrivial attracting closed invariant set which is located in the phase space of a system inside some absorbing domain, which includes all those orbits that cross its boundary. Unlike the dissipative chaos, conservative chaos spreads over the whole phase space and is associated with the inextricable interrelations between nonuniformly hyperbolic and elliptical behavior of orbits.

Mixed dynamics is a new third type of dynamical chaos which is characterized by the fact that asymptotically stable elements of dynamics (attractors) coexist with completely unstable ones (repellers), and, moreover, they are principally inseparable from each other and form conservative elements of dynamics (so-called reversible cores), see more details in [Gonchenko, 2016; Gonchenko & Turaev, 2017].

Recall that the Conley's theorem [Conley, 1978] implies that any smooth system on a compact manifold $M$ has an attractor $\mathcal{A}$ and a repeller $\mathcal{R}$ (i.e. attractor at time reversal). Thus, we have here the following formalized scheme: $\mathcal{A} \cap \mathcal{R} = \emptyset$ for dissipative chaos; $\mathcal{A} = \mathcal{R} = M$ for conservative chaos; $\mathcal{A} \cap \mathcal{R} \neq \emptyset$ and $\mathcal{A} \neq \mathcal{R}$ for mixed dynamics. This evidently implies that any fourth type of chaos does not exist.

The phenomenon of mixed dynamics was discovered, in fact, in the paper [Gonchenko *et al.*, 1997a], where, in particular, it was proved that, in the space of two-dimensional diffeomorphisms, there are open regions (the so-called Newhouse regions) in which diffeomorphisms with infinitely many stable, completely unstable and saddle periodic orbits are dense and, moreover, in this case the closures of sets of hyperbolic periodic orbits of all possible types have nonempty intersections.

Recall, that *Newhouse regions* are open regions from the space of $C^r$-smooth systems, $r \geq 2$, where systems with homoclinic tangencies are dense, i.e. such systems which have saddle periodic orbits whose invariant manifolds intersect nontransversely. In turn, Newhouse regions exist in any neighborhood of any system with homoclinic tangency [Newhouse, 1979; Gonchenko *et al.*, 1993b; Palis & Viana, 1994; Romero, 1995].

Naturally, for any definition of attractor, it must be a stable closed invariant set which should contain all stable periodic orbits, if they exist. The same (for completely unstable periodic orbits) must also hold for repellers. Thus, in [Gonchenko *et al.*, 1997a] it was shown that an attractor can intersect with repeller and this property can be generic for systems from Newhouse regions. The mathematical justification of this phenomenon was given quite recently, see e.g. [Gonchenko, 2016; Gonchenko & Turaev, 2017].

As for strange attractors to which this paper is devoted, their generally accepted definition, which would be suitable for all occasions, does not exist up to now. The only exceptions are the so-called *genuine strange attractors* whose definition includes two main conditions: (1) the existence of an absorbing region in the phase space, where the attractor exists, and (2) the instability of orbits of the attractor, which means that each orbit of the attractor has a positive maximal Lyapunov exponent. It is also assumed that properties (1) and (2) are satisfied for all nearby systems.

On the other hand, the so-called *quasiattractors* are also referred to the strange attractors, with due reason, see discussions in [Afraimovich & Shilnikov, 1983a; Shilnikov, 1997; Gonchenko *et al.*, 1997b]. Recall that a quasiattractor is defined as a nontrivial attracting invariant set that either contains stable periodic orbits of very large periods (and with very narrow domains of attraction), or such orbits appear under arbitrarily small smooth perturbations [Afraimovich & Shilnikov, 1983a]. This is connected with the fact that quasiattractors admit the existence of such homoclinic tangencies whose bifurcations lead to the birth of asymptotically stable periodic orbits. Note that for these homoclinic tangencies certain conditions-criterion are fulfilled [Gavrilov & Shilnikov, 1972, 1973; Gonchenko, 1983; Gonchenko *et al.*, 1993c, 1996a, 2008]. Besides attracting invariant tori, small Hénon-like attractors [Mora & Viana, 1993; Palis & Viana, 1994; Gonchenko *et al.*, 1996b; Colli, 1998; Homburg, 2002], and even small Lorenz-like attractors [Gonchenko *et al.*, 2006, 2009, 2014b; Gonchenko & Ovsyannikov, 2013, 2017] can appear here.

*Remark 1.1.* In dissipative systems with special structures, strange attractors of other types may exist, which do not formally fit into this scheme. For example, nonsmooth or discontinuous systems may have attractors with a singularly hyperbolic behavior of orbits, in the sense that Lyapunov exponents are not defined for some orbits, although, for such systems, both conditions (the existence of absorbing regions and the instability of orbits on the attractor) are fulfilled. Examples of such attractors are the Lozi attractor [Lozi, 1978] and the Belykh attractor [Belykh, 1995]. A completely different type of complex nonperiodic behavior of orbits is demonstrated by the so-called strange nonchaotic attractors that arise in special models that have the structure of direct product of a nonspecific dynamical system and a quasiperiodic system. They are characterized by the fact that one of Lyapunov exponents is zero for any orbit, and the remaining ones are less than zero (also it is allowed the existence of a set of zero measure of orbits with the positive Lyapunov exponent). For more details, see e.g. [Feudel *et al.*, 2006].

One can say that the most known strange attractors of smooth dynamical systems, in particular, many such attractors in systems from applications, are, in fact, quasiattractors. Examples are numerous "torus-chaos" attractors arising under the breakdown of two-dimensional tori [Afraimovich & Shilnikov, 1974, 1983b]; attractors in Chua circuits [Chua *et al.*, 1986; Shilnikov, 1994]; the Hénon attractor [Hénon, 1976; Benedicks & Carleson, 1991]; attractors in periodically perturbed two-dimensional systems with a homoclinic figure-eight of a saddle [Gonchenko *et al.*, 2013c] and many others.

A very important class of quasiattractors is composed by *spiral attractors* that are related to the existence of homoclinic orbits to a saddle-focus equilibrium. Such attractors are often found in applications, and many of their examples are well known. In particular, there are spiral attractors of three-dimensional flows, such as the Rössler attractor [Rössler, 1976a, 1976b], attractors in the Arneodo–Coullet–Tresser models [Arneodo *et al.*, 1980, 1981, 1982] (called also as ACT-attractors), in the Rosenzweig–MacArthur models [Rosenzweig & MacArthur, 1963; Kuznetsov *et al.*, 2001; Bakhanova *et al.*, 2018] etc. It is quite interesting that all the above attractors appear in flows according to a fairly simple and universal phenomenological scenario proposed by Shilnikov [1986]. The main specificity of such spiral attractors is that they are organized around a saddle-focus equilibrium of type $(1, 2)$, i.e. with one-dimensional stable and two-dimensional unstable invariant manifolds, and, for certain values of parameters, they have homoclinic orbits to this equilibrium. With some natural modifications, the Shilnikov scenario was also transferred to the case of three-dimensional maps [Gonchenko *et al.*, 2012]. Therefore, for such spiral attractors, we proposed, in [Gonchenko *et al.*, 2014a], the generalizing term: "Shilnikov attractor" for flows (these include, in particular, the above mentioned Rössler attractor and ACT-attractors); and "Shilnikov discrete attractor" for maps. Various examples of such discrete attractors were found in three-dimensional Hénon maps [Gonchenko *et al.*, 2012, 2014a, 2013b; Gonchenko & Gonchenko, 2016] and in several models from the rigid body dynamics, e.g. in a nonholonomic model of Chaplygin top [Borisov *et al.*, 2016]. We plan to give a detailed review on a topic of spiral (and, in particular, Shilnikov) attractors in part II of our paper.

Until recently, only the hyperbolic attractors and Lorenz-like attractors could be considered as the genuine strange attractors. However, the situation has changed after the paper [Turaev & Shilnikov, 1998], where a new class of genuine strange attractors was introduced, the so-called *wild hyperbolic attractors*. These attractors, unlike hyperbolic and Lorenz ones, admit the existence of homoclinic tangencies, but they do not contain stable periodic orbits and any other stable invariant subsets that do not arise also for small smooth perturbations. Although systems with wild hyperbolic attractors belong to Newhouse domains,[1] bifurcations of their homoclinic tangencies, in contrast to systems with quasiattractors, do not lead to the birth of stable periodic orbits [Gonchenko *et al.*, 1993c, 1996a, 2008], see also Sec. 2.

Recall that in [Turaev & Shilnikov, 1998], a geometrical model of four-dimensional flow with a wild spiral attractor containing a saddle-focus equilibrium was also constructed. One of the main features of the Turaev–Shilnikov spiral attractor is that it possesses a *pseudohyperbolic structure*, see

[1]The term "wild" goes back to the Newhouse paper [Newhouse, 1979], in which the concept of "wild hyperbolic set" was introduced, i.e. such a uniformly hyperbolic invariant set, in which, among its stable and unstable invariant manifolds, there are always those that intersect nontransversely, and this property is preserved for all small $C^2$-smooth perturbations.

Definition 2.1. Speaking shortly, this is a "weak" version of hyperbolicity for which an exponential contraction along certain directions takes place, while transversally there is an exponential expansion of the volume.

We note that if the pseudohyperbolicity conditions are fulfilled (for all points of some absorbing domain $D$ of attractor), then, as was shown in [Turaev & Shilnikov, 1998], the attractor exists and is unique, and each of its orbit has the positive maximal Lyapunov exponent. In this case the attractor is, in fact, the Ruelle attractor [Ruelle, 1981], i.e. closed, invariant, (asymptotically) stable and chain-transitive set, see [Gonchenko & Turaev, 2017] for more details.

In fact, in the paper [Turaev & Shilnikov, 1998] can be seen the foundation of a very promising theory of pseudohyperbolic strange attractors. New examples of such attractors were also found shortly. Thus, in the paper [Gonchenko *et al.*, 2005] it was shown that in three-dimensional Hénon maps of the form

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = M_1 + Bx + M_2 y - z^2, \qquad (1)$$

where $M_1$, $M_2$, $B$ are parameters ($B$ is the Jacobian), in a certain domain of parameter values adjoining the point $A^* = (M_1 = 1/4, M_2 = 1, B = 1)$, there exist discrete Lorenz attractors.

The pseudohyperbolicity of such attractors was claimed in [Gonchenko *et al.*, 2005] due to the fact that, for values of parameters close to $A^*$, the second power of map (1) locally (in a small neighborhood of the fixed point with the triplet $(-1, -1, 1)$ of multipliers) can be represented as the Poincaré map of a periodically perturbed Shimizu–Morioka system, which, in turn, possesses the Lorenz attractor [Shilnikov, 1991, 1993]. If the perturbation is sufficiently small (which is determined by the closeness of the values of parameters to $A^*$), then the desired pseudohyperbolicity should be naturally inherited from the pseudohyperbolicity of the Lorenz attractor [Tucker, 1999; Ovsyannikov & Turaev, 2017]. We recall that, in [Turaev & Shilnikov, 2008] it was shown that the property of pseudohyperbolicity of flows is also preserved for their Poincaré maps for small periodic perturbations.

In Fig. 1 (from [Gonchenko *et al.*, 2005]) we show two examples of discrete Lorenz attractors of map (1). We note that the phase portraits of these attractors are very similar to portraits of the Lorenz attractors for flow. However, the corresponding values of the parameters [$M_1 = 0, M_2 = 0.85, B = 0.7$ in the case of Fig. 1(a) and $M_1 = 0, M_2 = 0.825$, $B = 0.7$ in the case of Fig. 1(b)] are not nearly close to $A^*$. Therefore, the conditions of pseudohyperbolicity of such attractors need to be verified additionally.

Such a task seems very complicated. Strictly speaking, here it is necessary to use methods of mathematical proofs based on numerics — the so-called "computer assisted proofs", as it was done

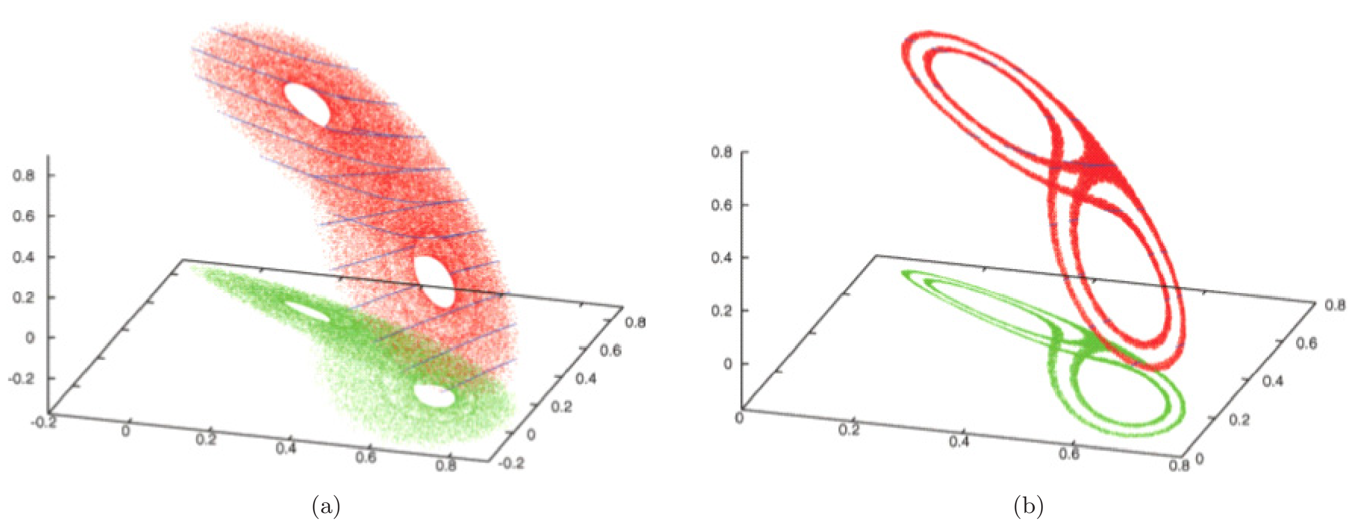



Fig. 1. Portraits of discrete Lorenz attractors in the case of map (1) with (a) $M_1 = 0$, $M_2 = 0.85$, $B = 0.7$ and (b) $M_1 = 0$, $M_2 = 0.825$, $B = 0.7$. In both cases, about $10^5$ iterations of a single initial point on the attractor are shown. The projections of attractors onto the plane $(x, y)$ and some slices of the attractor by the plane $z =$ const are also shown (although these slices look like lines, in fact they have a complex Cantor structure).

in the famous work [Tucker, 1999] for the Lorenz model. However, even the most common computer methods can greatly help here. So, at the very first stage, we can check some necessary conditions. For example, the Lyapunov exponents $\Lambda_1 > \Lambda_2 > \Lambda_3$ of any pseudohyperbolic attractor of three-dimensional maps should satisfy the following conditions

$$\Lambda_1 > 0, \quad \Lambda_1 + \Lambda_2 > 0, \quad \Lambda_1 + \Lambda_2 + \Lambda_3 < 0. \qquad (2)$$

Here the first and third inequalities indicate that the observed attractor is strange, and the second one should mean that two-dimensional areas are expanded on the attractor.

However, the Lyapunov exponents are certain average characteristics of orbits on attractor and, therefore, it is possible that an attractor has very small "windows" (whose sizes may be less than any reasonable accuracy of calculations), where conditions (2) for the corresponding orbits are violated. At our request, conditions of pseudohyperbolicity of the attractors from Fig. 1 were verified (by means of "computer assisted proof" methods), by Figueros and Tucker, who have obtained very interesting and fine results. Namely, they have shown that an asymptotically stable periodic orbit with extremely small basin of attraction (which diameter has an order $10^{-40}$) exists inside the attractor from Fig. 1(a), while, the attractor from Fig. 1(b) is the genuine pseudohyperbolic attractor.[2] Besides, independently and by other methods, similar results were obtained by Kuznetsov and Kuptsov, who have also verified pseudohyperbolicity of some other attractors found in our paper [Gonchenko & Gonchenko, 2016]. We hope that these interesting results will be published in the nearest future.

The existence of homoclinic tangencies naturally kills the hyperbolicity, however, in general, the pseudohyperbolicity is not violated. For example, both attractors from Fig. 1 contain a saddle fixed point with multipliers $\lambda_1$, $\lambda_2$, $\lambda_3$ such that $\lambda_1 < -1, 0 < \lambda_2 < 1, -1 < \lambda_3 < 0$ and $|\lambda_2| > |\lambda_3|$, where $\lambda_1\lambda_2\lambda_3 = B = 0.7 < 1$ and the saddle value $\sigma = |\lambda_1\lambda_2|$ is greater than 1 (for exact values of $\lambda_i$ see Fig. 8). Then the homoclinic tangencies, that inevitably occur here, will be typically such as in Fig. 3. In the general case, such (quadratic) homoclinic tangencies are called simple [Gonchenko *et al.*, 1993c], and in the case $\sigma > 1$ they do not destroy the pseudohyperbolicity (if the fixed point itself is pseudohyperbolic), although they are certain indicators of wild hyperbolicity (for more detail see [Gonchenko *et al.*, 1993b, 1993c, 2008]).

*Remark 1.2.* On the other hand, stable periodic orbits are necessarily born if $\sigma < 1$, or if a fixed (periodic) point of an attractor is a saddle-focus (whether with one-dimensional or with two-dimensional unstable manifold). In particular, the spiral attractors of three-dimensional smooth maps or flows are always quasiattractors. In this connection, the following problem seems to be very interesting: *let a three-dimensional diffeomorphism have a strange attractor containing a saddle fixed point with the two-dimensional unstable invariant manifold, then this attractor is the quasiattractor.*[3]

Taking into account this remark, we will consider in the present paper only such strange attractors of three-dimensional maps that contain saddle fixed points with one-dimensional unstable manifolds and with saddle value $\sigma$ greater than 1. Moreover, we pay the main attention to the so-called homoclinic attractors that contain *exactly one* saddle fixed point. As shown in the papers [Gonchenko *et al.*, 2005, 2012, 2014a; Gonchenko & Gonchenko, 2016], this direction is rather promising.

The content of the paper is organized as follows.

In Sec. 2, we give a definition of pseudohyperbolic attractors both for flows (Definition 2.1) and maps (Definition 2.2) and describe some of their basic properties. Thus, unlike hyperbolic attractors, pseudohyperbolic ones admit homoclinic

[2] We note that these results are in good agreement with our results, see Secs. 3.2 and 4.1, obtained by means of a quite simple and original numerical method (the LMP-method), which looks like some trick performed within the standard procedure for calculating Lyapunov exponents.

[3] This problem seems to be very difficult, and its solution is connected, for example, with the proof of the existence of the so-called nonsimple homoclinic tangencies [Tatjer, 2001; Gonchenko *et al.*, 2007a, 2014b], examples of which are shown in Fig. 4 — only here the direction of the arrows should be reversed, so that the unstable manifold of the point $O$ becomes two-dimensional. We note that bifurcations of such tangencies lead to the birth of stable periodic orbits [Tatjer, 2001]. In turn, the appearance of nonsimple tangencies in the case under consideration is to be very expected, because the two-dimensional unstable manifold should fold infinitely many times in different directions (it is as if we tried to "package" the two-dimensional plane into a three-dimensional cube, while avoiding the appearance of sharp corners).

tangencies, therefore, we give a brief classification of homoclinic tangencies and specify those types (the so-called simple homoclinic tangencies) that do not destroy pseudohyperbolicity.

In Sec. 3, we give a review of some modern qualitative and numerical methods for searching strange attractors and for the verification of their pseudohyperbolicity. First, we describe new phenomenological scenarios leading to the emergence of strange (pseudohyperbolic) attractors under a series of local and global bifurcations in one-parameter families of three-dimensional maps. Second, we observe some searching methods (of saddle charts and modified Lyapunov diagrams) which are designed to find attractors in specific models. Third, we represent the so-called LMP-method for verification of pseudohyperbolicity of attractors.

In Sec. 4, we demonstrate examples of (pseudohyperbolic) attractors in concrete models: in three-dimensional generalized Hénon maps and in two models from the rigid body dynamics (nonholonomic models of Celtic stone and Chaplygin top). We also check by the LMP-method the pseudohyperbolicity of these attractors.

In Sec. 5, we observe absolutely a new example of four-dimensional flow possessing the wild spiral attractor of Turaev–Shilnikov type. We also verify its pseudohyperbolicity using the LMP-method.

In the Appendix, we give a detailed definition of a pseudohyperbolic multidimensional map, in terms of Lyapunov exponents, and deduce from it some important consequences.

## 2. Pseudohyperbolicity and Homoclinic Tangencies

In this section we discuss basic concepts of the theory of pseudohyperbolic strange attractors. The notion of pseudohyperbolic system was introduced by Turaev and Shilnikov: the flow case was considered in [Turaev & Shilnikov, 1998] and the case of diffeomorphisms in [Turaev & Shilnikov, 2008], see also [Sataev, 2010; Gonchenko *et al.*, 2013a] and the Appendix. Thus, following [Turaev & Shilnikov, 1998, 2008], we will adhere to the following definitions for pseudohyperbolic attractors.

**Definition 2.1.** The flow case. An attractor of an $n$-dimensional flow $F$ is called pseudohyperbolic if it possesses the following properties.

(1) For each point of some absorbing domain $\mathcal{D}$ of the attractor, there exist two linear subspaces $E^{ss}$ with $\dim E^{ss} = k$ and $E^{cu}$ with $\dim E^{cu} = n-k$, where $k \geq 1$, which are invariant with respect to the differential $DF$ of the flow and such that $DF$ is exponentially contracting along all directions in $E^{ss}$ and it expands exponentially all $(n-k)$-dimensional volumes in $E^{cu}$.
(2) The subspaces $E^{ss}$ and $E^{cu}$ depend continuously on a point from $D$.
(3) The corresponding coefficients of contraction and expansion are uniformly bounded from 1.
(4) The angles between any tangent vector to $E^{ss}$ and any tangent vector to $E^{cu}$ are uniformly separated from zero.
(5) Any possible contractions in $E^{cu}$ are uniformly weaker than any contraction in $E^{ss}$.

The definition for discrete pseudohyperbolic attractors of diffeomorphisms is quite similar and we give it a shorter form.

**Definition 2.2.** The discrete (diffeomorphism) case. An attractor of an $n$-dimensional diffeomorphism $f$ is called pseudohyperbolic if, for each point of some absorbing domain $\tilde{\mathcal{D}}$ of the attractor, there exist two linear subspaces $E^{ss}$ with $\dim E^{ss} = k$ and $E^{cu}$ with $\dim E^{cu} = n-k$, where $k \geq 1$, which are invariant with respect to the differential $Df$ of $f$ and such that the properties (1)–(5) of Definition 2.1 are fulfilled for $Df$.

For the sake of completeness and for the interested reader, we also give in the Appendix a more detailed definition in terms of Lyapunov exponents.

Thus, unlike hyperbolicity, here is not required the existence of uniform expansions in $E^{cu}$ along all directions. Nevertheless, the pseudohyperbolicity is preserved under small smooth perturbations [Turaev & Shilnikov, 1998, 2008]. Therefore, if a system has a pseudohyperbolic attractor, then this attractor is strange, since the expansion of volumes in $E^{cu}$ guarantees the existence of positive maximal Lyapunov exponent for any orbit. In other words, pseudohyperbolic attractors are genuine attractors.

However, in contrast to the hyperbolic and Lorenz attractors, pseudohyperbolic attractors can possess *homoclinic tangencies*. Moreover, if it is unknown in advance that the strange attractor is hyperbolic, then in addition to transverse homoclinic orbits (at points of which stable and unstable invariant manifolds of saddle periodic orbits intersect transversally), there should also exist nontransversal ones — homoclinic tangencies. By itself, the appearance of a homoclinic tangency is not something exceptional: this is a codimension-one bifurcation phenomenon when a quadratic homoclinic tangency appears. However, as was shown in [Newhouse, 1979], this single bifurcation can imply very complicated structure of the bifurcation set. In particular, arbitrarily close to any two-dimensional diffeomorphism with a homoclinic tangency, there are open regions (Newhouse regions) in which diffeomorphisms with homoclinic tangencies are dense.

Dynamics of systems from Newhouse regions is extremely rich. So, as established in [Gonchenko *et al.*, 1993a, 2001], in these regions there are dense systems with infinitely many homoclinic tangencies of any orders and arbitrarily degenerate periodic orbits, etc. All this implies that bifurcations of homoclinic tangencies cannot be studied completely, for example, by means of finite-parameter families — the traditional apparatus of the classical bifurcation theory, see more discussions in [Gonchenko *et al.*, 1991, 2007b]. Therefore, here, naturally, the problems of a completely different kind come to the fore. For example, such problems are those connected with the study of the basic bifurcations and basic characteristic properties of systems from Newhouse regions. Moreover, what is very important and interesting, is that the question of which bifurcations and which characteristic properties are the main ones could be decided by the researcher himself.

In the theory of strange attractors, one of the most important problems relates to an identification whether a given attractor is the quasiattractor or the genuine attractor (in particular, pseudohyperbolic one). Sometimes, we can easily identify that the attractor under consideration is quasiattractor. So, in the case of strange attractors of two-dimensional diffeomorphisms (if they are not hyperbolic), bifurcations of inevitable homoclinic tangencies lead to the appearance of stable periodic orbits of quite large periods and, accordingly, any such attractor should be considered as quasiattractor.[4]

In the case of strange attractors of three-dimensional diffeomorphisms, which are one of the main topics of the present paper, the problem of identification of their types (quasiattractor or genuine attractor) is much more complicated. However, even here, homoclinic tangencies found in attractors can be considered as peculiar indicators. Thus, if an attractor allows homoclinic tangencies to a fixed or periodic point such as in Fig. 2, then it is definitely the quasiattractor. In the first case, Fig. 2(a), the fixed point is a saddle with the saddle value $\sigma$ less than 1, and in the second case, Fig. 2(b), it is a saddle-focus. The birth of stable periodic orbits under bifurcations of such homoclinic tangencies was established e.g. in [Gonchenko *et al.*, 1993c, 1996a, 2008] — here it is only required that the Jacobian $J$ of the fixed point is less than one, and in the case of a saddle its unstable manifold is one-dimensional (in the case of a saddle-focus, there is no meaning whether the manifold is one-dimensional or two-dimensional).

On the other hand, it is very important that there are homoclinic tangencies that do

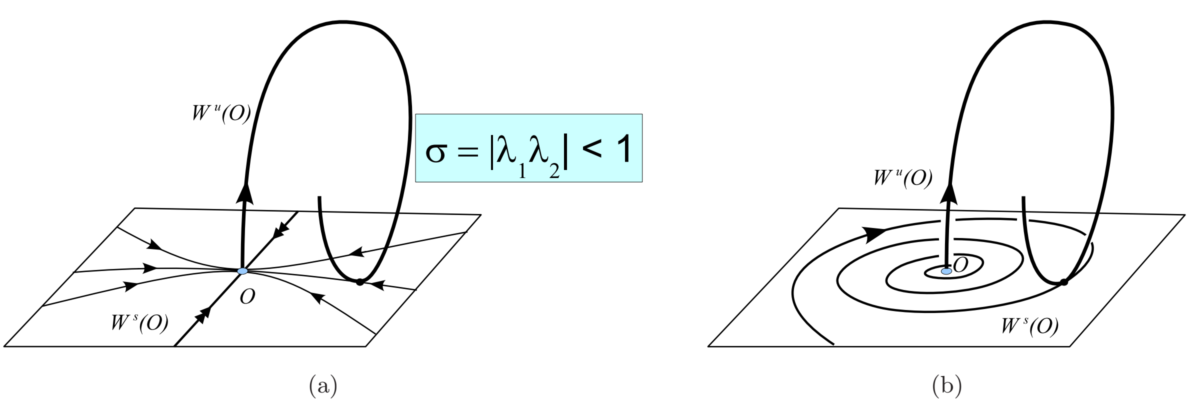



Fig. 2. Examples of homoclinic tangencies whose bifurcations lead to the birth of stable periodic orbits.

[4]This is true, for example, for the Hénon attractors, for which stable periodic orbits arise under arbitrarily small perturbations, although they may be absent (for parameter values forming a nowhere dense set of positive measure, according to the Benedicks–Carleson theory [Benedicks & Carleson, 1991]).

not destroy pseudohyperbolicity. In the case of three-dimensional diffeomorphisms, these are *simple homoclinic tangencies* [Gonchenko *et al.*, 1993b, 1993c] provided that $\sigma > 1$.

Let, for example, a diffeomorphism $f$ have a saddle fixed point $O$ with real multipliers $\lambda_1, \lambda_2, \lambda_3$ such that $|\lambda_1| > 1 > |\lambda_2| > |\lambda_3| > 0$ provided that $\sigma = |\lambda_1||\lambda_2| > 1$. For such tangencies, the point $O$ itself is pseudohyperbolic: it has $N^{ss}(O)$ as the line passing through $O$ in the direction of the eigenvector of the linearization matrix $A$ corresponding to its strong stable multiplier $\lambda_3$, and $N^{cu}(O)$ is the plane containing the eigenvectors of the matrix $A$ corresponding to the multipliers $\lambda_1$ and $\lambda_2$. Obviously, for any point $p$ from a small neighborhood $U(O)$ of the saddle $O$, there will be the same invariant decompositions into the spaces $N^{ss}(p)$ and $N^{cu}(p)$. Similar invariant expansions near the entire homoclinic orbits can also be obtained if the homoclinic tangency is simple [Gonchenko *et al.*, 2008].

The simplicity of a homoclinic tangency in the case of the diffeomorphism $f$ can be defined as follows [Gonchenko *et al.*, 1993c, 2008]. Choose any two homoclinic points $p$ and $q$ in $U(O)$ such that $p \in W^u_{\text{loc}}(O)$, $q \in W^s_{\text{loc}}(O)$, and $f^s(p) = q$ for some integer $s$. Define the so-called global map $T_1$ that is constructed along orbits of $f$ and acts from a small neighborhood $V(p)$ of the point $p$ to a small neighborhood of the point $q$. Let $q = f^s(p)$ for some natural $s$, then we can write $T_1 = f^s|_{V(p)}$.[5] It is then required that

- the plane $DT_1(N^{cu}(p))$ intersects transversally with $N^{ss}(q)$.

Note that the curve $T_1(W^u_{\text{loc}}(O))$ touches the two-dimensional plane $W^s_{\text{loc}}(O)$ along the vector $\ell_{\tan}$, which, in turn, has a nonzero angle with the line $N^{ss}(q)$, see Fig. 3.

If an attractor of a three-dimensional smooth map is pseudohyperbolic, then it can contain only simple homoclinic tangencies.[6] For any small smooth perturbations, pseudohyperbolicity is preserved [Turaev & Shilnikov, 1998, 2008]. However, if these perturbations are not too small, it can be broken. In this case, the destruction itself can be caused by the appearance of such homoclinic tangencies as in Fig. 2 (for example, the fixed point, initially with $\sigma > 1$, in the process of evolution can become a saddle point with $\sigma < 1$, or, otherwise, a saddle-focus). A more delicate mechanism for the destruction of pseudohyperbolicity is associated with the emergence of the so-called nonsimple

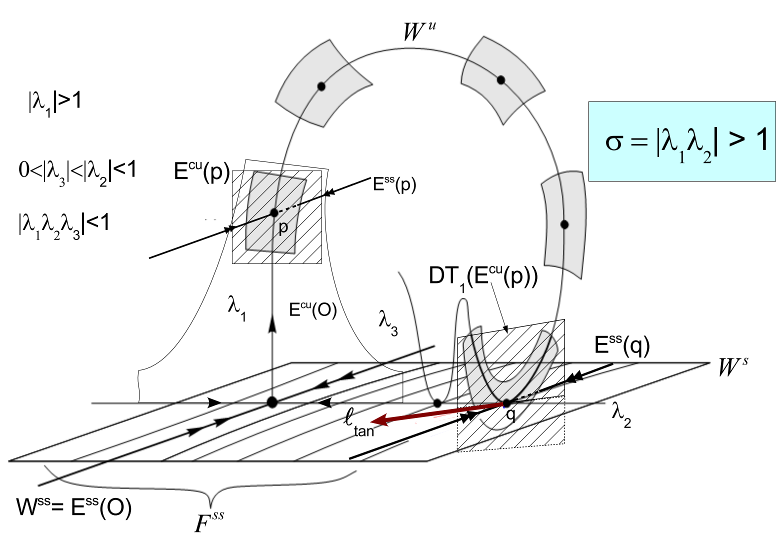



Fig. 3. The definition of simple homoclinic tangency.

[5]Note that the local invariant manifolds $W^u_{\text{loc}}(O)$ and $W^s_{\text{loc}}(O)$ can always be straightened by introducing in $U(O)$ such $C^r$-coordinates $(x, y, z)$ in which $W^u_{\text{loc}}(O) = \{x = 0, y = 0\}$ and $W^s_{\text{loc}}(O) = \{z = 0\}$ [Shilnikov *et al.*, 1998; Gonchenko *et al.*, 2008].

[6]Moreover, except for quadratic tangencies, there can exist homoclinic tangencies of arbitrarily large orders [Gonchenko *et al.*, 1993a, 2001], but they all should be simple (in the sense that, at any homoclinic point $p$, the subspaces $N_2(p)$ and $N_1(p)$ intersect transversally, see more detail in [Gonchenko *et al.*, 2018]).

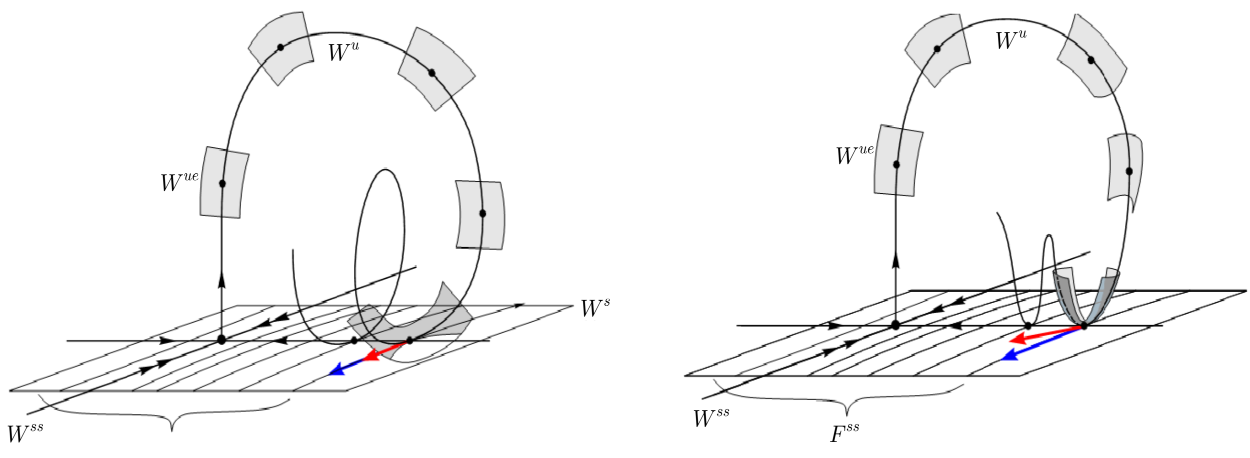



Fig. 4. Two types of nonsimple homoclinic tangencies: (a) when the surface $T_1(N_2(p))$ intersects transversally with $W^s_{\text{loc}}(O)$ but the vector $\ell_{\tan}$ belongs to $N_1(q)$ and (b) when the surface $T_1(N_2(p))$ touches $W^s_{\text{loc}}(O)$.

homoclinic tangencies, examples of which are shown in Fig. 4. In this case, as was established in [Tatjer, 2001; Gonchenko *et al.*, 2007a, 2014b], stable periodic orbits, closed invariant curves and even nontrivial attracting invariant sets, e.g. small Lorenz-like attractors, can be born at bifurcations of such homoclinic tangencies.

From this fact we can draw an important conclusion for the theory of strange attractors of three-dimensional smooth maps: if such an attractor is genuine, then it should be either hyperbolic or pseudohyperbolic. As for hyperbolic attractors, their mathematical theory is rather well developed. We note, however, that, for a long time, the hyperbolic attractors had only purely mathematical interest, as no bifurcation mechanisms of their appearance in applications were known. The situation changed after the papers by Shilnikov and Turaev [1995, 1997], where it was shown that hyperbolic attractors (e.g. Smale–Williams and Anosov attractors) can be born due to the loss of stability of a stable periodic orbit under bifurcations of "blue sky catastrophe" type. From the papers [Kuznetsov, 2005; Kuznetsov & Seleznev, 2006; Kuznetsov & Pikovsky, 2007; Kuznetsov, 2007] it became known that such attractors are also found in physical models. Note that there are powerful analytical and computer methods for proving hyperbolicity of attractors. Nowadays, similar methods are developed for detecting and verifying of pseudohyperbolic attractors.

Some such methods have been developed in our recent papers [Gonchenko *et al.*, 2012, 2014a; Gonchenko & Gonchenko, 2016; Gonchenko *et al.*, 2018]. In particular, in [Gonchenko *et al.*, 2012, 2014a] an effective qualitative method of new phenomenological bifurcation scenarios of the appearance of pseudohyperbolic attractor was proposed. Here also it is worth noting the paper [Gonchenko & Gonchenko, 2016] where a new search method of saddle charts and modified Lyapunov diagrams was developed. The above methods help to find attractors which can be pseudohyperbolic. Recently, in the paper [Gonchenko *et al.*, 2018] a quite effective method for verifying pseudohyperbolicity of found attractors was proposed. All the above methods give a powerful toolkit both to search attractors which are good candidates to be genuine attractors and to verify their pseudohyperbolicity. We give a review on these methods in the following section.

## 3. Methods for Searching Strange Attractors and Verification of Their Pseudohyperbolicity

In this section, we give a review of quite new qualitative and numeric methods which help to search for strange attractors and verify their pseudohyperbolicity. First, we observe, in one-parameter families of three-dimensional maps, phenomenological scenarios leading to the appearance of such pseudohypebolic attractors as discrete Lorenz and figure-eight attractors. Second, we describe rather effective numerical search methods such as the methods of saddle chart and modified Lyapunov diagrams. These methods help to find strange attractors that can be considered as good candidates of pseudohyperbolic ones. Finally, we present a new numerical

LMP-method for verifications of pseudohyperbolicity of found (using, for example, the above methods) strange attractors.

### 3.1. *On phenomenological scenarios leading to the emergence of pseudohyperbolic attractors in three-dimensional maps*

In this section, we discuss questions of qualitative study of strange attractors of three-dimensional maps. Moreover, the main attention is focused on those attractors that can be pseudohyperbolic. Evidently, the spectrum $\Lambda_1, \Lambda_2, \Lambda_3$ of Lyapunov exponents for orbits in such attractors must satisfy the necessary condition (2), see also Remark 1.2. Besides, we restrict ourselves to the study of the so-called *homoclinic attractors*, i.e. those that contain only one fixed point together with its unstable manifold.

Here under the attractor of a map $f$, following Ruelle [1981], we mean *closed*, *invariant*, *stable*, *and chain-transitive set* $\mathcal{A}$. By stability, we mean the standard asymptotic one, which indicates that the attractor lies inside some absorbing domain $\mathcal{D}$, all points of which tend to $\mathcal{A}$ under positive iterations of the map $f$. Recall that the chain-transitivity, see e.g. [Anosov & Bronshtein, 1985; Turaev & Shilnikov, 1998], means that any two points on the attractor can be connected by an $\varepsilon$-orbit for any $\varepsilon > 0$. The latter means that, for any two points $a, b \in \mathcal{A}$ and any $\varepsilon > 0$, in $\mathcal{A}$ there exist points $a = x_1, x_2, \ldots, x_{N-1}, x_N = b$, where $N = N(\varepsilon)$ is such that $x_i \in \mathcal{A}$ and $\text{dist}(x_{i+1}, f(x_i)) < \varepsilon$, $i = 1, \ldots, N-1$. The sequence of points $\{x_i\}$ is called an $\varepsilon$-orbit of the point $x_0$ of length $N$, and the point $b$ is said to be $\varepsilon$-accessible from the point $a$. Then we define the homoclinic attractor $\mathcal{A}$ with a fixed (periodic) point $O$ as a closed, invariant set consisting of all points $\varepsilon$-accessible from the point $O$ for any $\varepsilon > 0$, i.e. $\mathcal{A}$ is the prolongation of $O$. On the concept of prolongation in dynamical systems, see more details e.g. in [Anosov & Bronshtein, 1985].

In this case, geometrically the attractor $\mathcal{A}$, as a set in $R^3$, can be considered as the closure of the unstable manifold of its fixed point $O$.[7] From this quite obvious observation, it is concluded that geometrical and dynamical properties of the homoclinic attractor depend on its homoclinic structure, i.e. on a set of intersections of the stable and unstable invariant manifolds of the point $O$ belonging to this attractor. In this connection, we proposed in [Gonchenko *et al.*, 2012] rather simple phenomenological scenarios for the emergence of discrete homoclinic attractors of certain types in one-parameter families of maps. Two such scenarios are represented schematically in Fig. 5.

We note two main features of these scenarios. The first one is that, when the parameter changes, the stable fixed point $O$ loses stability under the supercritical period-doubling bifurcation. Immediately after this bifurcation, the point $O$ becomes a saddle with a one-dimensional unstable manifold, and in its neighborhood a stable cycle $(p_1, p_2)$ of period two is born (i.e. $f(p_1) = p_2$ and $f(p_2) = p_1$), which becomes an attractor now. Besides, the saddle point $O$ will have multipliers $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_1 < -1$, $|\lambda_{2,3}| < 1$ and $\lambda_2\lambda_3 < 0$ (since $f$ is orientable). We assume that, at further change of parameter, the point $O$ no longer undergoes bifurcations and the cycle $(p_1, p_2)$ loses its stability. How this happens — does not yet matter, but what is important — and this is the second main feature of these scenarios — there is a global bifurcation associated with the creation of homoclinic intersections of the one-dimensional unstable $W^u$ and two-dimensional stable $W^s$ invariant manifolds of $O$. A configuration of these manifolds will be similar to what we see in Figs. 5(c) and 5(d).

To explain how two such different configurations are created, suppose, for definiteness, that $\lambda_2 > 0$ and $\lambda_3 < 0$ (here also $\lambda_1 < -1$). Then $W^u$ is divided by a point $O$ into two connected components, separatrices $W^{u+}$ and $W^{u-}$, which are invariant for $f^2$ and such that $f(W^{u+}) = W^{u-}$ and $f(W^{u-}) = W^{u+}$. Then, if $W^{u+}$ intersects $W^s_{\text{loc}}(O)$ at a point $h_1$, then $W^{u-}$ should intersect $W^s_{\text{loc}}(O)$ at the point $h_2 = f(h_1)$. The map $f$ in the restriction to $W^s_{\text{loc}}(O)$ is very simple: it has a stable fixed point $O$ of the type of a nonorientable node, since $\lambda_2\lambda_3 < 0$.

In the case $|\lambda_2| > |\lambda_3|$, as in Fig. 5(c), in $W^s_{\text{loc}}(O)$ there exists a strongly stable invariant

[7]This is true for the genuine attractor. However, we cannot know this in advance. Nevertheless, such definition agrees well with the computer study of attractors, when we cannot see very small stable invariant subsets inside the attractor (e.g. stable periodic orbits of very large periods). Sometimes (e.g. for quasiattractors), these subsets can be visible and this corresponds to the well-known phenomenon of appearance of "windows of stability".

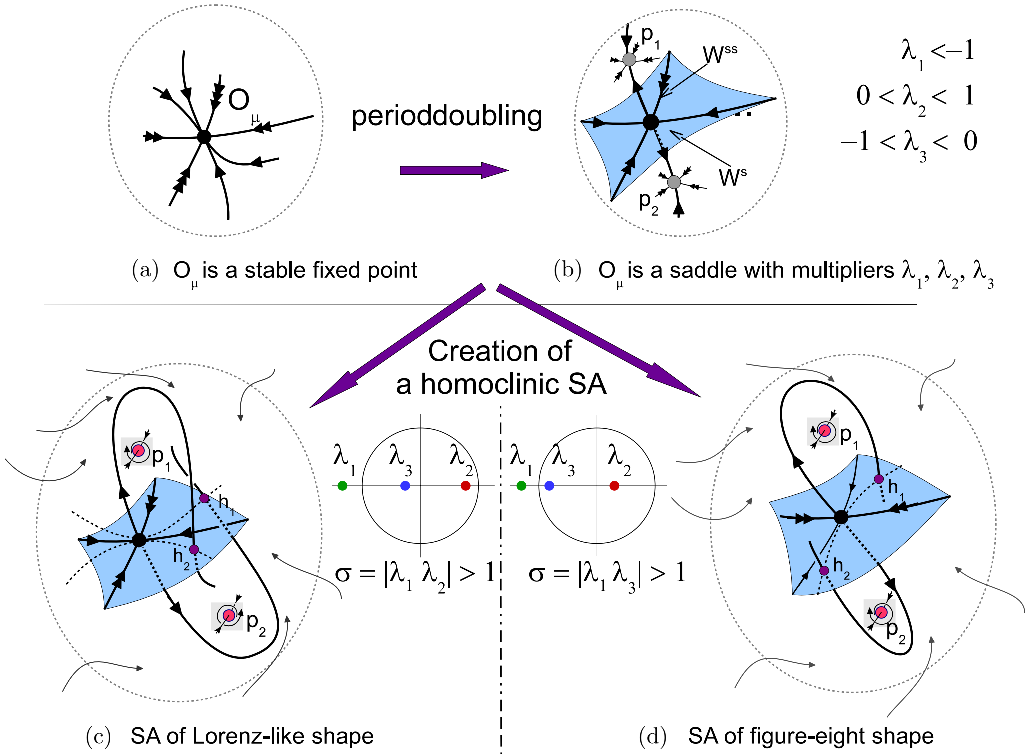



Fig. 5. Schematic pictures of two phenomenological scenarios for the appearance of discrete homoclinic attractors: the case of Lorenz-like attractor — the path (a) → (b) → (c); the case of figure-eight attractor — the path (a) → (b) → (d).

manifold $W^{ss}$ that is an $f$-invariant curve tangent at the point $O$ to the eigendirection corresponding to the negative multiplier $\lambda_3$. The curve $W^{ss}$ splits the disk $W^s_{\text{loc}}(O)$ into two connected components $W^s_1$ and $W^s_2$. Since $\lambda_2 > 0$ (and $|\lambda_2| > |\lambda_3|$), each of these components is invariant under $f$, i.e. points from $W^s_1$ cannot get into $W^s_2$ under iterations of $f$, and vice versa. There is also a continuous family of smooth invariant curves on $W^s_{\text{loc}}$ that all enter the point $O$ touching the eigendirection corresponding to (positive) multiplier $\lambda_2$. Let the point $h_1$ belong to one of these curves, say $l_1$. Then the curve $l_2 = f(l_1)$ is also an invariant curve from this family, and $h_2 \in l_2$. The curves $l_1$ and $l_2$ lie exactly in one component, either in $W^s_1$ or in $W^s_2$, and enter $O$, forming a "zero-angle wedge" configuration. Correspondingly, the configuration of unstable separatrices of the point $O$, see Fig. 5(c), will resemble that typical for the unstable separatrices of the equilibrium of Lorenz attractor. Therefore, the attractor arising here was named in [Gonchenko *et al.*, 2013b] as "discrete Lorenz attractor".

Similar simple geometric arguments for the case $|\lambda_2| < |\lambda_3|$, as in Fig. 5(d), show that here the configuration of unstable separatrices of the point $O$ will be completely different. It is more like the configuration of separatrices in the attractor of the Poincaré map of a periodically perturbed two-dimensional system with a homoclinic figure-eight of a saddle equilibrium [Gonchenko *et al.*, 2013c]. Therefore, an attractor arising in this case was named "discrete figure-eight attractor" (see Figs. 1 and 8 which give an idea of the typical form of such an attractor).

We note that, for both such types of attractors, the condition $\sigma > 1$ (here $\sigma = |\lambda_1\lambda_2|$ in the "Lorenz" and $\sigma = |\lambda_1\lambda_3|$ in the "figure-eight" case, respectively) is very important, as it is necessary for the attractor under consideration to be pseudohyperbolic. Otherwise, such attractors are quasiattractors of Lorenz or figure-eight type; or — another possibility — from a homoclinic configuration with $\sigma < 1$, a large attractor enclosing a stable closed invariant curve (torus) can be created which, can be further broken giving rise to strange attractor of completely different nature (for example, "torus-chaos"). Both these possibilities are well observed in computer experiments, see, for example, [Gonchenko *et al.*, 2012].

These obvious observations tell us that, in the cases of saddle fixed points of other types, one can also expect the existence of homoclinic attractors whose configuration will depend essentially on multipliers of these points. In particular, in the case when there are complex conjugate multipliers, we can also expect the existence of discrete attractors of the spiral type, see e.g. [Gonchenko *et al.*, 2012, 2014a].

*Remark 3.1.* Note that our discrete Lorenz and figure-eight attractors differ substantially from their analogues obtained in Poincaré maps from periodically perturbed three-dimensional flows. Thus, for a small periodic perturbation of the system with the Lorenz attractor, we obtain a pseudohyperbolic attractor [Turaev & Shilnikov, 2008] which has a saddle fixed point with all positive multipliers, and fixed points lie in the "holes" of the attractor. In the case of the Lorenz discrete attractor, the fixed point has two negative multipliers, and a period-2 orbit lies in "holes". As the discrete figure-eight attractors, it seems that they have no flow analogues at all. This is due to the fact that if the corresponding system would have a homoclinic figure-eight of saddle, then either this figure-eight is stable (attractor) and then the attractor obtained will be with $\sigma < 1$, or, if $\sigma > 1$, the homoclinic figure-eight is not an attractor. This allows us to say that both the Lorenz discrete attractor and the discrete figure-eight attractor are new.

The problem of *study and classification of homoclinic attractors for three-dimensional diffeomorphisms* was first explicitly formulated in the paper [Gonchenko *et al.*, 2012], although the first results on this subject were obtained in the paper [Gonchenko *et al.*, 2005], in which discrete Lorenz attractors were found in three-dimensional Hénon maps. We note that a possibility of the birth of such attractors at local bifurcations of triply degenerate fixed points (e.g. having multipliers $(-1, -1, +1)$) was studied in the paper [Shilnikov *et al.*, 1993]. Since the three-dimensional Hénon map (1) contains three parameters, such a point exists in it, and moreover, as shown in [Gonchenko *et al.*, 2005], conditions from [Shilnikov *et al.*, 1993] are fulfilled in this case. Thus, the main idea of our work [Gonchenko *et al.*, 2005] was to apply knowledge about the properties of degenerate local bifurcations to a specific situation. Obviously, this approach can also be used in the study of various models containing at least three parameters.

In [Gonchenko *et al.*, 2012] another idea was proposed based on feasibility of phenomenological scenarios of the appearance of strange homoclinic attractors in one-parameter families of three-dimensional maps. Such scenarios, as those presented in Fig. 5, look quite realizable for concrete models and, moreover, they allow quite simple numeric analysis — here, for example, one does not need to know all subtleties of global bifurcations leading to the creation of homoclinic structures, but it is sufficient only to calculate/construct some basic simple characteristics (phase portrait, multipliers of the fixed point, Lyapunov exponents, etc.).

*Remark 3.2.* The idea of studying strange attractors by means of phenomenological scenarios involving two main bifurcation stages — the loss of stability of a simple attractor and the appearance of a homoclinic attractor — was first proposed by Shilnikov in the paper [Shilnikov, 1986], where such a scenario was represented to explain the phenomenon of emergence of spiral chaos in the case of multidimensional flows. We will discuss this scenario and its generalizations in Part II of the paper.

### 3.2. *On numerical methods for searching pseudohyperbolic attractors*

The fact that the configuration of discrete homoclinic attractors depends essentially on multipliers of their fixed points was used in the paper [Gonchenko & Gonchenko, 2016] for classification of such attractors in the case of orientable three-dimensional maps. If we restrict ourselves only to pseudohyperbolic homoclinic attractors, then such a classification problem turns out to be quite solvable for distinguishing attractors by types of their homoclinic structures. In this case, as shown in [Gonchenko & Gonchenko, 2016], five different types of such attractors are possible. All of them relate to the case when the fixed point is a saddle (all multipliers are real) with the one-dimensional unstable invariant manifold. Two of these types, the discrete Lorenz attractors and discrete figure-eight attractors, can be observed in the case when the unstable multiplier $\lambda_1$ is negative, i.e. $\lambda_1 < -1$; and three other types of discrete attractors (the so-called "double figure-eight", "super figure-eight" and "super Lorenz" attractors) relate to the case when $\lambda_1 > 1$, see [Gonchenko & Gonchenko, 2016].

In order to find such attractors in specific models, some fairly effective methods were proposed in

[Gonchenko & Gonchenko, 2016]. One of them is the so-called "method of saddle charts". We illustrate the essence of this method in the example of a *three-dimensional generalized Hénon map* of the form

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = Bx + Az + Cy + f(y, z), \tag{3}$$

where the nonlinearity $f$ depends only on coordinates $y$ and $z$ and, besides, $f(0,0) = 0, f'_y(0,0) = f'_z(0,0) = 0$. The map (3) depends on three parameters $A, B$, and $C$ and has constant Jacobian equal to $B$. We will assume that $0 < B < 1$, i.e. the map is orientable and volume contracting. Obviously, any map of form $\bar{x} = y, \bar{y} = z, \bar{z} = Bx + g(y, z)$ having a fixed point (for example, map (1) for $(1 + B - M_2)^2 + 4M_1 > 0$) can be written in the form (3), if to move this point into the origin.

The point $O(0,0,0)$ is fixed for map (3) and the characteristic equation of (3) at this point has the form

$$\chi(\lambda) \equiv \lambda^3 - A\lambda^2 - C\lambda - B = 0. \tag{4}$$

Thus, the multipliers of point $O$ are functions of only the parameters $A, B$ and $C$ and do not depend on the nonlinearities $f(y, z)$. Then we can split the space of parameters $A$, $B$ and $C$ into domains corresponding to various types of location of multipliers of the point $O$ with respect to the unit circle. We also distinguish the domains corresponding to $\sigma > 1$ and $\sigma < 1$ in the cases when the unstable manifold of $O$ is one-dimensional. Such a partition of the $(A, C)$-parameter plane for fixed $B$ is called the *saddle chart* [Gonchenko & Gonchenko, 2016].[8] An example of such a saddle chart, with $B = 0.5$, is shown in Fig. 6.

The domain IV, the so-called "stability triangle" (the domain $\{C > B^2 - 1 - BA\} \cap \{C < A + B + 1\} \cap \{C < 1 - BA\}$), corresponds to the case when the fixed point $O$ is asymptotically stable.

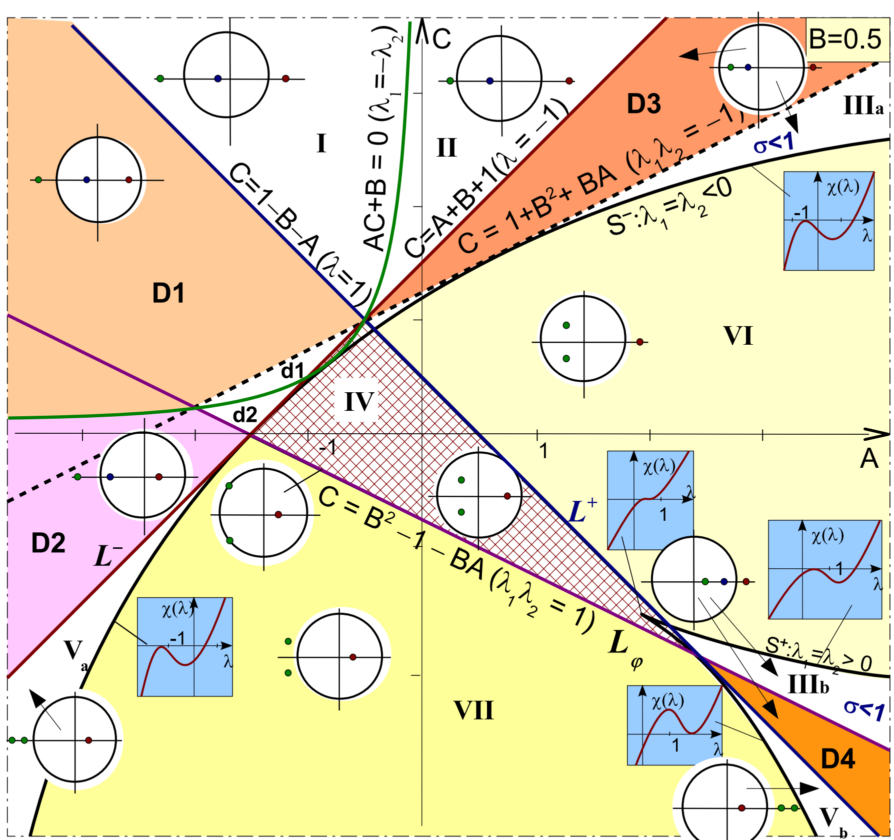



Fig. 6. Examples of the saddle chart for map (3) with $B = 0.5$.

[8] In the case of three-dimensional flows, a similar "saddle chart" for equilibrium states was proposed in [Shilnikov *et al.*, 2001] in the form of a table, see there Appendix C.2.

For all other values of $A$ and $C$ (except for the bifurcation curves), the point $O$ is of saddle type — it has multipliers both inside and outside the unit circle. Depending on the location of multipliers we select in the chart several domains shown in Fig. 6. The boundaries of the domains consist of seven main curves. First, there are three bifurcation curves, where the point $O$ has multipliers on the unit circle:

(a) the curve $L_+ : C = 1 - B - A$ (when $\lambda = +1$);
(b) the curve $L_- : C = 1 + B + A$ (when $\lambda = -1$);
(c) the curve $L_\varphi : C = B^2 - 1 - BA$ at $-2 < A - B < 2$ (when $\lambda_{1,2} = e^{\pm i\varphi}$).

We note that the curve $C = B^2 - 1 - BA$ belongs entirely to the boundaries of the domains, but for $|A - B| > 2$ it is not a bifurcation curve: here the point $O$ has multipliers $(B, -|\lambda|, -|\lambda|^{-1})$ at $A - B < -2$ and $(B, |\lambda|, |\lambda|^{-1})$ at $A - B > 2$. Besides, the saddle chart contains four additional curves:

(a) "the resonant curve", $AC+B = 0$, $A < 0$ (when $\lambda_1 = -\lambda_2$);
(b) the curve "$\sigma = 1$", $C = 1 + B^2 + AB$ (when $\lambda_1\lambda_2 = -1$);

and two curves of "double roots"

(a) $S^-$ (when $\lambda_1 = \lambda_2 < 0$),
(b) $S^+$ (when $\lambda_1 = \lambda_2 > 0$).

The latter two curves separate domains where $O$ has a type of "node" and "focus" as well as "saddle" and "saddle-focus". These curves have the following equations

$$S^\pm : (\lambda_\pm)^3 - A(\lambda_\pm)^2 - C\lambda_\pm - B = 0,$$

where

$$\lambda_\pm = \frac{A \pm \sqrt{A^2 + 3C}}{3}$$

at $A^2 + 3C > 0$ (i.e. $\lambda_\pm$ are the roots of equation $3\lambda^2 - 2A\lambda - C = 0$).

We remark especially four regions D1, D2, D3 and D4 of the saddle chart, see Fig. 6, where the point $O$ has real multipliers $\lambda_1, \lambda_2, \lambda_3$ such that $|\lambda_1| > 1, |\lambda_{2,3}| < 1$, and the saddle value $\sigma = |\lambda_1| \cdot \max\{|\lambda_2|, |\lambda_3|\}$ is greater than 1. As we propose, only for values $A$ and $C$ from these domains, homoclinic attractors under consideration (containing the point $O$) can be pseudohyperbolic. In other domains, except for the domain IV (the stability triangle), the point $O$ is either a saddle-focus, or a saddle with two-dimensional unstable manifold, or has $\sigma < 1$. As we suppose, if the map has a homoclinic attractor for the values of parameters from one of these last regions, then it is, in our opinion, a quasiattractor, see Remark 1.2.

Using saddle charts in numerical experiments is very convenient in combining with the Lyapunov diagram method. However, here we also modify the latter method. As a standard, the Lyapunov diagram is a chart (in the $(A, C)$-parameter plane, when $B$ is fixed) consisting of colored areas corresponding to domains of parameters with different spectra of the Lyapunov exponents $\Lambda_1 > \Lambda_2 > \Lambda_3$. We use, in particular, the green color (also denoted by the number "1" in black and white drawings) — for stable periodic regimes ($\Lambda_1 < 0$); light blue color (number "2") — for quasiperiodic regimes ($\Lambda_1 = 0$); yellow color (number "3") when $\Lambda_1 > 0, \Lambda_2 < 0$, red color (number "4") when $\Lambda_1 > 0, \Lambda_2 \sim 0$, and blue color (number "5") when $\Lambda_1 > \Lambda_2 > 0$ — for strange attractors.[9] To these five colors, we added one more — dark gray (number "6") to indicate *regions with homoclinic attractors*, when the numerically obtained points on the attractor approach the point $O$ to a very small distance (less than $10^{-4}$ during at least $10^6$ iterations of the map).

As was noted in the Introduction, the conditions (2) for the spectrum $\Lambda_1, \Lambda_2, \Lambda_3$ of numerically obtained Lyapunov exponents should be considered as one of necessary conditions for pseudohyperbolicity of strange attractors in three-dimensional maps. Moreover, this condition is evidently fulfilled for three-dimensional flows: here $\Lambda_2 = 0$ and, hence, $\Lambda_1 > 0$ automatically implies $\Lambda_1 + \Lambda_2 > 0$; the inequality $\Lambda_1 + \Lambda_2 + \Lambda_3 < 0$ follows from volume contracting properties of a flow near attractor. However, it is well known that not all chaotic attractors for three-dimensional flows, and more so

[9]The red domains (where $\Lambda_1 > 0, \Lambda_2 \sim 0$) were especially highlighted in [Gonchenko *et al.*, 2005] — these are such domains where the value of $\Lambda_2$ either always fluctuate very close to zero, or differ from zero by an amount (of the order of $10^{-5}$ or $10^{-6}$), comparable to the accuracy of calculating the exponents. Surprisingly, such domains turned out to be quite large, and this phenomenon (apparently related to the fact that the mapping on the attractor turned out to be very close to the time-discretization of some flow, for example, with the Lorenz attractor) was discussed in [Gonchenko *et al.*, 2005].

for three-dimensional maps, are pseudohyperbolic. Thus, we strongly need some additional numerical methods that give more confidence that the attractor is genuine. Such methods exist, for example, as the Tucker methods of rigorous numerics [Tucker, 1999] based on interval arithmetic. However, the Tucker method is very difficult and too time-consuming for using it in our simple and standard numerics directed more for searching attractors than for their delicate studying. Instead, we use a sufficiently simple but quite effective "light method" proposed in [Gonchenko *et al.*, 2018] for verifying pseudohyperbolicity (LMP-method) of strange attractors of three-dimensional maps and flows. It can be easily modified for $n$-dimensional maps and flows at which the sum of the first $(n-1)$ Lyapunov exponents is positive, whereas, the sum of all $n$ exponents is negative. Further, for simplicity, we describe this method for the case of three-dimensional flows and maps and test this method on the well-known Lorenz attractor.

### 3.3. *LMP-method for verification of pseudohyperbolicity of attractors*

The essence of the LMP-method consists of the fact that we pay more attention for checking sufficient conditions for pseudohyperbolicity, see Definitions 2.1 and 2.2 and the Appendix. In the case of three-dimensional maps (as well as flows), we consider conditions (2) to be necessary and assume that they hold. Then, the sufficient condition is related to the existence of, at every point $x$ near the attractor, two transversal linear subspaces $E^{ss}(x)$ and $E^{cu}(x)$ such that (see Definition 2.2)

(i) $\dim E^{ss} = 1, \dim E^{cu} = 2$;
(ii) $E^{ss}(x)$ and $E^{cu}(x)$ depend continuously on $x$;
(iii) $E^{ss}(x)$ and $E^{cu}(x)$ are invariant with respect to the differential $DT$ of the map $T$, i.e.

$$Df(E^{ss}(x)) = E^{ss}(f(x)),$$

$$Df(E^{cu}(x)) = E^{cu}(f(x));$$

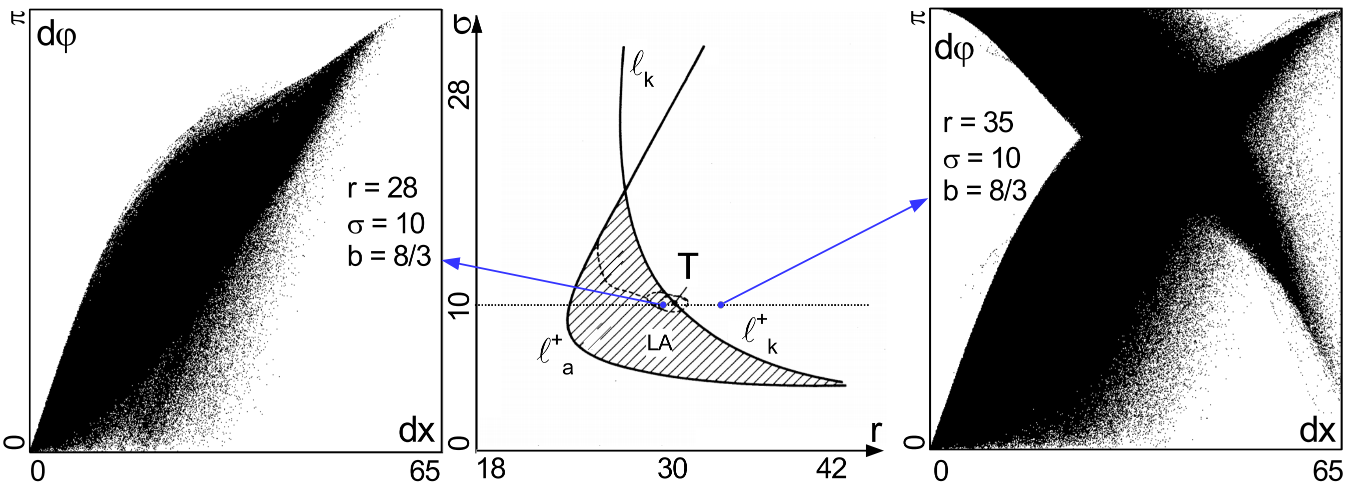
(iv) the map $T$ in the restriction to $E^{ss}$ is uniformly contracting, and in the restriction to $E^{cu}$ extends exponentially by two-dimensional volumes, and if in $E^{cu}$ there is a contraction, then it is uniformly weaker than the contraction in $E^{ss}$.

We note that the strongly contracting space $E^{ss}(x)$ is one-dimensional (for attractors under consideration) and it depends continuously on the point $x$. This means that angles $d\varphi$ between any vectors $E^{ss}(x)$ and $E^{cu}(y)$ should be close for nearby $x$ and $y$ (theoretically, $d\varphi \to 0$ as $x \to y$). In fact, the LMP-method allows us to calculate these angles and, thus, to verify the continuity of the field $E^{ss}$ of strong contacting directions at points of the attractor. The process of calculations consists of two stages. The first stage is standard: we calculate the spectrum of Lyapunov exponents $\Lambda_1, \Lambda_2, \Lambda_3$ (if conditions (2) are not valid, we can stop calculations) and, in parallel, we store an array of data $\mathcal{N} = \{x_n\}$, where $x_{n+1} = f(x_n)$ and $n = 1, \ldots, k$, containing information about points $x_n$ on the attractor. The second step is not quite standard: we calculate the maximal Lyapunov exponent for backward iterations of the map using essentially the information obtained in the first stage. In particular, our backward iterations are forcibly attached to those points of the attractor that were obtained in the first stage. Evidently, if we take any point on the attractor, then its backward iterations, sooner or later, depart far from the attractor and we can lose any information on the attractor.

Note that the maximal Lyapunov exponent for backward iterations is equal with a minus sign to the minimal Lyapunov exponent $\Lambda_3$, and during these calculations we find vectors $E^{ss}(x_n)$. As the final result of calculations, we construct the LMP-graph on the $(dx, d\varphi)$-coordinate plane, where $dx$ is the distance between two points $x$ and $y$ of the attractor and $d\varphi$ is the angle between vectors $E^{ss}(x)$ and $E^{ss}(y)$ (in fact, we construct the graph knowing points $x_i$ and $x_j$ and vectors $E^{ss}(x_i)$ and $E^{ss}(x_j)$ for all possible $i$ and $j$).

We note that if the attractor is pseudohyperbolic, which implies that the field $E^{ss}(x)$ is continuous, the LMP-graph has to intersect the $d\varphi$-axis only at the origin ($dx = 0, d\varphi = 0$) or, if $E^{ss}$ is nonorientable, at the points $d\varphi = 0$ and $d\varphi = \pi$. Thus, if the constructed LMP-graph satisfies this property, we can conclude that our attractor should be surely pseudohyperbolic. On the other hand, if the LMP-graph intersects the $d\varphi$-axis at other points or there is no visible gap between the points of graph and the $d\varphi$-axis, we say that the attractor is a quasiattractor.

In order to demonstrate that the LMP-method indeed works we consider, as a simple and




(a) (b) (c)

Fig. 7. An illustration of results obtained by the LMP-method: (a) LMP-graph for the classical Lorenz values $\sigma = 10, r = 28, b = 8/3$, (b) the domain $A$ of the $(\sigma, r)$-parameter plane (for $b = 8/3$) corresponding to the existence of the genuine Lorenz attractor — this figure is taken from [Bykov & Shilnikov, 1992] and (c) LMP-graph for values $\sigma = 10, r = 35, b = 8/3$ from the right of the domain $A$.

illustrative example, the well-known Lorenz model

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = x(r - z) - y, \\ \dot{z} = xy - bz \end{cases} \tag{5}$$

and verify the pseudohyperbolicity of some attractors in this model.

It is well-known from the paper [Tucker, 1999] that, for the classical values of parameters ($\sigma = 10$, $r = 28, b = 8/3$), "the Lorenz attractor exists", i.e. it satisfies conditions of the Afraimovich–Bykov–Shilnikov geometrical model [Afraimovich *et al.*, 1977; Afraimovich *et al.*, 1982]. In other words, it is pseudohyperbolic in terms of [Turaev & Shilnikov, 1998]. Numerics from [Bykov & Shilnikov, 1992] show that this property holds for some region $A$, see Fig. 7(b), of the $(\sigma, r)$-parameter plane (for $b = 8/3$). The right boundary $l_k^+$ of $A$ corresponds to vanishing of the so-called separatrix value for homoclinic loops (automatically this means violation of conditions from [Afraimovich *et al.*, 1977; Afraimovich *et al.*, 1982]) and, as result, the attractor becomes a quasiattractor in the domain to the right of $l_k^+$. In Fig. 7(a) are also presented the LMP-graphs for the classical values of the parameters $\sigma = 10, r = 28, b = 8/3$ and in Fig. 7(c) — for values $\sigma = 10, r = 35, b = 8/3$ from the right of $l_k^+$. Thus, our LMP-test confirms that the first attractor is genuine, indeed. In the second case we see that the LMP-graph intersects the axis $d\varphi$ at the points $d\varphi = 0$ and $d\varphi = \pi$. However, the field $E^{ss}(x)$ is orientable here as it should always be in the case of a flow. Thus, the second attractor is certainly a quasiattractor.

In Sec. 4, we consider some examples of three-dimensional maps such as generalized Hénon maps and nonholonomic models of Celtic stone and Chaplygin top and verify some attractors of these models for pseudohyperbolicity using LMP-method. In Sec. 5, we apply LMP-method to the attractors of four-dimensional flow system, which is a four-dimensional extension of the Lorenz model, and show that the attractors in this model can be pseudohyperbolic spiral attractors of Turaev–Shilnikov type.

## 4. On Examples of Strange Attractors in Various Three-Dimensional Diffeomorphisms

In this section, we discuss several examples of strange attractors in three-dimensional maps including three-dimensional Hénon maps and nonholonomic models of rigid body dynamics (we observe attractors in these models in three-dimensional Poincaré maps). We consider only such attractors for which necessary conditions for pseudohyperbolicity (expressed by numerically obtained

Lyapunov exponents) are satisfied. However, we show using the LMP-method that not all such attractors are genuine, some of them are, in fact, quasiattractors.

### 4.1. *Strange attractors in three-dimensional generalized Hénon maps*

We note that there are various methods to study chaotic dynamics in concrete models. One of the regular and reasonable approaches to this problem is related to the construction of diagrams of Lyapunov exponents. Namely in this way discrete Lorenz attractors were found in [Gonchenko *et al.*, 2005] for the three-dimensional Hénon map of form (1). Examples of such attractors are shown in Fig. 1. Now we can find such attractors, as they say, "purposefully", using our approach. To do this, we consider map (1) in the following "reduced to zero" form

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = Bx + Az + Cy - z^2, \qquad (6)$$

and take the saddle chart such as in Fig. 6(a) but constructed for the required fixed $B$, in our case for $B = 0.7$. Next, against the background

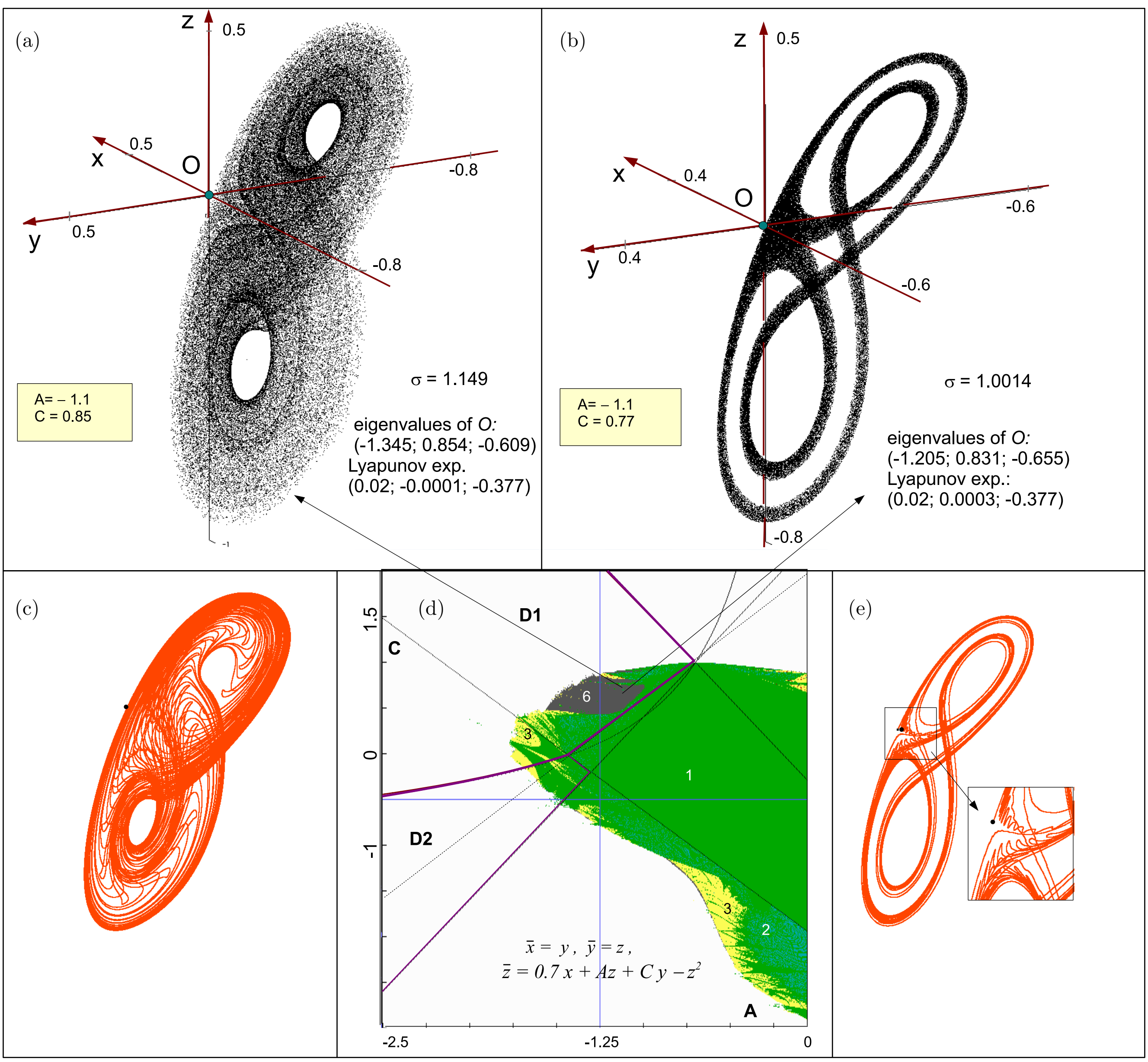



Fig. 8. Discrete Lorenz attractor for map (6) with $B = 0.7$.

of this chart, we numerically construct the modified diagram of Lyapunov exponents. As a result, we get such a picture as in Fig. 8(d), where, in particular, the region of "dark gray" chaos intersects the region D1. This suggests that, for the corresponding values of $A$ and $C$, a discrete Lorenz attractor can be observed. The numeric results are shown in Fig. 8(a) for $A = -1.1; C = 0.85$ and in Fig. 8(b) for $A = -1.11; C = 0.77$, where we point out also values of the multipliers of $O$, the saddle value $\sigma$ of $O$, and the values of Lyapunov exponents $\Lambda_1, \Lambda_2, \Lambda_3$. In both cases we have that $\sigma > 1$ and $\Lambda_1 > 0, \Lambda_1 + \Lambda_2 > 0$. In Figs. 8(c) and 8(e), we show also a behavior of one of the unstable separatrices of the point $O$ (the behavior of another separatrix is symmetric due to the unstable multiplier of $O$ being negative). We see that $W^u(O)$ has in both cases a homoclinic intersection with $W^s(O)$, although, in the first case, typical zigzags in $W^u$ are seen more clearly than for the second case.

In Fig. 9, we represented the LMP-graphs for the discrete Lorenz attractors of Figs. 8(a)(left) and 8(b)(right). The two upper figures show the LMP-graphs obtained by plotting every second iteration of the map, whereas, in the lower figures the graphs are plotted for each iteration. In principle, there is no difference between Figs. 9(a) and 9(c), both of them look quite "chaotic" and show that the field $E^{ss}$ in the case of attractors of Fig. 8(a) is not continuous. Thus, this attractor is certainly a quasiattractor. It is not the case for Figs. 9(b) and 9(d). The evident difference between them can be explained by the fact that the field $E^{ss}$ of strong contracting directions is nonorientable here (this is inherited by the fact that a strongly stable multiplier of the fixed point $O$ is negative), and $E^{ss}$

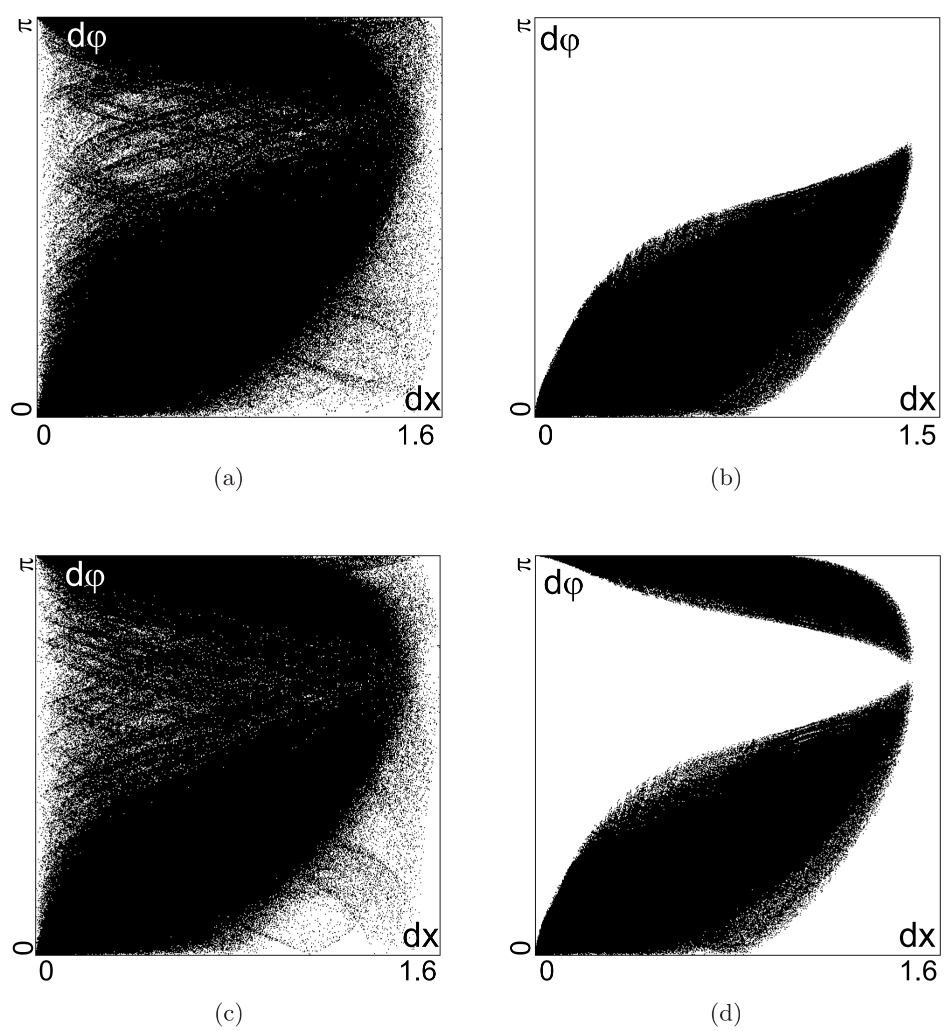



Fig. 9. LMP-graphs for attractors of Figs. 8(a)(left) and 8(b)(right).

becomes orientable if we consider every second iteration of the map. Thus, we can conclude from the LMP-graph in Fig. 9(b) that the attractor from Fig. 8(b) is pseudohyperbolic. We can also remark that the LMP-graph in Fig. 9(d), when every iteration is plotted, carries some information about how close is our pseudohyperbolic attractor to its breaking down, when it becomes a quasiattractor. This can be estimated from the distance between the two "whale" components of the graph. When these components intersect, the field $E^{ss}$ is immediately destroyed and, as a consequence, nonsimple homoclinic tangencies like in Fig. 4 appear.

Obviously, in the case of map (3), the saddle chart does not depend on the nonlinear terms $f(y,z)$. At the same time, the form of Lyapunov diagram on the $(A,C)$-parameter plane is determined only by these terms. With modern computers, the calculation of Lyapunov exponents does not take much time, especially in the case of three-dimensional maps, and the saddle chart for map (3) is constructed "instantly" on the "search stage". In addition, as our experience shows, when varying nonlinearities one can see a certain tendency in changing the location of the "dark-gray spot" (when the attractor is homoclinic). If desired, this spot

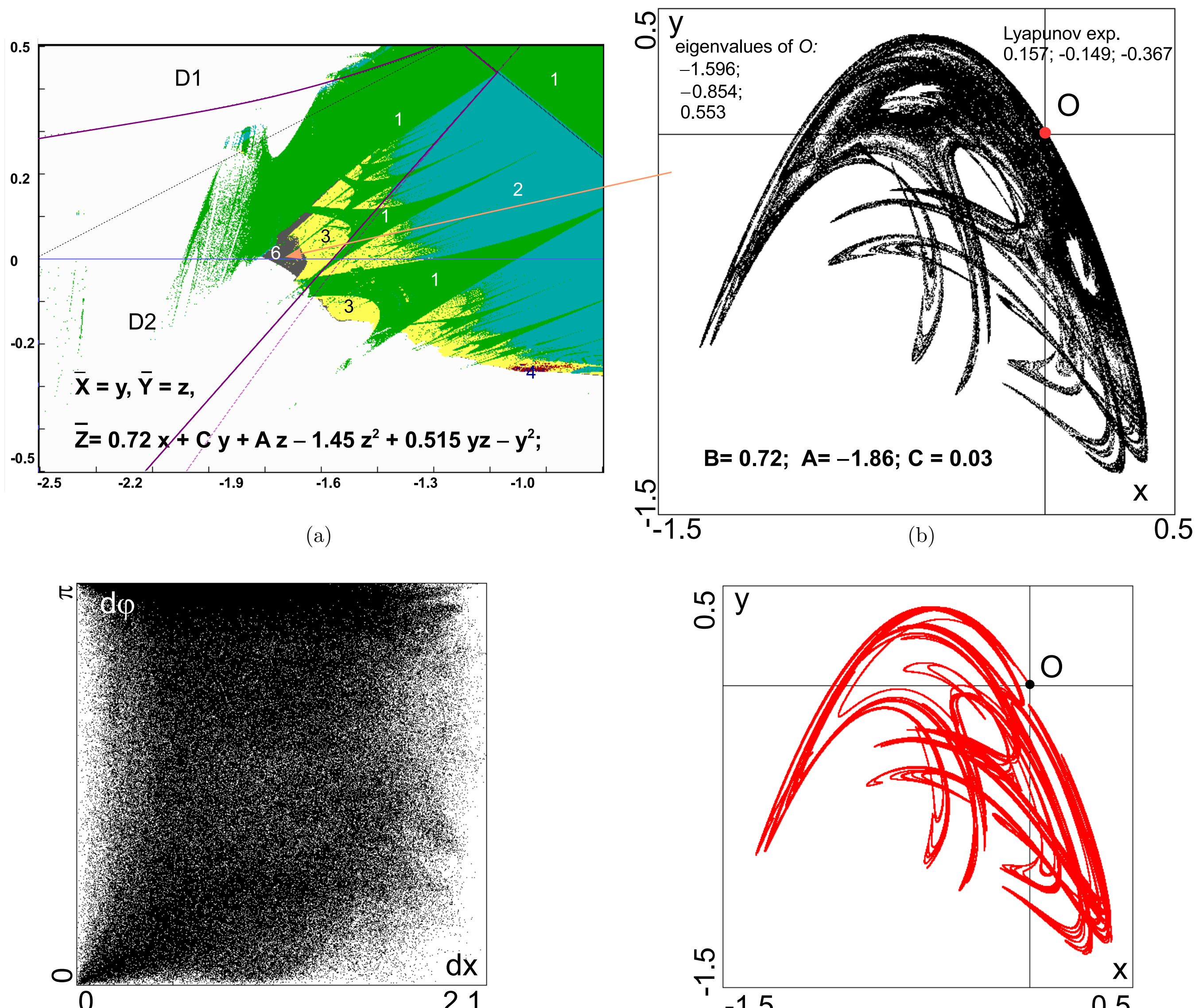



Fig. 10. An example of discrete figure-eight attractor in the corresponding generalized Hénon map: (a) a fragment of the saddle chart on a background of the Lyapunov diagram, (b) the projection of attractor on the $(x, y)$-plane, (c) the LMP-graph and (d) a behavior of one of the unstable separatrices of the fixed point $O$ is shown.

can be "driven" into any of the domains of the saddle chart (except for the "stability triangle"), and, accordingly, we can find the attractor of interest to us. By the same way, various homoclinic attractors of map (3) were found in [Gonchenko & Gonchenko, 2016]. Some of them (for the values of parameters $A$ and $C$ from the domains D1, D2, D3 and D4) were presented as candidates for pseudohyperbolic attractors (because the necessary conditions (2) are fulfilled for them).

In Fig. 10 illustrations are shown that relate to the discrete figure-eight attractor of map (3) with the nonlinearity $f(y,z) = -1.45z^2 + 0.515yz - y^2$ for values of parameters $B = 0.72; A = -1.86; C = 0.03$ belonging to the domain D2. Although the necessary conditions (2) are fulfilled for the attractor, it looks like a typical quasiattractor that its LMP-graph of Fig. 10(c) confirms. Unfortunately, we could not find good examples of discrete figure-eight attractors in the case of three-dimensional Hénon maps (however, we are sure that pseudohyperbolic attractors of such type exist there).

In Fig. 11 illustrations are shown that relate to the discrete double figure-eight attractor of map (3)

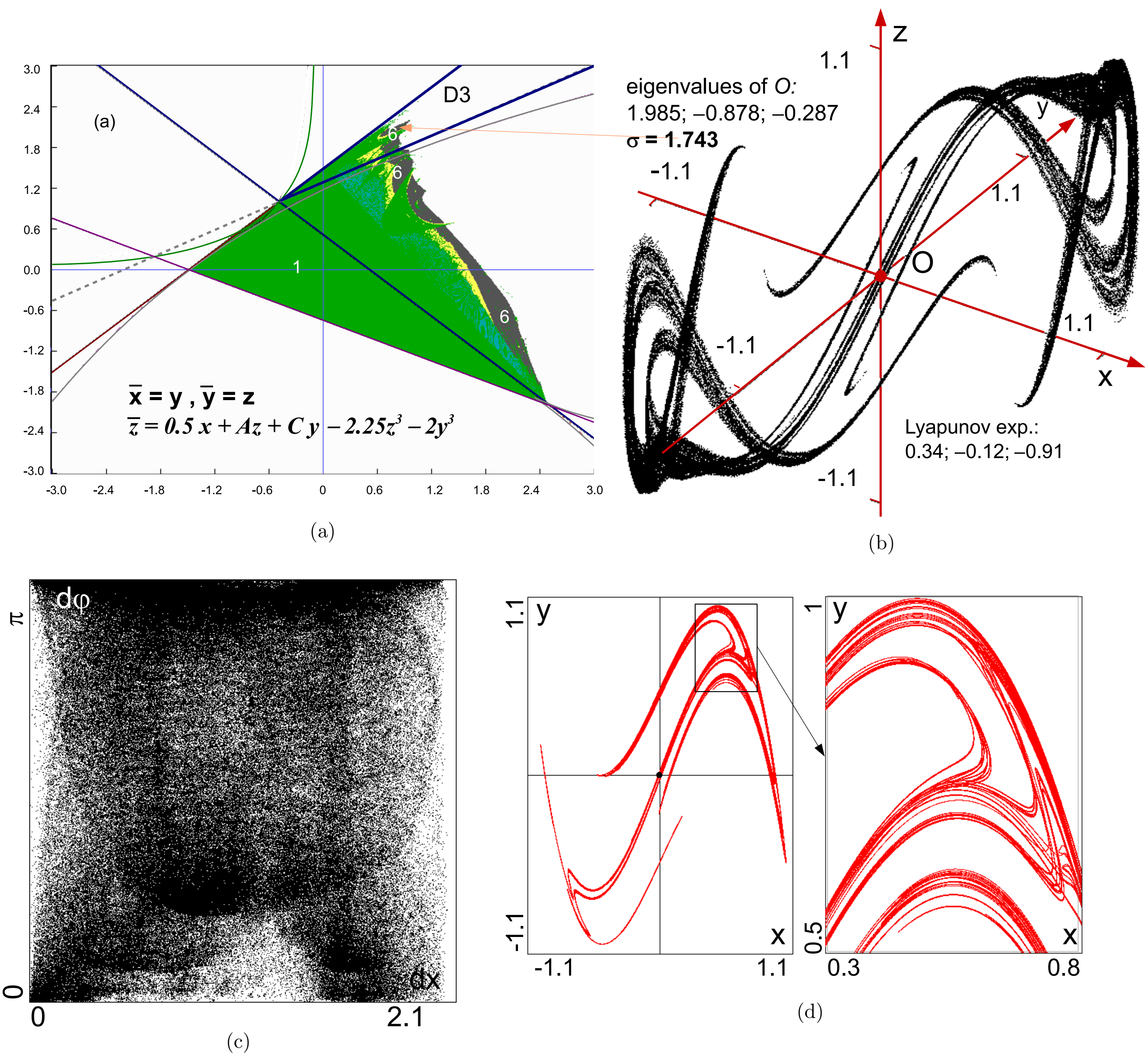



Fig. 11. An example of discrete double figure-eight attractor in the corresponding generalized Hénon map: (a) a fragment of the saddle chart on a background of the Lyapunov diagram, (b) the phase portrait of attractor, (c) the LMP-graph and (d) a behavior of the unstable separatrices of the fixed point $O$ is shown (left) and a magnification of some fragment (right).

with the cubic nonlinearity $f(y,z) = -2.25z^3 - 2y^3$ for values of parameters $B = 0.5; A = 0.82; C = 2.06$ belonging to the domain D3. The necessary conditions (2) are satisfied again, but LMP-graph of Fig. 11(c) shows that the attractor is certainly a quasiattractor.

In Fig. 12 illustrations related to one more homoclinic attractor are for the so-called discrete super figure-eight attractor. This attractor is observed in map (3) with the cubic nonlinearity $f(y,z) = y^3 - z^3$ for values of parameters $B = 0.05$; $A = 3.702$; $C = -2.749$ belonging to the domain D4. The necessary conditions (2) are satisfied again. Concerning the LMP-graph, see Fig. 12(c), we have here a very suspicious case, since near to the axis $dx = 0$ we see a thin strip-like area where there are practically no points of the graph. This favors the fact that the attractor is pseudohyperbolic, although it exists in very small domain of parameters.

### 4.2. *Pseudohyperbolic attractors in nonholonomic models of rigid body dynamics*

In this section, we review some results on strange attractors in two nonholonomic models of rigid body dynamics: the models of Celtic stone and Chaplygin top. Both these models describe motions of a rigid body along the plane without slipping (this is a nonholonomic constraint) and with conserving the full energy. These nonholonomic approximations of the real motion of a rigid body along the plane allow us to write the equations of its dynamics

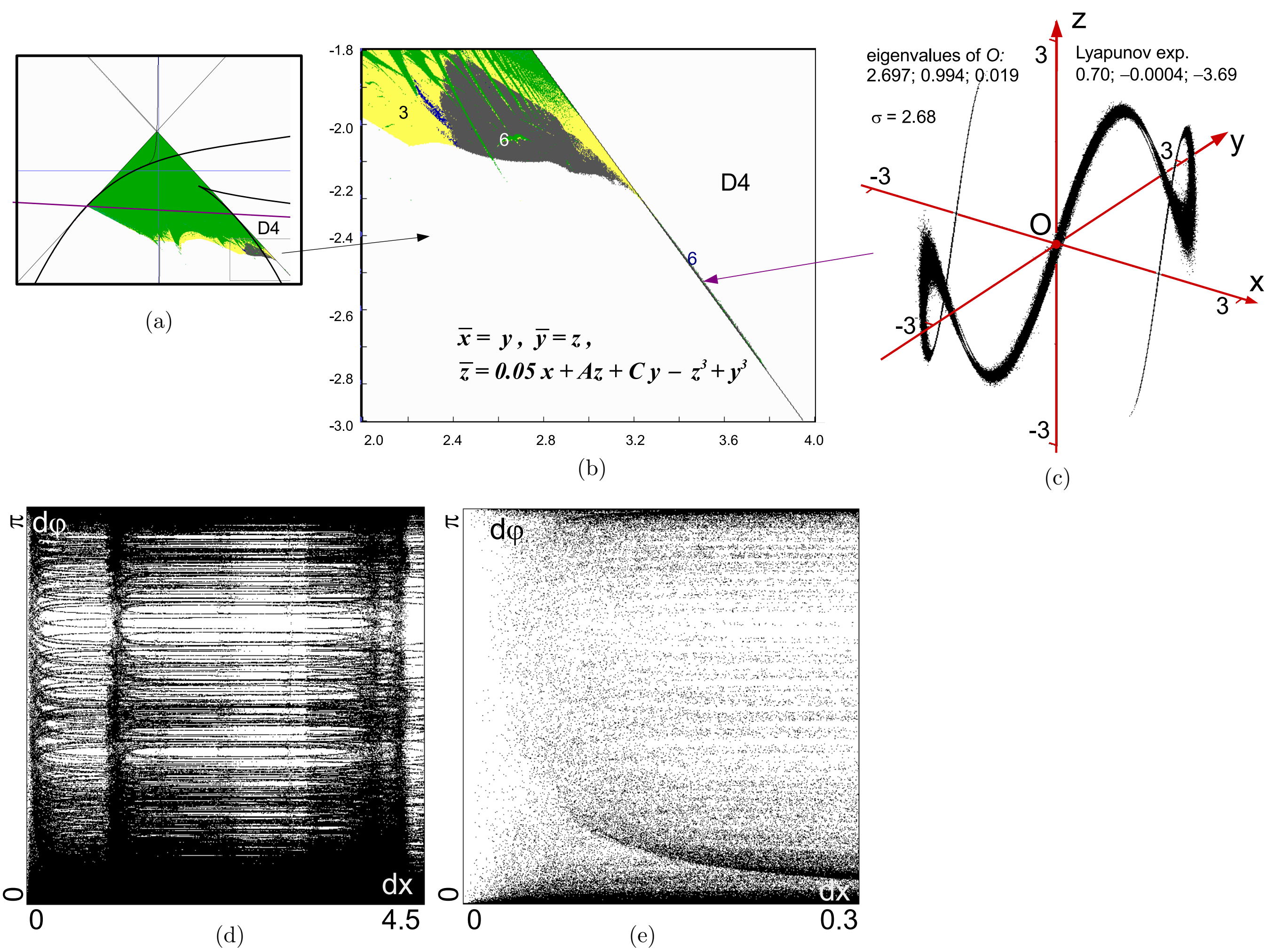



Fig. 12. An example of discrete super Lorenz attractor in the corresponding generalized Hénon map: (a) a fragment of the saddle chart on a background of the Lyapunov diagram, (b) a magnification of some parts of the Lyapunov diagram (a very thin dark-gray strip is seen in domain D4), (c) the phase portrait of attractor, (d) the LMP-graph and (e) a part of the LMP-graph near the axis $d\varphi$.

in the form of a five-dimensional system [Markeev, 1992] (three coordinates determine the projections of angular momenta and two others are two Euler angles) with the first integral that is the full energy $E$. Thus, due to the restriction on a level of the integral, the system becomes four-dimensional and we can study its chaotic dynamics on the corresponding three-dimensional Poincaré section. Further, we discuss some results from [Gonchenko *et al.*, 2013a; Borisov *et al.*, 2014] concerning strange homoclinic attractors using the above models.

Our first model is the Celtic stone model from [Gonchenko *et al.*, 2013a]. This model depends on a lot of parameters characterizing physical and geometrical properties of the stone. As a natural control parameter we consider the value of the full energy $E$. When varying $E$ in an appropriate interval of its values we can observe a sequence of bifurcations, in the corresponding one-parameter family $\mathcal{F}_E$ of three-dimensional Poincaré maps, which lead to the evolution from the asymptotically stable fixed point to the discrete Lorenz attractor.

The main stages of appearance of such attractor are shown in Fig. 13 (at the top panel) when the parameter $E$ increases from $E = 747$ to $E = E^* = 752$. At first, for $E < E_1 \simeq 747.61$, the attractor is a stable fixed point $O$, see Fig. 13 for $E = 747$. Then this point undergoes a period-doubling bifurcation at $E = E_1$ and the stable cycle $P = (p_1, p_2)$ of period two becomes an attractor, see Fig. 13 for $E = 748.4$. Note that the point $O$ is now a saddle fixed point (with multipliers $\lambda_1 < -1, 0 < \lambda_2 < 1, -1 < \lambda_3 < 0$, where $|\lambda_3| < |\lambda_2|$ and $|\lambda_1||\lambda_2| > 1$). At $E = E_2 \simeq 748.4395$ a "homoclinic figure-eight-butterfly" of the unstable manifolds (separatrices) of the saddle $O$ is created, which gives rise then to a saddle closed invariant curve $L = (L_1, L_2)$ of period two (where $\mathcal{F}_E(L_1) = L_2, \mathcal{F}_E(L_2) = L_1$), the

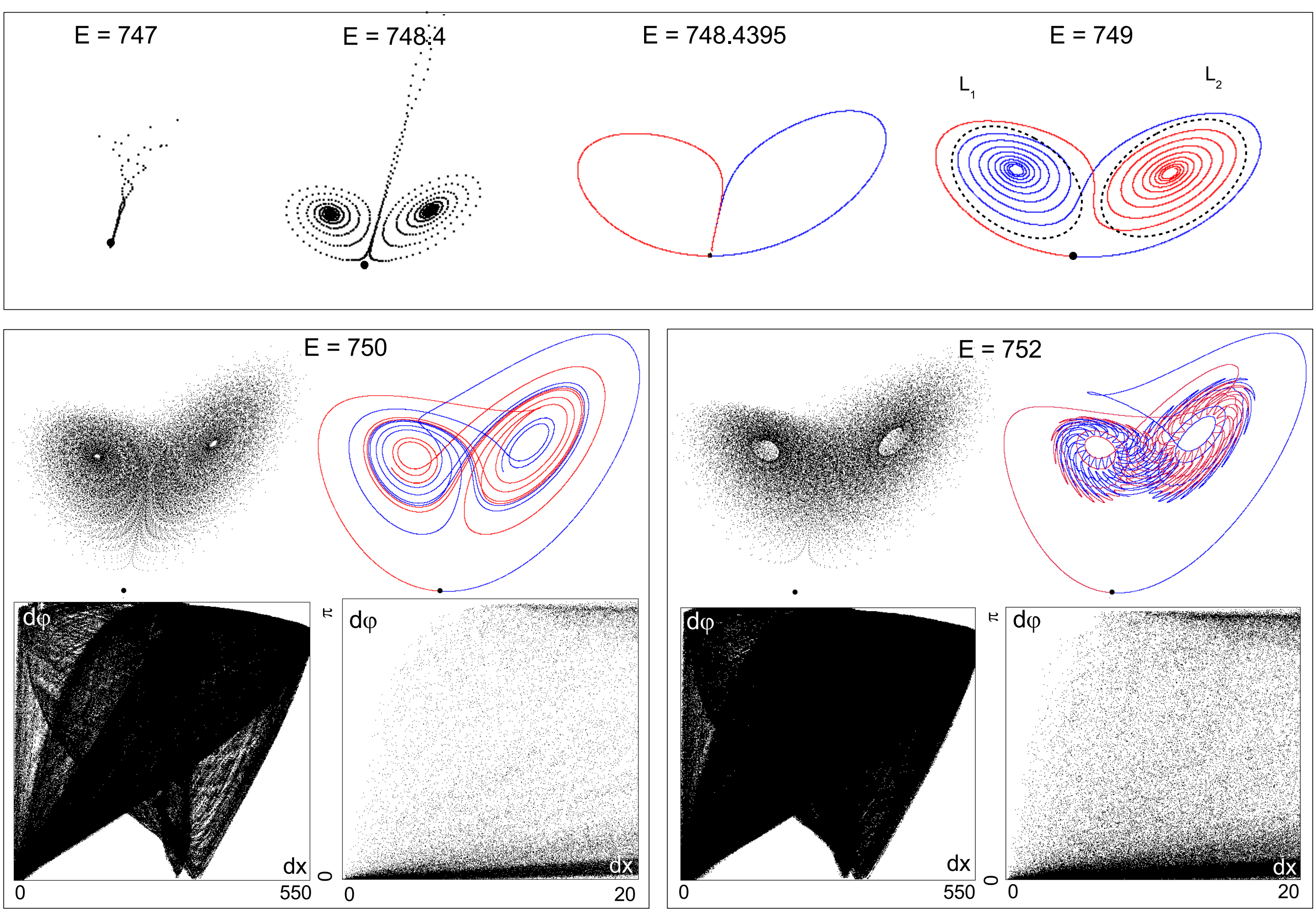



Fig. 13. The upper line: first stages (from $E = 747$ to $E = 749$) in developing the dynamics to the discrete Lorenz attractors. The middle line: $E = 750$ and $E = 752$, projections of attractors (left) and unstable separatrices of $O$ (right) on some two-dimensional plane. The bottom line: the corresponding LMP-graphs (left) and their magnifications (right).

curves $L_1$ and $L_2$ surround the points $p_1$ and $p_2$, respectively, see Fig. 13 for $E = 749$. At the same time, the unstable separatrices of $O$ are rebuilt and now, for $E_2 < E < E_3$, the left (the right) point wounds to the right (the left) point of the cycle $P$. Moreover, together with the closed period-two invariant curve $L$, the birth of an invariant limit set $\Omega$ occurs here, which is not attracting yet. As the numerical calculations show, for $E = E_3 \sim 748.97$ the separatrices "lie" onto the stable manifold of the curve $L$ and then leave it. Almost after that, at $E = E_4 \sim 748.98$, the period-two cycle $P$ loses sharply the stability under a subcritical discrete Andronov–Hopf bifurcation: the period-two closed invariant curve $L$ merges with the cycle $P$ after the cycle becomes of saddle-focus type. The value of $E = E_4$ is the exact bifurcation moment of the discrete Lorenz attractor creation. We show in Fig. 13 two examples of such attractors, for $E = 750$ (at the left panel) and $E = 752$ (at the right panel).

Thus, this bifurcation scenario in the Celtic stone model fits into the overall scheme of the appearance of discrete Lorenz attractors in Sec. 3.1. However, it has a certain specific feature. So, at the beginning, for $E$ close to $E_4$, this attractor is quite unusual. Despite that it is a discrete attractor, unstable invariant manifolds of the point $O$ behave very similar to the flow case: here homoclinic intersections are invisible, since a splitting of the corresponding manifolds is comparable with the accuracy of calculations, see Fig. 13 for $E = 750$. However, with increasing $E$ homoclinic intersection becomes more visible and typical zigzags appear in the unstable manifolds of $O$, see Fig. 13 for $E = 752$. Another specific is that the sequence of bifurcations when creating an attractor is strikingly similar (one can say "one-to-one" for $\mathcal{F}_E^2$) to what happens in the Lorenz model [Shilnikov, 1980]. Besides, this attractor is pseudohyperbolic as seen from the analysis of its LMP-graph, see Fig. 13 where the LMP-graphs

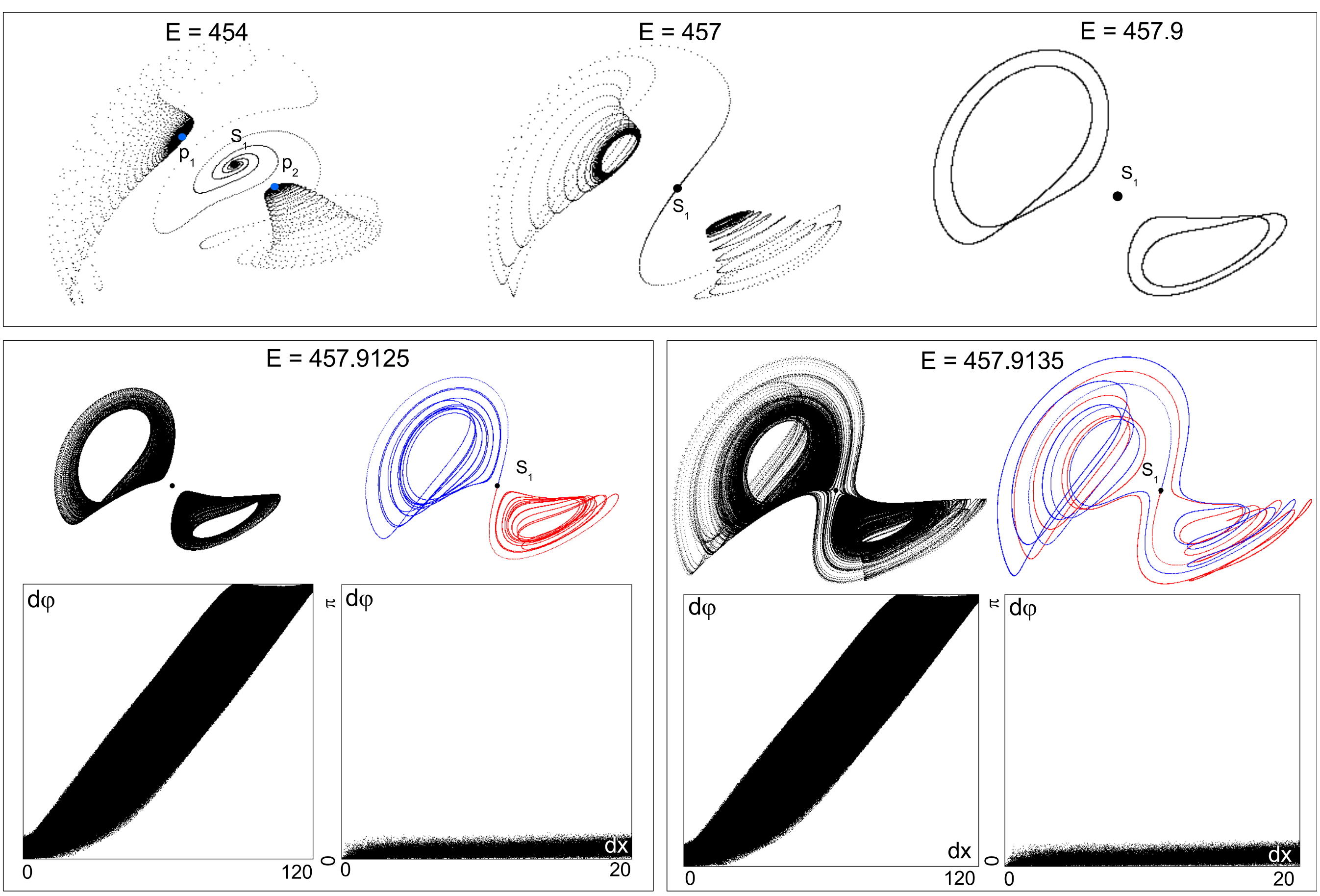



Fig. 14. The upper line: first stages (from $E = 454$ to $E = 457.9$) in developing chaotic dynamics. The middle line: a two-component torus-chaos for $E = 457.9125$ and a discrete figure-eight attractor for $E = 457.9135$; in both pictures is shown a two-dimensional projection for attractor (left) and for unstable separatrices of $S_1$ (right). The bottom line: the corresponding LMP-graphs and their magnifications.

are shown for $E = 750$ and $E = 752$, both full-scale ones (left) and their enlarged fragments near the axis $d\varphi$ (right). In the later figures we see that some neighborhoods of the axis $d\varphi$ do not contain any points on the graph. It follows surely that the attractors for $E = 750$ and $E = 752$ are pseudohyperbolic.

The second example is the nonholonomic model of Chaplygin top from the paper [Borisov *et al.*, 2014]. Figure 14 (top panel) shows some steps in the development of the attractor in Poincaré map of the model as the energy $E$ grows from $E = 454$. At first, for $E_1 \simeq 417.5 < E < E_2 \simeq 455.95$, the attractor is a period-two orbit $(p_1, p_2)$ that emerges at $E = E_1$ along with a saddle period-two orbit $S = (s_1, s_2)$ as a result of a saddle-node bifurcation (we do not show this moment). Simultaneously, the map has a saddle fixed point $S_1$: this point, a saddle-focus then a saddle, has a two-dimensional unstable manifold; then at $E > E_3 \simeq 456.15$ the fixed point becomes a saddle with one-dimensional unstable manifold as a result of a subcritical period-doubling bifurcation when the saddle orbit $(s_1, s_2)$ merges with $S_1$. At $E = E_2 \simeq 455.95$ the orbit $(p_1, p_2)$ loses the stability under the supercritical Andronov–Hopf bifurcation and a stable period-two closed invariant curve appears. Thus, at $E > E_3$ the one-dimensional unstable separatrices of the saddle fixed point $S_1$ (with multipliers $\lambda_1 < -1, |\lambda_{2,3}| < 1$ and $\lambda_2\lambda_3 < 0$) wind up onto a stable closed invariant curve of period-two, see Fig. 14 for $E = 457$. Further, several doublings of the invariant curve occur after the first doubling, see Fig. 14 for $E = 457.9$. Further growth of $E$ leads to a strange attractor appearance: at first a two-component "torus-chaos", see Fig. 14 for $E = 457.9125$, and then a discrete figure-eight attractor, see Fig. 14 for $E = 457.9135$.

Note that at $E = 457.9135$, the fixed point $S_1$ has the multipliers $\lambda_1 \simeq -1.00907, \lambda_2 \simeq -0.99732, \lambda_3 \simeq 0.98885$. Thus, the area-expansion condition $|\lambda_1\lambda_2| > 1$ is fulfilled. Moreover, the Lyapunov exponents for a random trajectory in the attractor are as follows: $\Lambda_1 \simeq 0.00063, \Lambda_2 \simeq -0.00003, \Lambda_3 \simeq -0.00492$, which gives $\Lambda_1 + \Lambda_2 > 0$ and hints at the pseudohyperbolicity. Besides, the corresponding LMP-graph in Fig. 14 (bottom line) confirms this. However, speaking strongly, we can

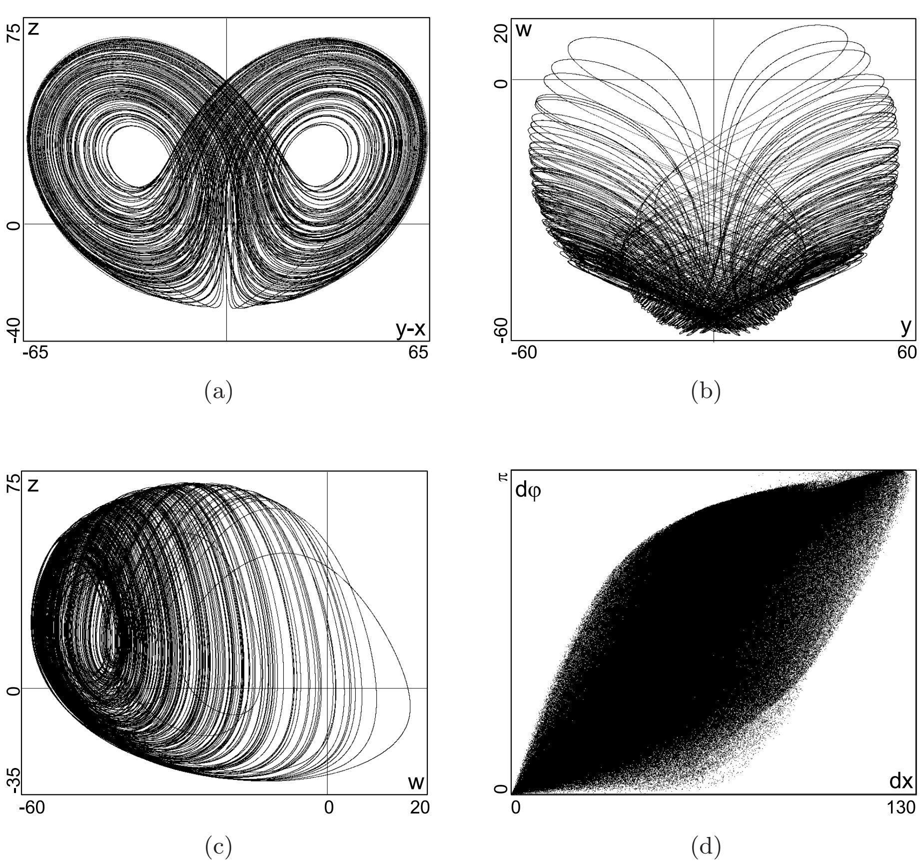



Fig. 15. (a)–(c) Projections of the attractor onto different two-dimensional planes and (d) LMP-graph for the attractor.

assume here the pseudohyperbolicity only "visually", since the problem under consideration is too delicate to blindly trust the numerical results. Also, note that the "torus-chaos" of Fig. 14 for $E = 457.9125$ looks pseudohyperbolic, which is quite doubtful.

## 5. On Wild Hyperbolic Attractor of Turaev–Shilnikov

Here we consider the example of the attractor found in [Gonchenko *et al.*, 2018], which is, in fact, wild spiral attractor of Turaev–Shilnikov type, i.e. a pseudohyperbolic attractor containing an equilibrium of saddle-focus type. Recall that the base of theory of pseudohyperbolic attractors was laid out in the paper [Turaev & Shilnikov, 1998], in which a geometric model of wild spiral attractor for a four-dimensional flow was constructed. This attractor contains a saddle-focus equilibrium $O$ with eigenvalues $\gamma, \lambda \pm i\omega, \tilde{\lambda}$, where $\gamma > 0 > \lambda > \tilde{\lambda}$ and, besides, $\gamma + 2\lambda > 0$ and the divergence in $O$ is negative, i.e. $\gamma + 2\lambda + \tilde{\lambda} < 0$. Thus, the point $O$ is pseudohyperbolic and $\dim E^{ss}(O) = 1, \dim E^{cu}(O) = 3$; here the vector $E^{ss}(O)$ is collinear to the eigenvector corresponding to the strong stable eigenvalue $\tilde{\lambda}$ and the three-dimensional plane $E^{cu}(O)$ contains eigenvectors corresponding to three other eigenvectors of $O$ (thus, the plane $E^{cu}(O)$ touches at $O$ the central unstable invariant manifold of the flow).

The geometric model constructed in [Turaev & Shilnikov, 1998] is in a sense similar to the Afraimovich–Bykov–Shilnikov model from [Afraimovich *et al.*, 1977; Afraimovich *et al.*, 1982], only the saddle is replaced by the saddle-focus and the flow under consideration is four-dimensional and, accordingly, the conditions of pseudohyperbolicity appear to be more complicated. Until recently, the question on the existence of a concrete four-dimensional flow with wild hyperbolic attractor was open. In the paper [Gonchenko *et al.*, 2018] examples of wild spiral attractors were found in the extended Lorenz system of the form

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = x(r - z) - y, \\ \dot{z} = xy - bz + \mu w, \\ \dot{w} = -bw - \mu z, \end{cases} \tag{7}$$

where $\sigma, r, b$ and $\mu$ are parameters. Note that at $\mu = 0$ the system has an invariant three-dimensional plane $w = 0$, in the restriction on which system (7) coincides with the Lorenz system. When $\mu$ is nonzero this structure is broken and the Lorenz attractor existing e.g. at $\mu = 0$ can evolute when $\mu$ varies. It would be quite interesting to track this evolution (e.g. when varying $\mu$ for fixed $\sigma, r, b$). However, we illustrate only one result from [Gonchenko *et al.*, 2018].

For the values of parameters

$$r = 25, \quad \sigma = 10, \quad b = \frac{8}{3}, \quad \mu = 7,$$

system (7) has an attractor, whose projections onto two-dimensional planes (a) $\{w = 0, x + y = 0\}$; (b) $\{x = z = 0\}$ and (c) $\{x = y = 0\}$ are shown in Fig. 15. This attractor is spiral, since it contains the saddle-focus equilibrium $O(0, 0, 0, 0)$ with the eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(\sqrt{(\sigma - 1)^2 + 4\sigma r} - \sigma - 1), \\ \lambda_{2,3} &= -b \pm i\mu, \\ \lambda_4 &= -\frac{1}{2}(\sqrt{(\sigma - 1)^2 + 4\sigma r} + \sigma + 1), \end{aligned}$$

i.e. $\lambda_1 = 10.93, \lambda_{2,3} = -8/3 \pm 7i, \lambda_4 = -21.93$ for given values of parameters. Thus, $O$ is a saddle-focus of type (3,1), i.e. with three-dimensional stable and one-dimensional unstable invariant manifolds, and, hence, $\dim E^{ss}(O) = 1, \dim E^{cu}(O) = 3$. The necessary conditions $\Lambda_1 > 0$, $\Lambda_1 + \Lambda_2 + \Lambda_3 > 0$ and $\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 < 0$ are also fulfilled here for numerically obtained Lyapunov exponents $\Lambda_1 = 2.19, \Lambda_2 = 0, \Lambda_3 = -1.96, \Lambda_4 = -16.56$. Moreover, it is verified in [Gonchenko *et al.*, 2018] that the field $E^{ss}$ of strong stable directions at the points of attractor is continuous: the corresponding LMP-graph is shown in Fig. 15(d). It is quite similar to that for the Lorenz attractor [compare Figs. 15(d) and 7].

## Acknowledgments

This paper was supported by the RSF grant 17-11-01041. Numerical experiments in Sec. 4 were supported by the RSF grant 14-12-00811. The authors thank the RFBR (grants 16-01-00364 and 18-31-00431), the Russian Ministry of Science and Education (project 1.3287.2017, target part), and the Basic Research Program at National Research University Higher School of Economics in 2018 for support of scientific researches.

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## Appendix A

## To the Definition of Pseudohyperbolic Map

We consider an $m$-dimensional diffeomorphism $f$. Let $Df$ be its differential. Recall that the differential of the map $f : \mathbb{R}^m \to \mathbb{R}^m$ in a point $x_0$ is a linear operator $A = \frac{\partial f}{\partial x}|_{x=x_0}$ that maps a vector $\ell_{x_0}$ at $x_0$ to the vector $\ell_{x_1} = A\ell_{x_0}$ at the point $x_1 = f(x_0)$. An open domain $\mathcal{D} \subset \mathbb{R}^m$ is called *absorbing domain* of the diffeomorphism $f$ if $f(\overline{\mathcal{D}}) \subset \mathcal{D}$.

**Definition A.1.** A diffeomorphism $f$ is called pseudohyperbolic on $\mathcal{D}$ if the following conditions are satisfied.

(1) Each point of $\mathcal{D}$ has two transversal linear subspaces $N_1$ and $N_2$, which have complementary dimensions ($\dim N_1 = k \geq 1, \dim N_2 = m - k \geq 2$), and are continuously dependent on the point and invariant under $Df$, i.e.

$$Df(N_1(x)) = N_1(f(x)),$$

$$Df(N_2(x)) = N_2(f(x)),$$

such that, for every orbit $L : \{x_i \,|\, x_{i+1} = f(x_i),\ i = 0, 1, \ldots; x_0 \in \mathcal{D}\}$, its maximal Lyapunov exponent for $Df|_{N_1}$ is strictly less than the minimal Lyapunov exponent for $Df|_{N_2}$, i.e. the following inequality holds:

$$\limsup_{n\to\infty} \frac{1}{n} \ln \left( \sup_{\substack{u\in N_1(x_0)\\ \|u\|=1}} \|Df^n(x_0)u\| \right) < \liminf_{n\to\infty} \frac{1}{n} \ln \left( \inf_{\substack{v\in N_2(x_0)\\ \|v\|=1}} \|Df^n(x_0)v\| \right), \tag{A.1}$$

where $Df^n$ is an $(m \times m)$-matrix defined as

$$Df^n = Df_{x_{n-1}} \cdot \ldots \cdot Df_{x_1} \cdot Df_{x_0},$$

and $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ are the superior and inferior limits, respectively.

(2) Diffeomorphism $f$ in the restriction to $N_1$ is uniformly contractive, that is, there exist constants $\lambda > 0$ and $C_1 > 0$ such that

$$\|Df^n(N_1)\| \le C_1 e^{-\lambda n}. \tag{A.2}$$

(3) Diffeomorphism $f$ in the restriction to $N_2$ extends exponentially $(m-k)$-dimensional volumes, that is, there exist constants $\sigma > 0$ and $C_2 > 0$ such that[10]

$$|\det Df^n(N_2)| \ge C_2 e^{\sigma n}. \tag{A.3}$$

From Definition A.1 it immediately follows that:

(1*) All orbits in $\mathcal{D}$ are unstable: each orbit has the positive maximal Lyapunov exponent

$$\Lambda_{\max}(x) = \limsup_{n\to\infty} \frac{1}{n} \ln \|Df^n(x)\| > 0.$$

Note that the conditions of pseudohyperbolicity mean that whole ($(m-k)$-dimensional) volumes in $N_2$ are expanded under forward iterations of $f$. This does not prohibit the existence of contraction directions in $N_2$, but any contractions along them should be uniformly weaker than any contraction in $N_1$. Thus, the uniform hyperbolicity can be considered as a specific case of pseudohyperbolicity, when all directions in $N_2$ are uniformly expanding, i.e. when the inequality $\|Df^{-n}(N_2)\| < Ce^{-\sigma n}$ holds. Nevertheless, the same as in the case of hyperbolic systems [Anosov, 1967; Turaev & Shilnikov, 1998], the following result is standardly proved here.

(2*) The pseudohyperbolicity conditions are preserved for all sufficiently small $C^r$-perturbations of the system. Moreover, the spaces $N_1$ and $N_2$ vary continuously.

It follows from statement (1*) that if a pseudohyperbolic diffeomorphism $f$ has an attractor in $\mathcal{D}$, then this attractor is strange and does not contain stable periodic orbits, which, as follows from the condition (2*), do not appear also for small smooth perturbations. In other words, pseudohyperbolic attractors are genuine attractors.

[10] If $\dim N_2 = 1$, then the usual definition of uniform hyperbolicity is obtained, therefore we require in the definition that $\dim N_2 \ge 2$.